Mar 30

Grad Student Top. Conf this weekend.

Why are we doing cohomology?

Product structures.

e.g. $\mathbb{C}P^\infty$ 1 cell in each even dim.

$H_i = \mathbb{Z} \quad \forall\ i$ even.

$H^i(\mathbb{C}P^\infty) = \mathbb{Z}[\alpha]$ $\alpha \in H^2(\mathbb{C}P^\infty)$

$\alpha^k \in H^{2k}(\mathbb{C}P^\infty)$.

So all elts of H^* are $\mathbb{Z}$-multiples of powers of a single elt $\alpha \in H^2(\mathbb{C}P^\infty)$.

Also: pairing b/w H_* & H^*

Poincare duality...

Cohomology theory

Reduced groups, relative cohomology, long ex seq of pairs, excision, Mayer-Vietoris all work for cohomology.

Induced homomorphisms — contravariant

Given $f: X \to Y$ get

$$f^\#: C^n(Y,G) \to C^n(X,G)$$

$$\varphi \mapsto \varphi(\sigma) = \varphi(f(\sigma))$$

σ: n-simp in X

$\delta f^\# = f^\# \delta \Rightarrow f^\#$ pres. cocycle / cobound.

$\rightsquigarrow f^*: H^*(Y,G) \to H^*(X,G)$.

cocycles to cocycles:

Say $\delta\varphi = 0$ $\varphi \in C^n(Y,G)$

Want $\delta f^\# \varphi = 0$.

$$\overset{\shortparallel}{f^\# \delta \varphi}$$

Also: $(fg)^* = g^* f^*$, $id^* = id$

Say $X \mapsto H^n(X,G)$ is a contravariant functor.

Homotopy invariance same as before

$$f \simeq g \Rightarrow f^* = g^*$$

In homology: $g_\# - f_\# = \partial P + P\partial$

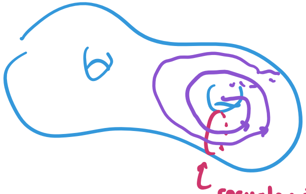
dualize: $g^\# - f^\# = \partial P^* + P^* \partial$.

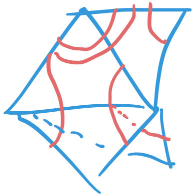
Product Structures

① Evaluation pairing

$$H^k(X) \times H_k(X) \longrightarrow \mathbb{Z}$$

can use to show (co)cycles are nontrivial.





② Cup product

$$H^p(X) \times H^q(X) \longrightarrow H^{p+q}(X)$$

$$(\varphi, \psi) \longmapsto \varphi \cup \psi$$

$\rightsquigarrow$ $H^*(X)$ is a graded ring.

e.g. polynomials

③ Cap product dim $X = n$

$$H^p(X) \times H_n(X) \longrightarrow H_{n-p}(X)$$

$$(\varphi, \alpha) \longmapsto \varphi \frown \alpha$$

Big goal:

Poincaré Duality Thm

M: compact, conn, oriented n-manifold

Then $H^p(M) \longrightarrow H_{n-p}(M)$

$$\varphi \longmapsto \varphi \frown [M]$$

is an isomorphism.

So all p-cocycles are: intersect with $n-p$ manifold.

i.e. all cocycles are nice.

Cup Product

For $\varphi \in C^k(X,R)$, $\psi \in C^\ell(X,R)$ R ring.

the cup product $\varphi \cup \psi \in C^{k+\ell}(X,R)$

$$(\varphi \cup \psi)(\sigma) = \varphi(\sigma|_{[v_0,\dots,v_k]})\psi(\sigma|_{[v_k,\dots,v_{k+\ell}]})$$

for $\sigma : \Delta^{k+\ell} \longrightarrow X$ a simplex.

We will see: cup product is intersection.

Need to show we get a product on level of cohomology: ① $\delta\varphi = \delta\psi = 0 \Rightarrow \delta(\varphi\cup\psi) = 0$

& ② φ or ψ cobound $\Rightarrow \varphi \cup \psi$ is.

So we'll get $H^k(X,R) \times H^\ell(X,R) \xrightarrow{\cup} H^{k+\ell}(X,R)$

Lemma. $\delta(\varphi\cup\psi) = \delta\varphi \cup \psi + (-1)^k \varphi \cup \delta\psi$

Lemma $\Rightarrow$ ① & ②.

Pf of Lemma

Say $\varphi \in C^k$, $\psi \in C^\ell$

$\sigma : \Delta^{k+\ell+1} \longrightarrow X$

$$(\delta\varphi \cup \psi)(\sigma) = \sum_{i=0}^{k+1} (-1)^i \varphi(\sigma|_{[v_0,\dots,\hat{v}_i,\dots v_{k+1}]}) \cdot \psi(\sigma|_{[v_{k+1},\dots,v_{k+\ell+1}]})$$

$$(-1)^k \varphi \cup \delta\psi(\sigma) = \sum_{i=k}^{k+\ell+1} (-1)^i \varphi(\sigma|_{[v_0,\dots,v_k]}) \cdot \psi(\sigma|_{[v_k,\dots,\hat{v}_i,\dots,v_{k+\ell+1}]})$$

Cup product is: assoc, distrib.

(since it is for cochains)

If R has 1 then $H^*(X,R)$ has 1

namely 1 in $H^0(X,R)$

Next time: Cup product on

orientable & nonorientable

surfaces.