Differential Topology

Bjørn Ian Dundas

January 10, 2013

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Chapter 1

Preface

1.0.1 What version is this, and how stable is it?

The version you are looking at right now is a β-release resulting from the major revision on Kistrand, Northern Norway in June 2012. The last stable manuscript: August 2007. If you have any comments or suggestion, I will be more than happy to hear from you so that the next stable release of these notes will be maximally helpful.

The plan is to keep the text available on the net, also in the future, and I have occasionally allowed myself to provide links to interesting sites. If any of these links are dead, please inform me so that I can change them in the next edition.

1.0.2 Acknowledgments

First and foremost, I am indebted to the students who have used these notes and given me invaluable feedback. Special thanks go to Håvard Berland, Elise Klaveness and Karen Sofie Ronæss. I owe a couple of anonymous referees much for their diligent reading and many helpful comments. I am also grateful to the Department of Mathematics for allowing me to do the 2012 revision in an inspiring environment.

1.0.3 The history of manifolds

Although a fairly young branch of mathematics, the history behind the theory of manifolds is rich and fascinating. The reader should take the opportunity to check up some of the biographies at The MacTutor History of Mathematics archive or at the Wikipedia of the mathematicians that actually are mentioned by name in the text (I have occasionally provided direct links). There is also a page called History Topics: Geometry and Topology Index which is worthwhile spending some time with. Of printed books, I have found Jean Dieudonné's book [4] especially helpful (although it is mainly concerned with topics beyond the scope of this book).

1.0.4 Notation

We let $\mathbf{N} = \{0, 1, 2, \dots\}$, $\mathbf{Z} = \{\dots, -1, 0, 1, \dots\}$, $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{C}$ be the sets of natural numbers, integers, rational numbers, real numbers and complex numbers. If X and Y are two sets, $X \times Y$ is the set of ordered pairs (x, y) with x an element in X and y an element in Y. If n is a natural number, we let $\mathbf{R}^n$ and $\mathbf{C}^n$ be the vector space of ordered n-tuples of real or complex numbers. Occasionally we may identify $\mathbf{C}^n$ with $\mathbf{R}^{2n}$. If $x = (x_1, \dots, x_n) \in \mathbf{R}^n$, we let $|x|$ be the norm $\sqrt{x_1^2 + \cdots + x_n^2}$. The sphere of dimension n is the subset $S^n \subseteq \mathbf{R}^{n+1}$ of all $x = (x_0, \dots, x_n) \in \mathbf{R}^{n+1}$ with $|x| = 1$ (so that $S^0 = \{-1, 1\} \subseteq \mathbf{R}$, and S^1 can be viewed as all the complex numbers $e^{i\theta}$ of unit length). Given functions $f\colon X \to Y$ and $g\colon Y \to Z$, we write gf for the composite, and $g \circ f$ only if the notation is cluttered and the $\circ$ improves readability. The constellation $g \cdot f$ will occur in the situation where f and g are functions with the same source and target, and where multiplication makes sense in the target.

1.0.5 How to start reading

The text proper starts with chapter 3 on smooth manifolds. If you are weak on point set topology, you will probably want to read the appendix 10 in parallel with chapter 3. The introduction 2 is not strictly necessary for highly motivated readers who can not wait to get to the theory, but provides some informal examples and discussions that may put the theme of these notes in some perspective. You should also be aware of the fact that chapter 6 and 5 are largely independent, and apart from a few exercises can be read in any order. Also, at the cost of removing some exercises and examples, the section on derivations 4.4, the section on the cotangent space/bundle 4.3/6.6 can be removed from the curriculum without disrupting the logical development of ideas.

Do the exercises, and only peek at the hints if you really need to.

Kistranda January 10, 2013

Chapter 2

Introduction

A globe

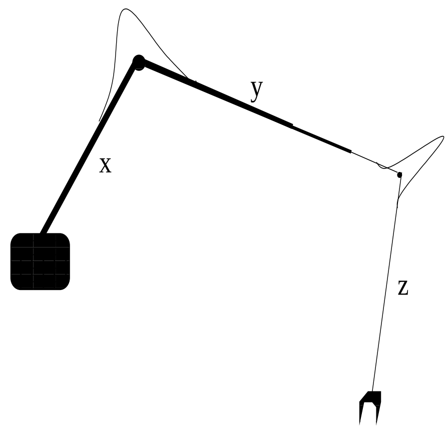
The earth is round. This may at one point have been hard to believe, but we have grown accustomed to it even though our everyday experience is that the earth is (fairly) flat. Still, the most effective way to illustrate it is by means of maps: a globe is a very neat device, but its global(!) character makes it less than practical if you want to represent fine details.

This phenomenon is quite common: locally you can represent things by means of "charts", but the global character can't be represented by one single chart. You need an entire atlas, and you need to know how the charts are to be assembled, or even better: the charts overlap so that we know how they all fit together. The mathematical framework for working with such situations is manifold theory. These notes are about manifold theory, but before we start off with the details, let us take an informal look at some examples illustrating the basic structure.

2.1 A robot's arm:

To illustrate a few points which will be important later on, we discuss a concrete situation in some detail. The features that appear are special cases of general phenomena, and hopefully the example will provide the reader with some deja vue experiences later on, when things are somewhat more obscure.

Consider a robot's arm. For simplicity, assume that it moves in the plane, has three joints, with a telescopic middle arm (see figure).



Call the vector defining the inner arm x, the second arm y and the third arm z. Assume $|x| = |z| = 1$ and $|y| \in [1,5]$. Then the robot can reach anywhere inside a circle of radius 7. But most of these positions can be reached in several different ways.

In order to control the robot optimally, we need to understand the various configurations, and how they relate to each other.

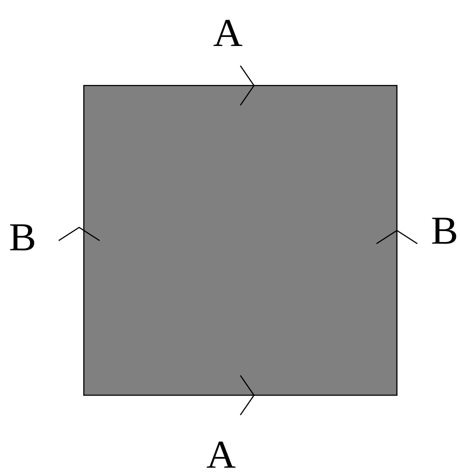
As an example, place the robot in the origin and consider all the possible positions of the arm that reach the point $P = (3,0) \in \mathbf{R}^2$, i.e., look at the set T of all triples $(x,y,z) \in \mathbf{R}^2 \times \mathbf{R}^2 \times \mathbf{R}^2$ such that

$$x + y + z = (3,0), \qquad |x| = |z| = 1, \qquad \text{and} \qquad |y| \in [1,5].$$

We see that, under the restriction $|x| = |z| = 1$, x and z can be chosen arbitrarily, and determine y uniquely. So T is "the same as" the set

$$\{(x,z) \in \mathbf{R}^2 \times \mathbf{R}^2 \,|\, |x| = |z| = 1\}.$$

Seemingly, our space T of configurations resides in four-dimensional space $\mathbf{R}^2 \times \mathbf{R}^2 \cong \mathbf{R}^4$, but that is an illusion – the space is two-dimensional and turns out to be a familiar shape. We can parametrize x and z by angles if we remember to identify the angles 0 and 2π. So T is what you get if you consider the square $[0,2\pi] \times [0,2\pi]$ and identify the edges as in the picture below.

