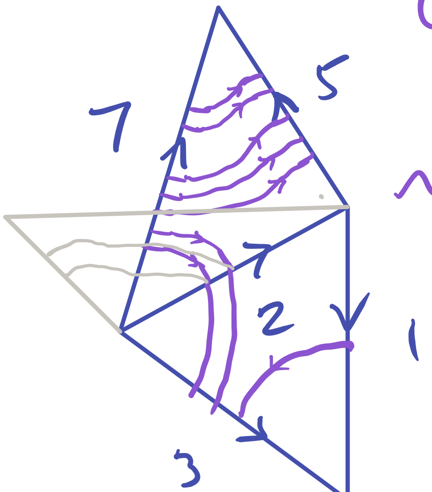
Geometric interpretation of 1-cocycles.

Mar 18

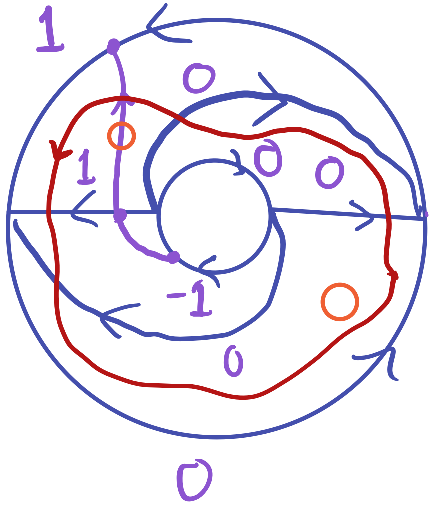
$\dim X = 2$



Cocycle condition
$(2+5=7)$

$\rightsquigarrow$ collection of arcs in each triangle

$\rightsquigarrow$ "curves" in X.



$n=2$
$k=1$

The cohomology class is intersect with purple.

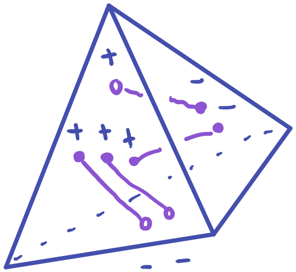
We'll see pairing

k-homology, $(n-k)$-cohomology

$(\bullet , \bullet) = 1 \implies$ both the hom & co cl. are $\neq 0$

The $n=3$ case

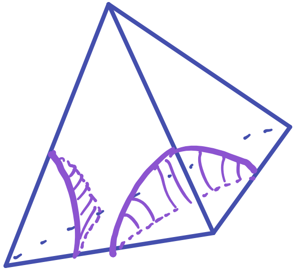
2- cocycles.



Cocycle condition: same # of incoming/outgoing dots.

The cohomology class is: intersect with purple.

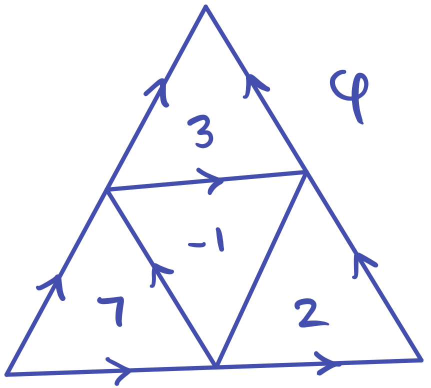
1- cocycles



This "eats" 1-chains.

Examples of 2-cocycles

① $X = D^2$



$$\Delta^0 \longrightarrow \Delta^1 \xrightarrow{\delta_1} \Delta^2 \longrightarrow 0$$

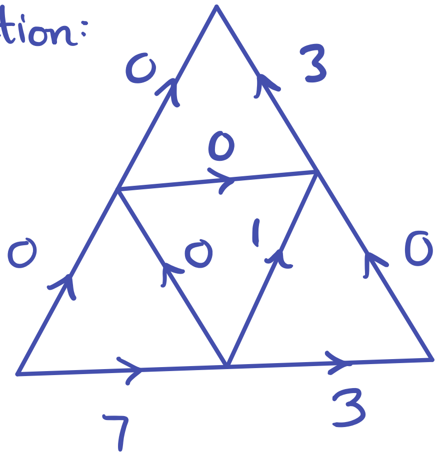
Any elt of Δ^2 is a cocycle.
Is it trivial (in $\operatorname{Im} \delta_1$)

We know $H^2(D^2; \mathbb{Z}) = 0$.

So $\varphi = \delta\psi \qquad \psi \in \Delta^1$

What is ψ?

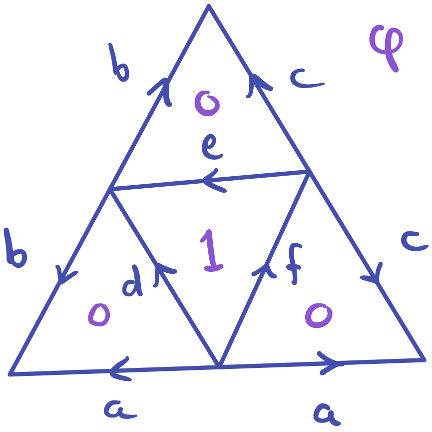
Solution:



antiderivative.

No obstruction.

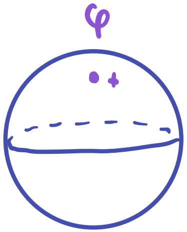
② $X = S^2$



Want to show

$[\varphi] \neq 0$ in $H^2(S^2)$

i.e. no antideriv. ψ



(Again every elt of Δ^2 is a 2-cocycle since $\Delta^3 = 0$).

Any ψ can be thought of as 6 nums $a, \dots, f$

If we want $\delta\psi = \varphi$ then

Each Δ gives an equation

$$\left.\begin{aligned} b+d &= a \\ e+c &= a \\ b+f &= c \\ e+f &= d+1 \end{aligned}\right\} \Rightarrow \underbrace{(b+d)}_{=a} - \underbrace{(e+c)}_{=a} = 1$$

contrad.

How are homology & cohomology related?

We'll see (as abelian gps)

$$H^n(X) \cong H_n(X)/T_n(X) \oplus T_{n-1}(X)$$

where T_n = torsion part of H_n.

For example. $X = \mathbb{RP}^2$

$H_0(X) = \mathbb{Z}$

$H_1(X) = \mathbb{Z}/2$

$H_2(X) = 0$.

$H^0(X) = \mathbb{Z}$

$H^1(X) = 0$

$H^2(X) = \mathbb{Z}/2$

find it!

We'll also see:

$$H^1(X) = \mathrm{Hom}(H_1(X), \mathbb{Z})$$

Note: $\mathrm{Hom}(\mathbb{Z}/2, \mathbb{Z}) = 0$.

Example of a chain complex

$$C: 0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

$$\rightsquigarrow H_0 = \mathbb{Z} \quad H_1 = \mathbb{Z}/2 \quad H_2 = 0 \quad H_3 = \mathbb{Z}$$

$$C^*: 0 \leftarrow \underset{\substack{\parallel \\ \mathrm{Hom}(\mathbb{Z},\mathbb{Z})}}{\mathbb{Z}} \xleftarrow{0} \underset{\substack{\parallel \\ \mathrm{Hom}(\mathbb{Z},\mathbb{Z})}}{\mathbb{Z}} \xleftarrow{2} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \leftarrow 0$$

verify this!

$$\rightsquigarrow H^0 = \mathbb{Z} \quad H^1 = 0 \quad H^2 = \mathbb{Z}/2 \quad H^3 = \mathbb{Z}$$