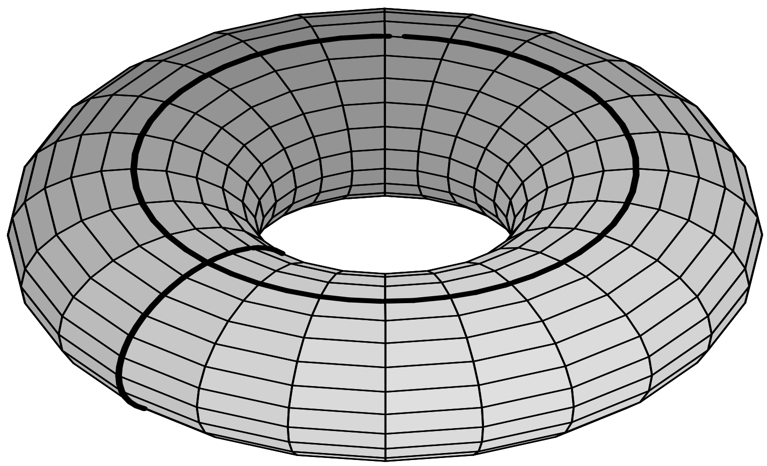


See

http://www.it.brighton.ac.uk/staff/jt40/MapleAnimations/Torus.html

for a nice animation of how the plane model gets glued.

In other words: The set T of all positions such that the robot reaches $P = (3, 0)$ may be identified with the torus.

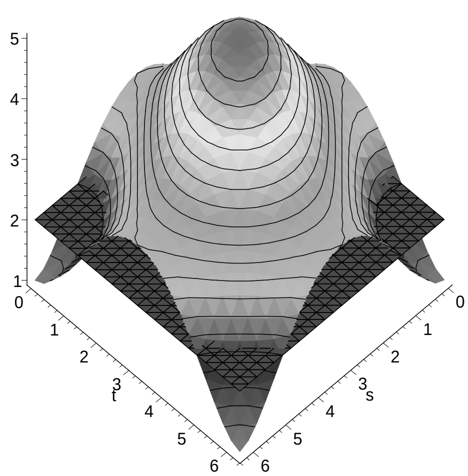


This is also true topologically in the sense that "close configurations" of the robot's arm correspond to points close to each other on the torus.

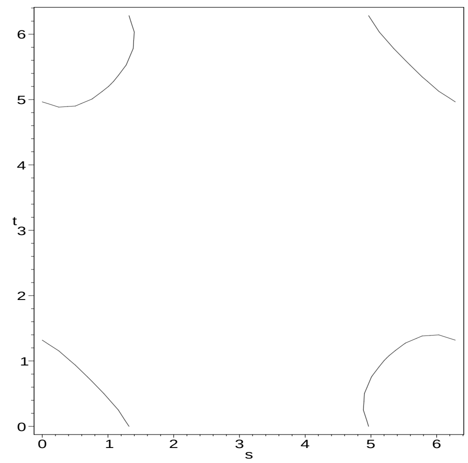
2.1.1 Question

What would the space S of positions look like if the telescope got stuck at $|y| = 2$?

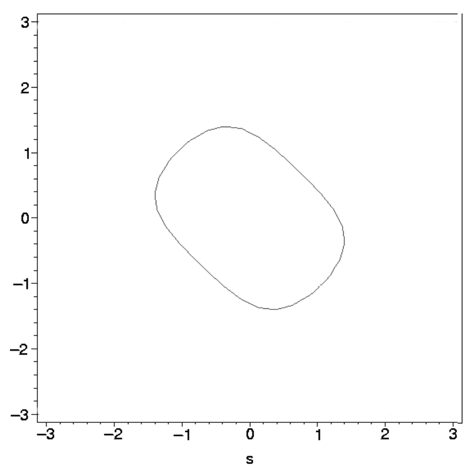
Partial answer to the question: since $y = (3, 0) - x - z$ we could try to get an idea of what points of T satisfy $|y| = 2$ by means of inspection of the graph of $|y|$. Below is an illustration showing $|y|$ as a function of T given as a graph over $[0, 2\pi] \times [0, 2\pi]$, and also the plane $|y| = 2$.



The desired set S should then be the intersection:



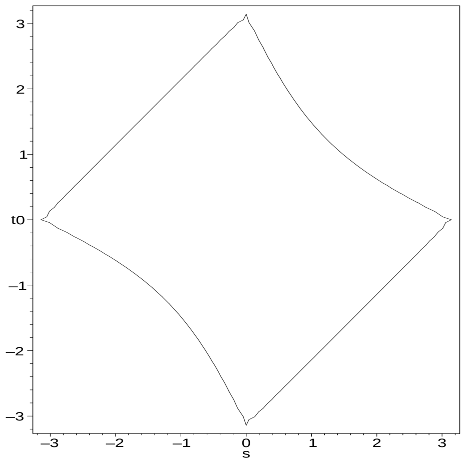
It looks a bit weird before we remember that the edges of $[0, 2\pi] \times [0, 2\pi]$ should be identified. On the torus it looks perfectly fine; and we can see this if we change our perspective a bit. In order to view T we chose $[0, 2\pi] \times [0, 2\pi]$ with identifications along the boundary. We could just as well have chosen $[-\pi, \pi] \times [-\pi, \pi]$, and then the picture would have looked like the following:



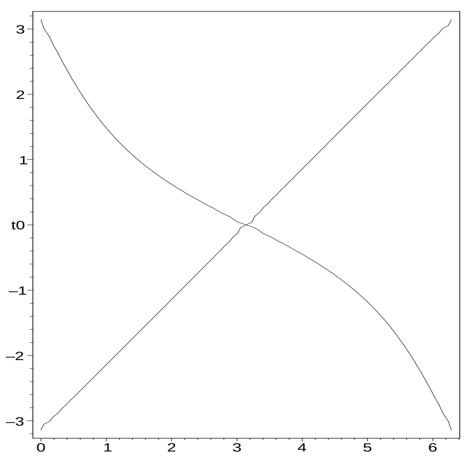
It does not touch the boundary, so we do not need to worry about the identifications. As a matter of fact, S is homeomorphic to the circle (*homeomorphic* means that there is a bijection between the two sets, and both the map from the circle to S and its inverse are continuous. See 10.2.8).

2.1.2 Dependence on the telescope's length

Even more is true: we notice that S looks like a smooth and nice picture. This will not happen for all values of $|y|$. The exceptions are $|y| = 1$, $|y| = 3$ and $|y| = 5$. The values 1 and 5 correspond to one-point solutions. When $|y| = 3$ we get a picture like the one below (it really ought to touch the boundary):



In the course we will learn to distinguish between such circumstances. They are qualitatively different in many aspects, one of which becomes apparent if we view the example with $|y| = 3$ with one of the angles varying in $[0, 2\pi]$ while the other varies in $[-\pi, \pi]$:



With this "cross" there is no way our solution space is homeomorphic to the circle. You can give an interpretation of the picture above: the straight line is the movement you get if you let $x = z$ (like two wheels of equal radius connected by a coupling rod y on an old fashioned train), while on the other x and z rotates in opposite directions (very unhealthy for wheels on a train).

Actually, this cross comes from a "saddle point" in the graph of $|y|$ as a function of T: it is a "critical" value where all sorts of bad things can happen.

2.1.3 Moral

The configuration space T is smooth and nice, and we get different views on it by changing our "coordinates". By considering a function on T (in our case the length of y) and restricting to the subset of T corresponding to a given value of our function, we get qualitatively different situations according to what values we are looking at. However, away from the "critical values" we get smooth and nice subspaces, see in particular 5.4.3.

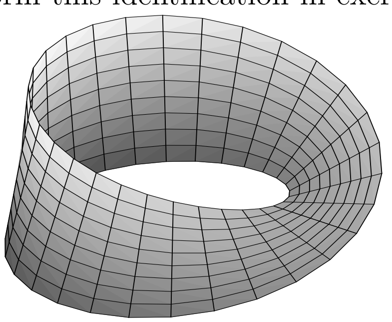
2.2 The configuration space of two electrons

Consider the situation where two electrons (with the same spin) are lonesome in space. To simplify matters, place the origin at the center of mass. The Pauli exclusion principle dictates that the two electrons can not be at the same place, so the electrons are somewhere outside the origin diametrically opposite of each other (assume they are point particles). However, you can't distinguish the two electrons, so the only thing you can tell is what line they are on, and how far they are from the origin (you can't give a vector v saying that this points at a chosen electron: $-v$ is just as good).

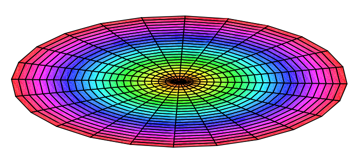
Disregarding the information telling you how far the electrons are from each other (which anyhow is just a matter of scale) we get that the space of possible positions may be identified with the *space of all lines through the origin* in $\mathbf{R}^3$. This space is called the (real) projective plane $\mathbf{R}\mathrm{P}^2$. A line intersects the unit sphere $S^2 = \{p \in \mathbf{R}^3 \mid |p| = 1\}$ in exactly two (antipodal) points, and so we get that $\mathbf{R}\mathrm{P}^2$ can be viewed as the sphere S^2 **but** with $p \in S^2$ identified with $-p$. A point in $\mathbf{R}\mathrm{P}^2$ represented by $p \in S^2$ (and $-p$) is written $[p]$.

The projective plane is obviously a "manifold" (i.e., can be described by means of charts), since a neighborhood around $[p]$ can be identified with a neighborhood around $p \in S^2$ – as long as they are small enough to fit on one hemisphere. However, I can not draw a picture of it in $\mathbf{R}^3$ without cheating.

On the other hand, there **is** a rather concrete representation of this space: it is what you get if you take a Möbius band and a disk, and glue them together along their boundary (both the Möbius band and the disk have boundaries a copy of the circle). You are asked to perform this identification in exercise 2.4.6.



A Möbius band: note that its boundary is a circle.



A disk: note that its boundary is a circle.

2.2.1 Moral

The moral in this subsection is this: configuration spaces are oftentimes manifolds that do not in any natural way live in Euclidean space. From a technical point of view they often are what called *quotient spaces* (although this example was a rather innocent one in this respect).

2.3 State spaces and fiber bundles

The following example illustrates a phenomenon often encountered in physics, and a tool of vital importance for many applications. It is also an illustration of a key result which we will work our way towards: Ehresmann's fibration theorem 9.5.6.

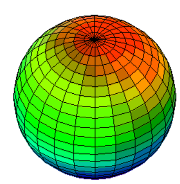
It is slightly more involved than the previous example, since it points forward to many concepts and results we will discuss more deeply later, so if you find the going a bit rough, I advice you not to worry too much about details right now, but come back to them when you are ready.

2.3.1 Qbits

In quantum computing one often talks about qbits. As opposed to an ordinary bit, which takes either the value 0 or 1 (representing "false" and "true" respectively), a *qbit*, or quantum bit, is represented by a complex linear combination ("superposition" in the physics parlance) of two states. The two possible states of a bit are then often called $|0\rangle$ and $|1\rangle$, and so a qbit is represented by the "pure qbit state" $\alpha|0\rangle+\beta|1\rangle$ where α, and β are complex numbers where $|\alpha|^2+|\beta|^2=1$ (since the total probability is 1: the numbers $|\alpha|^2$ and $|\beta|^2$ are interpreted as the probabilities that a measurement of the qbit will yield $|0\rangle$ and $|1\rangle$ respectively).

Note that the set of pairs $(\alpha,\beta)\in\mathbf{C}^2$ satisfying $|\alpha|^2+|\beta|^2=1$ is just another description of the sphere $S^3\subseteq\mathbf{R}^4=\mathbf{C}^2$. In other words, a pure qbit state is a point (α,β) on the sphere S^3.

However, for various reasons *phase changes* are not important. A phase change is the result of multiplying $(\alpha,\beta)\in S^3$ with a unit length complex number. That is, if $z=e^{i\theta}\in S^1\subseteq\mathbf{C}$, the pure qbit state $(z\alpha,z\beta)$ is a phase shift of (α,β), and these should be identified. The *state space* is what you get when you identify each pure qbit state with the other pure qbits states you get by phase change.



The state space S^2

So, what is the relation between the space S^3 of pure qbits states and the state space? It turns out that the state space may be identified with the two-dimensional sphere S^2, and the projection down to state space $\eta\colon S^3\to S^2$ may be given by

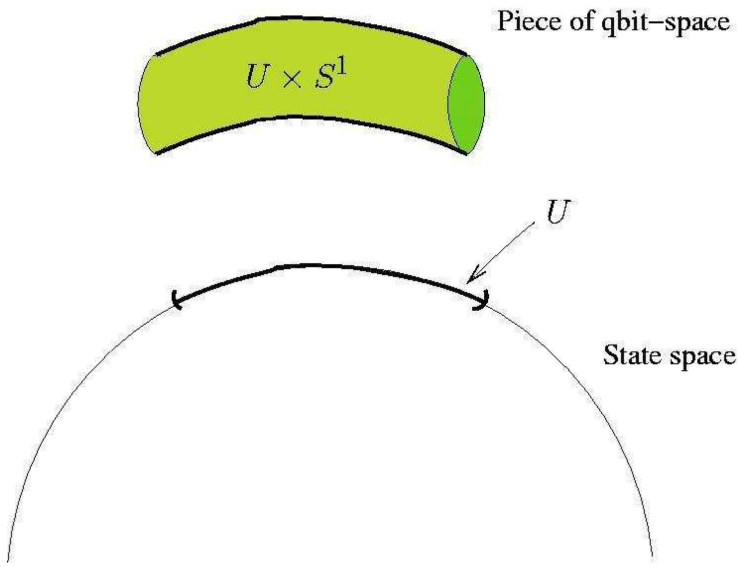
$$\eta(\alpha,\beta)=(|\alpha|^2-|\beta|^2,2\alpha\bar{\beta})\in S^2\subseteq\mathbf{R}^3=\mathbf{R}\times\mathbf{C}.$$

Note that $\eta(\alpha,\beta)=\eta(z\alpha,z\beta)$ if $z\in S^1$, and so η sends all the phase shifts of a given qbit to the same point in state space, and conversely, any qbit is represented by a point in state space.

Given a point in state space $p\in S^2$, the space of pure qbit states representing p can be identified with $S^1\subseteq\mathbf{C}$: choose a pure qbit state (α,β) representing p, and note that any other pure qbit state representing p is of the form $(z\alpha,z\beta)$ for some *unique* $z\in S^1$.

So, can a pure qbit be given uniquely by its associated qbit and some point on the circle, i.e., is the space of pure qbit states really $S^2 \times S^1$ (and not S^3 as I previously claimed)? Without more work it is not at all clear how these copies of S^1 lying over each point in S^2 are to be glued together: how does this "circle's worth" of pure qbit states change when we vary the position in state space slightly?

The answer comes through Ehresmann's fibration theorem 9.5.6. It turns out that $\eta\colon S^3 \to S^2$ is a *locally* trivial fibration, which means that in a small neighborhood U around any given point in state space, the space of pure qbit states *does* look like $U \times S^1$. On the other hand, the *global* structure is different. In fact, $\eta\colon S^3 \to S^2$ is an important mathematical object for many reasons, and is known as the *Hopf fibration*.



The pure qbit states represented in a small open neighborhood U in state space form a cylinder $U \times S^1$ (dimension reduced by one in the picture).

The input to Ehresmann's theorem comes in two types. First we have some point set information, which in our case is handled by the fact that S^3 is "compact" 10.7.1. Secondly there is a condition which only sees the linear approximations, and which in our case boils down to the fact that any "infinitesimal" movement on S^2 is the shadow of an "infinitesimal" movement in S^3. This is a question which is settled through a quick and concrete calculation of differentials. We'll be more precise about this later, but let saying that these conditions are easily checked given the right language it suffice for now (this is exercise 9.5.11).

2.3.2 Moral

The idea is the important thing: if you want to understand some complicated model through some simplification, it is often so that the complicated model *locally* (in the simple model) can be built out of the simple model through multiplying with some fixed space.

How these local pictures are glued together to give the global picture is another matter, and often requires other tools, for instance form algebraic topology. In the $S^3 \to S^2$ case, we see that S^3 and $S^2 \times S^1$ can not be identified since S^3 is simply connected (meaning that any closed loop in S^3 can be deformed continuously to a point) and $S^2 \times S^1$ is not.

An important class of examples (of which the above is an example) of locally trivial fibrations arise from symmetries: if M is some (configuration) space and you have a "group of symmetries" G (e.g., rotations) acting on M, then you can consider the space M/G of points in M where you have identified two points in M if they can be obtained from each other by letting G act (e.g., one is a rotated copy of the other). Under favorable circumstances M/G will be a manifold and the projection $M \to M/G$ will be a locally trivial fibration, so that M is built up of neighborhoods in M/G times G glued together appropriately.

2.4 Further examples

A short bestiary of manifolds available to us at the moment might look like this:

—The surface of the earth, S^2, and higher dimensional spheres, see 3.1.4;

—Space-time is a four dimensional manifold. It is not flat, and its curvature is determined by the mass distribution;

—Configuration spaces in physics (e.g., robot example 2.1, the two electrons of example 2.2 or the more abstract considerations at the very end of 2.3.2 above);

—If $f\colon \mathbf{R}^n \to \mathbf{R}$ is a map and y a real number, then the inverse image

$$f^{-1}(y) = \{x \in \mathbf{R}^n | f(x) = y\}$$

is often a manifold. For instance, if $f\colon \mathbf{R}^2 \to \mathbf{R}$ is the norm function $f(x) = |x|$, then $f^{-1}(1)$ is the unit circle S^1 (c.f. the submanifold chapter 5);

—The torus (c.f. the robot example 2.1);

—"The real projective plane" $\mathbf{R}\mathrm{P}^2 = \{$All lines in $\mathbf{R}^3$ through the origin$\}$ (see the two-electron example 2.2, but also exercise 2.4.6);

—The Klein bottle (see 2.4.3).

We end this introduction by studying surfaces a bit closer (since they are concrete, and drives home the familiar notion of charts in more exotic situations), and also come with some inadequate words about higher dimensional manifolds in general.

2.4.1 Charts

The space-time manifold brings home the fact that manifolds must be represented intrinsically: the surface of the earth is seen as a sphere "in space", but there is no space which should naturally harbor the universe, except the universe itself. This opens up the question of how one can determine the shape of the space in which we live.

One way of representing the surface of the earth as the two-dimensional space it is (not referring to some ambient three-dimensional space), is through an atlas. The shape of the earth's surface is then determined by how each map in the atlas is to be glued to the other maps in order to represent the entire surface.

Just like the surface of the earth is covered by maps, the torus in the robot's arm was viewed through flat representations. In the technical sense of the word, the representation was not a "chart" (see 3.1.1) since some points were covered twice (just as Siberia and Alaska have a tendency to show up twice on some European maps). It is allowed to have many charts covering Fairbanks in our atlas, but on each single chart it should show up at most once. We may fix this problem at the cost of having to use more overlapping charts. Also, in the robot example (as well as the two-electron and qbit examples) we saw that it was advantageous to operate with more charts.

Example 2.4.2 To drive home this point, please play Jeff Weeks' "Torus Games" on

`http://www.geometrygames.org/TorusGames/`

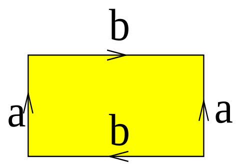
for a while.

2.4.3 Compact surfaces

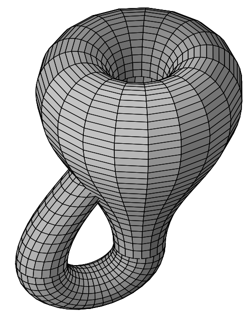
This section is rather autonomous, and may be read at leisure at a later stage to fill in the intuition on manifolds.

The Klein Bottle

To simplify we could imagine that we were two dimensional beings living in a static closed surface. The sphere and the torus are familiar surfaces, but there are many more. If you did example 2.4.2, you were exposed to another surface, namely the *Klein bottle*. This has a plane representation very similar to the Torus: just reverse the orientation of a single edge.



A plane representation of the Klein bottle: identify along the edges in the direction indicated.



A picture of the Klein bottle forced into our three-dimensional space: it is really just a shadow since it has self intersections. If you insist on putting this two-dimensional manifold into a flat space, you got to have at least four dimensions available.

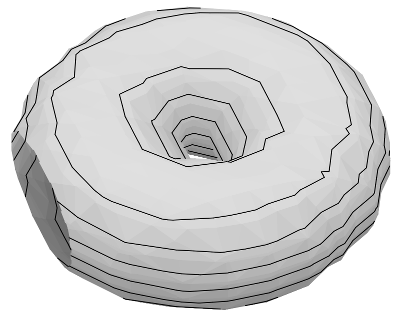
Although this is an easy surface to describe (but frustrating to play chess on), it is too complicated to fit inside our three-dimensional space: again a manifold is **not** a space inside a flat space. It is a locally Euclidean space. The best we can do is to give an "immersed" (i.e., allowing self-intersections) picture.

Speaking of pictures: the Klein bottle makes a surprising entré in image analysis. When analyzing the 9-dimensional space of all configuration of 3 by 3 gray-scale pixels, it is of importance – for instance if you want to implement some compression technique – to know what configurations occur most commonly. Carlsson, Ishkhanov, de Silva and Zomorodian show in the preprint `http://math.stanford.edu/comptop/preprints/mumford.pdf` that the subspace of "most common pixel configurations" actually "is" a Klein bottle (follow the url for a more precise description). Their results are currently being used in developing a compression algorithm based on a "Klein bottle dictionary".

Classification of compact surfaces

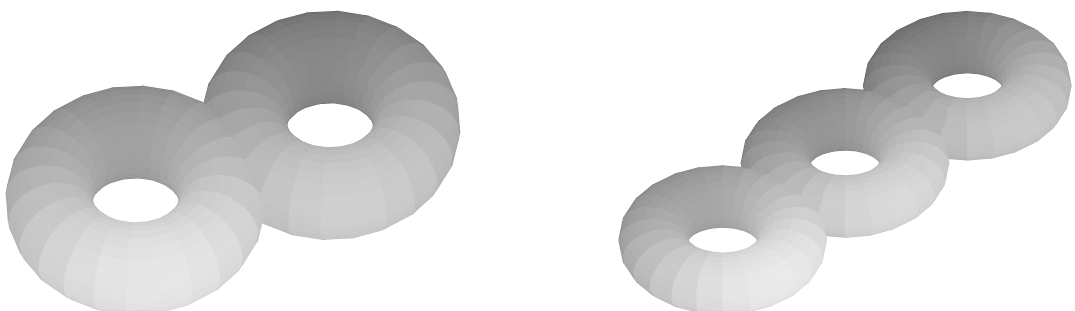
As a matter of fact, it turns out that we can write down a list of all compact surfaces (*compact* is defined in appendix 10, but informally should be thought of as "closed and of bounded size"). First of all, surfaces may be divided into those that are orientable and those that are not. *Orientable* means that there are no loops by which two dimensional beings living in the surface can travel and return home as their mirror images (is the universe non-orientable? is that why some people are left-handed?).

All connected compact orientable surfaces can be obtained by attaching a finite number of handles to a sphere. The number of handles attached is referred to as the *genus* of the surface.
A *handle* is a torus with a small disk removed (see the figure). Note that the boundary of the holes on the sphere and the boundary of the hole on each handle are all circles, so we glue the surfaces together in a smooth manner along their common boundary (the result of such a gluing process is called the *connected sum*, and some care is required).



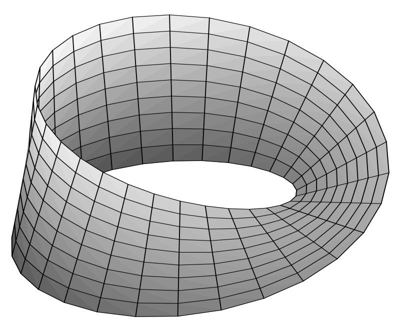
A handle: ready to be attached to another 2-manifold with a small disk removed.

Thus all orientable compact surfaces are surfaces of pretzels with many holes.



An orientable surface of genus g is obtained by gluing g handles (the smoothening out has yet to be performed in these pictures)

There are nonorientable surfaces too (e.g., the Klein bottle). To make them, consider a Möbius band. Its boundary is a circle, and so cutting a hole in a surface you may glue in a Möbius band. If you do this on a sphere you get the projective plane (this is exercise 2.4.6). If you do it twice you get the Klein bottle. Any nonorientable compact surface can be obtained by cutting a finite number of holes in a sphere and gluing in the corresponding number of Möbius bands.



A Möbius band: note that its boundary is a circle.

The reader might wonder what happens if we mix handles and Möbius bands, and it is a strange fact that if you glue g handles and $h > 0$ Möbius bands you get the same as if you had glued $h + 2g$ Möbius bands! For instance, the projective plane with a handle attached is the same as the Klein bottle with a Möbius band glued onto it. But fortunately this is it; there are no more identifications among the surfaces.

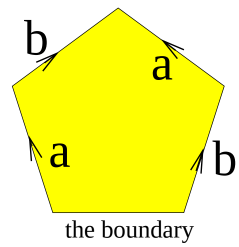
So, any (connected compact) surface can be obtained by cutting g holes in S^2 and *either* gluing in g handles *or* gluing in g Möbius bands. For a detailed discussion the reader may turn to Hirsch's book [5], chapter 9.

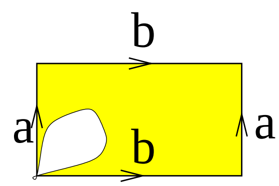
Plane models

If you find such descriptions elusive, you may find comfort in the fact that all compact surfaces can be described similarly to the way we described the torus. If we cut a hole in the torus we get a handle. This may be represented by plane models as to the right: identify the edges as indicated.
If you want more handles you just glue many of these together, so that a g-holed torus can be represented by a $4g$-gon where two and two edges are identified (see below for the case $g = 2$, the general case is similar. See also

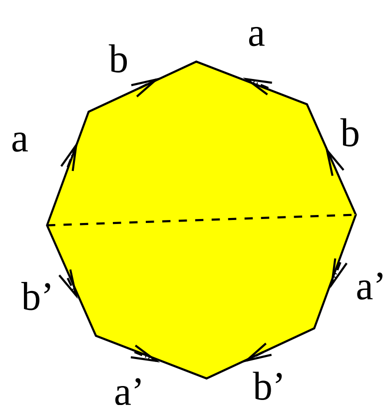
www.rogmann.org/math/tori/torus2en.html

for instruction on how to sew your own two and tree-holed torus).



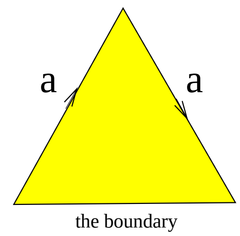


Two versions of a plane model for the handle: identify the edges as indicated to get a torus with a hole in.



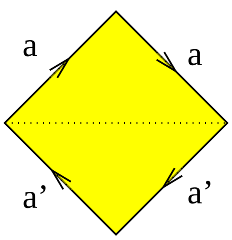
A plane model of the orientable surface of genus two. Glue corresponding edges together. The dotted line splits the surface up into two handles.

It is important to have in mind that the points on the edges in the plane models are in no way special: if we change our point of view slightly we can get them to be in the interior.
We have plane models for gluing in Möbius bands too (see picture to the right). So a surface obtained by gluing h Möbius bands to h holes on a sphere can be represented by a $2h$-gon, where two and two edges are identified.

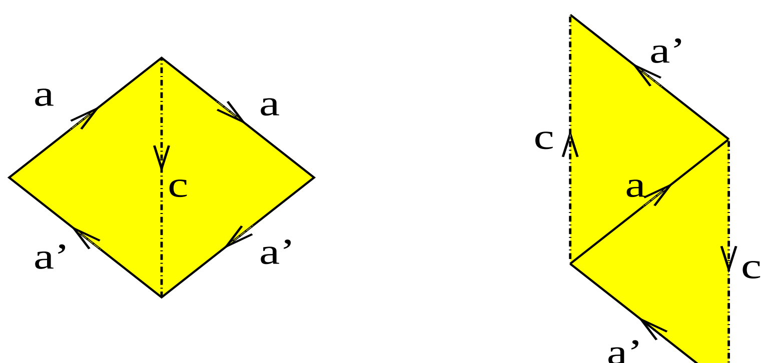


A plane model for the Möbius band: identify the edges as indicated. When gluing it onto something else, use the boundary.

Example 2.4.4 If you glue two plane models of the Möbius band along their boundaries you get the picture to the right. This represent the Klein bottle, but it is not exactly the same plane representation we used earlier.
To see that the two plane models give the same surface, cut along the line c in the figure to the left below. Then take the two copies of the line a and glue them together in accordance with their orientations (this requires that you flip one of your triangles). The resulting figure which is shown to the right below, is (a rotated and slanted version of) the plane model we used before for the Klein bottle.



Gluing two flat Möbius bands together. The dotted line marks where the bands were glued together.



Cutting along c shows that two Möbius bands glued together is the Klein bottle.

Exercise 2.4.5 Prove by a direct cut and paste argument that what you get by adding a handle to the projective plane is the same as what you get if you add a Möbius band to the Klein bottle.

Exercise 2.4.6 Prove that the real projective plane

$$\mathbf{R}\mathrm{P}^2 = \{\text{All lines in } \mathbf{R}^3 \text{ through the origin}\}$$

is the same as what you get by gluing a Möbius band to a sphere.

Exercise 2.4.7 See if you can find out what the "Euler number" (or Euler characteristic) is. Then calculate it for various surfaces using the plane models. Can you see that both the torus and the Klein bottle have Euler characteristic zero? The sphere has Euler number 2 (which leads to the famous theorem $V - E + F = 2$ for all surfaces bounding a "ball") and the projective plane has Euler number 1. The surface of exercise 2.4.5 has Euler number -1. In general, adding a handle reduces the Euler number by two, and adding a Möbius band reduces it by one.

Exercise 2.4.8 If you did exercise 2.4.7, design an (immensely expensive) experiment that could be performed by two-dimensional beings living in a compact orientable surface, determining the shape of their universe.

2.4.9 The Poincaré conjecture and Thurston's geometrization conjecture

In dimension tree, the last few years have seen a fascinating development. In 1904 H. Poincaré conjectured that any simply connected compact and closed 3-manifold is homeomorphic to the 3-sphere. This problem remained open for almost a hundred years, although the corresponding problem was resolved in higher dimensions by S. Smale (1961 for dimensions greater than 4) and M. Freedman (1982 in dimension 4).

In the academic year 2002/2003 G. Perelman posted a series of papers building on previous work by R. Hamilton, which by now are widely regarded as the core of a proof of the Poincaré conjecture. The proof relies on an analysis of the "Ricci flow" deforming the curvature of a manifold in a manner somehow analogous to the heat equation, smoothing out irregularities. Our encounter with flows will be much more elementary, but still prove essential in the proof of Ehresmann's fibration theorem 9.5.6.

Perelman was offered the Fields medal for his work in 2006, but spectacularly refused it. In this way he created much more publicity for the problem, mathematics and himself than would have otherwise been thinkable. It remains to be seen what he will do if offered a share in USD1M by the Clay Mathematics Institute. In 2006 several more thorough write-ups of the argument appeared (see e.g., the Wikipedia entry on the Poincaré conjecture for an updated account).

Of far greater consequence is Thurston's geometrization conjecture. This conjecture was proposed by W. Thurston in 1982. Any 3-manifold can be decomposed into *prime manifolds*, and the conjecture says that any prime manifold can be cut along tori, so that the interior of each of the resulting manifolds has one of *eight geometric structures* with finite volume. See e.g., the Wikipedia page for further discussion and references.

On the same page you will find asserted the belief that Perelman's work also implies the geometrization conjecture.

2.4.10 Higher dimensions

Although surfaces are fun and concrete, next to no real-life applications are 2 or 3-dimensional. Usually there are zillions of variables at play, and so our manifolds will be correspondingly complex. This means that we can't continue to be vague (the previous section indicated that even in three dimensions things become nasty). We need strict definitions to keep track of all the structure.

However, let it be mentioned at the informal level that we must not expect to have such a nice list of higher dimensional manifolds as we had for compact surfaces.

Classification problems for higher dimensional manifolds is an extremely complex and interesting business we will not have occasion to delve into. This study opens new fields of research using methods both from algebra and analysis that go far beyond the ambitions of this text.

Chapter 3

Smooth manifolds

3.1 Topological manifolds

Let us get straight to our object of study. The terms used in the definition are explained immediately below the box. If words like "open" and "topology" are new to you, you are advised to read the appendix 10 on point set topology in parallel with this chapter.

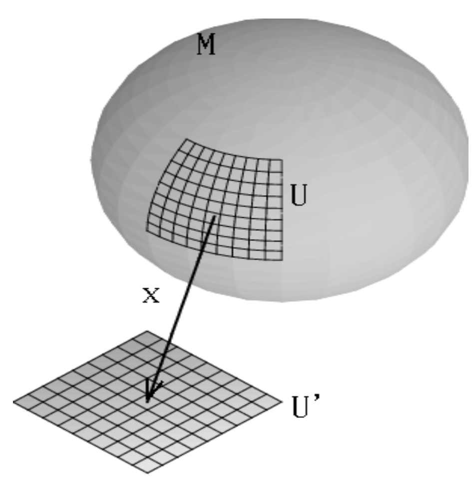
Definition 3.1.1 An *n-dimensional topological manifold M* is

> a Hausdorff topological space with a countable basis for the topology which is
>
> *locally homeomorphic* to $\mathbf{R}^n$.

The last point (*locally homeomorphic* to $\mathbf{R}^n$ – implicitly with the metric topology – also known as Euclidean space 10.1.10) means that for every point $p \in M$ there is

> an open neighborhood U of p in M,
>
> an open set $U' \subseteq \mathbf{R}^n$ and
>
> a homeomorphism (10.2.5) $\quad x\colon U \to U'$.

We call such an $x\colon U \to U'$ a *chart* and U a *chart domain*.
A collection of charts $\{x_\alpha\colon U_\alpha \to U'_\alpha\}$ covering M (i.e., such that the union $\bigcup U_\alpha$ of the chart domains is M) is called an *atlas*.



Note 3.1.2 The conditions that M should be "Hausdorff" (10.4.1) and have a "countable basis for its topology" (**??**) will not play an important rôle for us for quite a while. It is tempting to just skip these conditions, and come back to them later when they actually

are important. As a matter of fact, on a first reading I suggest you actually do this. Rest assured that all subsets of Euclidean spaces satisfy these conditions (see 10.5.6).

The conditions are there in order to exclude some pathological creatures that are locally homeomorphic to $\mathbf{R}^n$, but are so weird that we do not want to consider them. We include the conditions at once so as not to need to change our definition in the course of the book, and also to conform with usual language.

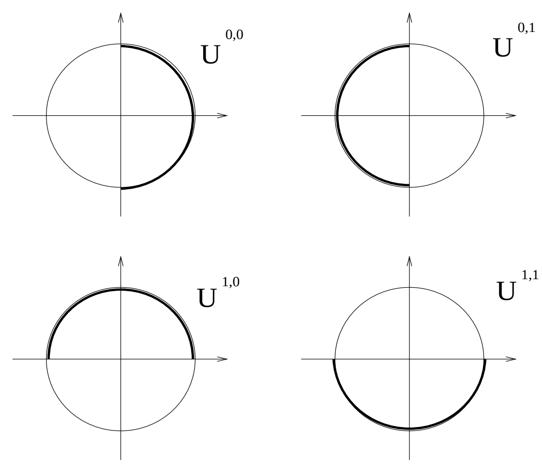
Example 3.1.3 Let $U \subseteq \mathbf{R}^n$ be an open subset. Then U is an n-manifold. Its atlas needs only have one chart, namely the identity map $id\colon U = U$. As a sub-example we have the open n-disk

$$E^n = \{p \in \mathbf{R}^n \,|\, |p| < 1\}.$$

Example 3.1.4 The n-sphere

$$S^n = \{p \in \mathbf{R}^{n+1} \,|\, |p| = 1\}$$

is an n-dimensional manifold.



To see that S^n is locally homeomorphic to $\mathbf{R}^n$ we may proceed as follows. Write a point in $\mathbf{R}^{n+1}$ as an $n+1$ tuple indexed from 0 to n: $p = (p_0, p_1, \dots, p_n)$. To give an atlas for S^n, consider the open sets

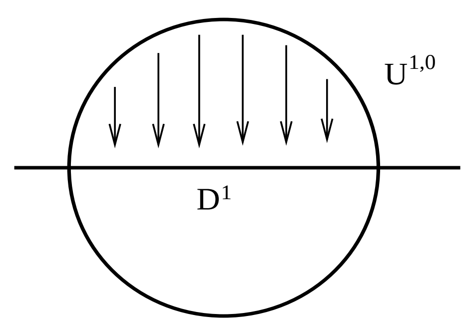
$$U^{k,0} = \{p \in S^n | p_k > 0\},$$
$$U^{k,1} = \{p \in S^n | p_k < 0\}$$

for $k = 0, \dots, n$, and let

$$x^{k,i}\colon U^{k,i} \to E^n$$

be the projection to the open n-disk E^n given by deleting the k-th coordinate:

$$(p_0, \dots, p_n) \mapsto (p_0, \dots, \widehat{p_k}, \dots, p_n) = (p_0, \dots, p_{k-1}, p_{k+1}, \dots, p_n)$$



(the "hat" in $\widehat{p_k}$ is a common way to indicate that this coordinate should be deleted).

[The n-sphere is Hausdorff and has a countable basis for its topology by corollary 10.5.6 simply because it is a subspace of $\mathbf{R}^{n+1}$.]

Exercise 3.1.5 Check that the proposed charts $x^{k,i}$ for S^n in the previous example really are homeomorphisms.

Exercise 3.1.6 We shall later see that an atlas with two charts suffice on the sphere. Why is there no atlas for S^n with only one chart?

Example 3.1.7 The real projective n-space $\mathbf{R}\mathrm{P}^n$ is the set of all straight lines through the origin in $\mathbf{R}^{n+1}$. As a topological space, it is the quotient space (see 10.6)

$$\mathbf{R}\mathrm{P}^n = (\mathbf{R}^{n+1} \setminus \{0\})/\sim$$

where the equivalence relation is given by $p \sim q$ if there is a nonzero real number λ such that $p = \lambda q$. Since each line through the origin intersects the unit sphere in two (antipodal) points, $\mathbf{R}\mathrm{P}^n$ can alternatively be described as

$$S^n/\sim$$

where the equivalence relation is $p \sim -p$. The real projective n-space is an n-dimensional manifold, as we shall see below. If $p = (p_0, \dots, p_n) \in \mathbf{R}^{n+1} \setminus \{0\}$ we write $[p]$ for its equivalence class considered as a point in $\mathbf{R}\mathrm{P}^n$.

For $0 \leq k \leq n$, let

$$U^k = \{[p] \in \mathbf{R}\mathrm{P}^n | p_k \neq 0\}.$$

Varying k, this gives an open cover of $\mathbf{R}\mathrm{P}^n$ (why is U^k open in $\mathbf{R}\mathrm{P}^n$?). Note that the projection $S^n \to \mathbf{R}\mathrm{P}^n$ when restricted to $U^{k,0} \cup U^{k,1} = \{p \in S^n | p_k \neq 0\}$ gives a two-to-one correspondence between $U^{k,0} \cup U^{k,1}$ and U^k. In fact, when restricted to $U^{k,0}$ the projection $S^n \to \mathbf{R}\mathrm{P}^n$ yields a homeomorphism $U^{k,0} \cong U^k$.

The homeomorphism $U^{k,0} \cong U^k$ together with the homeomorphism

$$x^{k,0} \colon U^{k,0} \to E^n = \{p \in \mathbf{R}^n | \, |p| < 1\}$$

of example 3.1.4 gives a chart $U^k \to E^n$ (the explicit formula is given by sending $[p] \in U^k$ to $\frac{|p_k|}{p_k|p|}(p_0, \dots, \widehat{p_k}, \dots, p_n)$). Letting k vary, we get an atlas for ^{n}.

We can simplify this somewhat: the following atlas will be referred to as the *standard atlas for* $\mathbf{R}\mathrm{P}^n$. Let

$$\begin{aligned} x^k \colon U^k &\to \mathbf{R}^n \\ [p] &\mapsto \frac{1}{p_k}(p_0, \dots, \widehat{p_k}, \dots, p_n) \, . \end{aligned}$$

Note that this is a well defined (since $\frac{1}{p_k}(p_0, \dots, \widehat{p_k}, \dots, p_n) = \frac{1}{\lambda p_k}(\lambda p_0, \dots, \widehat{\lambda p_k}, \dots, \lambda p_n)$). Furthermore x^k is a bijective function with inverse given by

$$\left(x^k\right)^{-1}(p_0, \dots, \widehat{p_k}, \dots, p_n) = [p_0, \dots, 1, \dots, p_n]$$

(note the convenient cheating in indexing the points in $\mathbf{R}^n$).

In fact, x^k is a homeomorphism: x^k is continuous since the composite $U^{k,0} \cong U^k \to \mathbf{R}^n$ is; and $\left(x^k\right)^{-1}$ is continuous since it is the composite $\mathbf{R}^n \to \{p \in \mathbf{R}^{n+1} | p_k \neq 0\} \to U^k$ where the first map is given by $(p_0, \dots, \widehat{p_k}, \dots, p_n) \mapsto (p_0, \dots, 1, \dots, p_n)$ and the second is the projection.

[That $\mathbf{R}\mathrm{P}^n$ is Hausdorff and has a countable basis for its topology is exercise 10.7.5.]

Note 3.1.8 It is not obvious at this point that $\mathbf{R}\mathrm{P}^n$ can be realized as a subspace of an Euclidean space (we will show it can in theorem 9.2.6).

Note 3.1.9 We will try to be consistent in letting the charts have names like x and y. This is sound practice since it reminds us that what charts are good for is to give "local coordinates" on our manifold: a point $p \in M$ corresponds to a point

$$x(p) = (x_1(p), \dots, x_n(p)) \in \mathbf{R}^n.$$

The general philosophy when studying manifolds is to refer back to properties of Euclidean space by means of charts. In this manner a successful theory is built up: whenever a definition is needed, we take the Euclidean version and require that the corresponding property for manifolds is the one you get by saying that it must hold true in "local coordinates".

Example 3.1.10 As we defined it, a topological manifold is a topological space with certain properties. We could have gone about this differently, minimizing the rôle of the space at the expense of talking more about the atlas.

For instance, given a set M a collection $\{U_\alpha\}_{\alpha \in A}$ of subsets of M such that $\bigcup_{\alpha \in A} U_\alpha = M$ (we say that $\{U_\alpha\}_{\alpha \in A}$ covers M) and a collection of injections (one-to-one functions) $\{x_\alpha \colon U_\alpha \to \mathbf{R}^n\}_{\alpha \in A}$, assume that if $\alpha, \beta \in A$ then the bijection $x_\alpha(U_\alpha \cap U_\beta) \to x_\beta(U_\alpha \cap U_\beta)$ sending q to $x_\beta {x_\alpha}^{-1}(q)$ is a continuous map between open subsets of $\mathbf{R}^n$.

The declaration that $U \subset M$ is open if for all $\alpha \in A$ we have that $x_\alpha(U \cap U_\alpha) \subseteq \mathbf{R}^n$ is open, determines a topology on M. If this topology is Hausdorff and has a countable basis for its topology, then M is a topological manifold. This can be achieved if, for instance, we have that

1. for $p, q \in M$, either there is an $\alpha \in A$ such that $p, q \in U_\alpha$ or there are $\alpha, \beta \in A$ such that U_α and U_β are disjoint with $p \in U_\alpha$ and $q \in U_\beta$ and

2. there is a countable subset $B \subseteq A$ such that $\bigcup_{\beta \in B} U_\beta = M$.

3.2 Smooth structures

We will have to wait until 3.3.5 for the official definition of a *smooth manifold*. The idea is simple enough: in order to do *differential* topology we need that the charts of the manifolds are glued smoothly together, so that we do not get different answers in different charts.

Again "smoothly" must be borrowed from the Euclidean world. We proceed to make this precise.

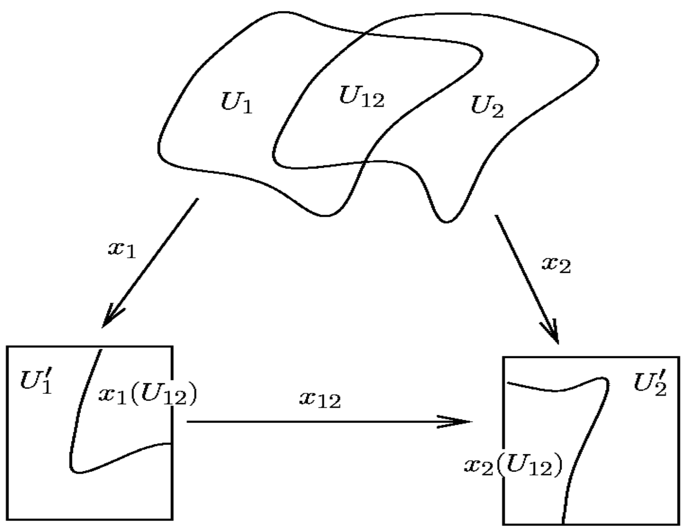
Let M be a topological manifold, and let $x_1 \colon U_1 \to U_1'$ and $x_2 \colon U_2 \to U_2'$ be two charts on M with U_1' and U_2' open subsets of $\mathbf{R}^n$. Assume that $U_{12} = U_1 \cap U_2$ is nonempty.

Then we may define a *chart transformation*

$$x_{12} \colon x_1(U_{12}) \to x_2(U_{12})$$

by sending $q \in x_1(U_{12})$ to

$$x_{12}(q) = x_2 x_1^{-1}(q)$$



(in function notation we get that

$$x_{12} = (x_2|_{U_{12}}) \circ (x_1|_{U_{12}})^{-1} : x_1(U_{12}) \to x_2(U_{12}),$$

where we recall that "$|_{U_{12}}$" means simply "restrict the domain of definition to U_{12}"). The picture of the chart transformation above will usually be recorded more succinctly as

$$\begin{array}{ccc} & U_{12} & \\ {}^{x_1|_{U_{12}}}\swarrow & & \searrow^{x_2|_{U_{12}}} \\ x_1(U_{12}) & & x_2(U_{12}) \end{array}$$

This makes things easier to remember than the occasionally awkward formulae.

The chart transformation x_{12} is a function from an open subset of $\mathbf{R}^n$ to another, and it makes sense to ask whether it is smooth or not.

Definition 3.2.1 An atlas on a manifold is *smooth* (or $\mathcal{C}^\infty$) if all the chart transformations are smooth (i.e., all the higher order partial derivatives exist and are continuous).

Definition 3.2.2 A smooth map f between open subsets of $\mathbf{R}^n$ is said to be a *diffeomorphism* if it has a smooth inverse f^{-1}.

Note 3.2.3 Note that if x_{12} is a chart transformation associated to a pair of charts in an atlas, then ${x_{12}}^{-1}$ is also a chart transformation. Hence, saying that an atlas is smooth is the same as saying that all the chart transformations are diffeomorphisms.

Note 3.2.4 We are only interested in the infinitely differentiable case, but in some situation it is sensible to ask for less. For instance, that all chart transformations are $\mathcal{C}^1$ (all the single partial differentials exist and are continuous). For a further discussion, see note 3.3.7 below.

One could also ask for more, for instance that all chart transformations are analytic functions. However, the difference between smooth and analytic is substantial as can be seen from Exercise 3.2.13.

Example 3.2.5 Let $U \subseteq \mathbf{R}^n$ be an open subset. Then the atlas whose only chart is the identity $id\colon U = U$ is smooth.

Example 3.2.6 The atlas

$$\mathcal{U} = \{(x^{k,i}, U^{k,i}) | 0 \leq k \leq n, 0 \leq i \leq 1\}$$

we gave on the n-sphere S^n is a smooth atlas. To see this, look at the example $U = U^{0,0} \cap U^{1,1}$ and consider the associated chart transformation

$$\left(x^{1,1}|_U\right) \circ \left(x^{0,0}|_U\right)^{-1} : x^{0,0}(U) \to x^{1,1}(U).$$

First we calculate the inverse of $x^{0,0}$: Let $p = (p_1, \ldots, p_n)$ be a point in the open disk E^n, then

$$\left(x^{0,0}\right)^{-1}(p) = \left(\sqrt{1 - |p|^2}, p_1, \ldots, p_n\right)$$

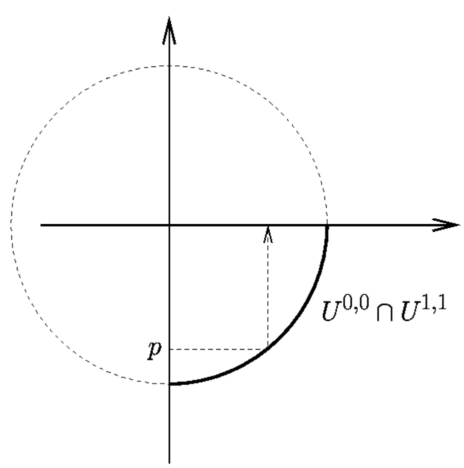
(we choose the positive square root, since we consider $x^{0,\mathbf{0}}$). Furthermore,

$$x^{0,0}(U) = \{(p_1, \ldots, p_n) \in E^n | p_1 < 0\}$$

Finally we get that if $p \in x^{0,0}(U)$ then

$$x^{1,1}\left(x^{0,0}\right)^{-1}(p) = \left(\sqrt{1 - |p|^2}, \widehat{p_1}, p_2, \ldots, p_n\right)$$

This is a smooth map, and generalizing to other indices we get that we have a smooth atlas for S^n.



How the point p in $x^{0,0}(U)$ is mapped to $x^{1,1}(x^{0,0})^{-1}(p)$.

Example 3.2.7 There is another useful smooth atlas on S^n, given by *stereographic projection*. It has only two charts.

The chart domains are

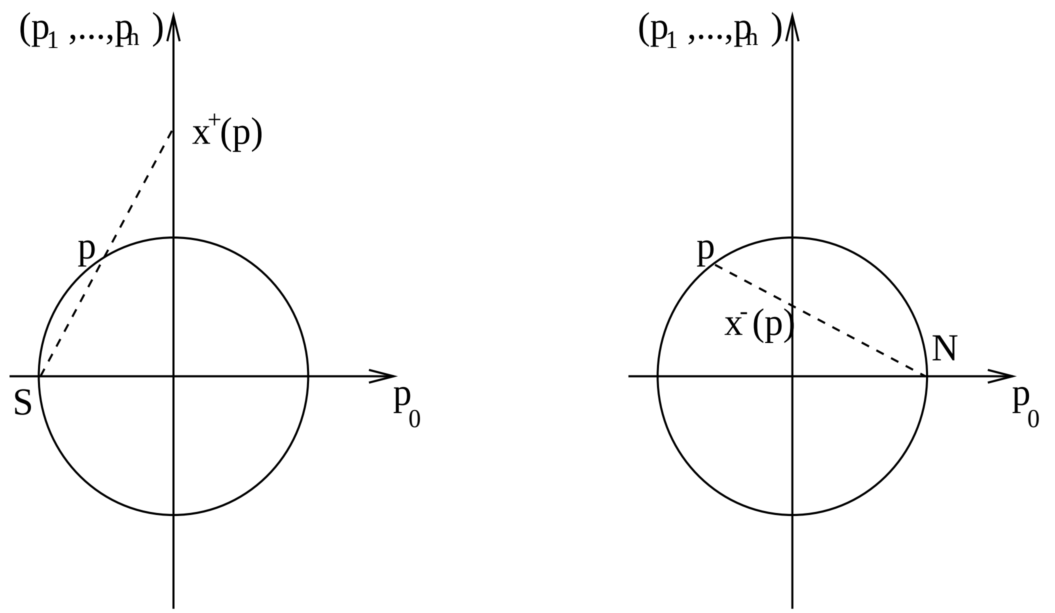
$$U^+ = \{p \in S^n | p_0 > -1\}$$
$$U^- = \{p \in S^n | p_0 < 1\}$$

and x^+ is given by sending a point on S^n to the intersection of the plane

$$\mathbf{R}^n = \{(0, p_1, \ldots, p_n) \in \mathbf{R}^{n+1}\}$$

and the straight line through the South pole $S = (-1, 0, \ldots, 0)$ and the point.

Similarly for x^-, using the North pole instead. Note that both maps are homeomorphisms onto all of $\mathbf{R}^n$



To check that there are no unpleasant surprises, one should write down the formulae:

$$x^+(p) = \frac{1}{1+p_0}(p_1, \ldots, p_n)$$
$$x^-(p) = \frac{1}{1-p_0}(p_1, \ldots, p_n).$$

We observe that this defines homeomorphisms $U^\pm \cong \mathbf{R}^n$. We need to check that the chart transformations are smooth. Consider the chart transformation $x^+ (x^-)^{-1}$ defined on $x^-(U^- \cap U^+) = \mathbf{R}^n \setminus \{0\}$. A small calculation gives that if $q \in \mathbf{R}^n$ then

$$\left(x^-\right)^{-1}(q) = \frac{1}{1+|q|^2}(|q|^2 - 1, 2q)$$

(solve the equation $x^-(p) = q$ with respect to p), and so

$$x^+ \left(x^-\right)^{-1}(q) = \frac{1}{|q|^2} q$$

which is smooth. A similar calculation for the other chart transformation yields that $\{x^-, x^+\}$ is a smooth atlas.

Exercise 3.2.8 Verify that the claims and formulae in the stereographic projection example are correct.

Note 3.2.9 The last two examples may be somewhat worrisome: the sphere is the sphere, and these two atlases are two manifestations of the "same" sphere, are they not? We address this kind of questions in the next chapter: "when do two different atlases describe the same smooth manifold?" You should, however, be aware that there **are** "exotic" smooth structures on spheres, i.e., smooth atlases on the topological manifold S^n which describe smooth structures essentially different from the one(s?) we have described (but only in high dimensions). See in particular exercise 3.3.9 and the note 3.3.6. Furthermore, there are topological manifolds which can not be given smooth structures.

Example 3.2.10 The atlas we gave the real projective space was smooth. As an example consider the chart transformation $x^2\left(x^0\right)^{-1}$: if $p_2 \neq 0$ then

$$x^2\left(x^0\right)^{-1}(p_1, \dots, p_n) = \frac{1}{p_2}(1, p_1, p_3, \dots, p_n)$$

Exercise 3.2.11 Show in all detail that the complex projective n-space

$$\mathbf{C}\mathrm{P}^n = (\mathbf{C}^{n+1} \setminus \{0\})/\sim$$

where $z \sim w$ if there exists a $\lambda \in \mathbf{C} \setminus \{0\}$ such that $z = \lambda w$, is a compact $2n$-dimensional manifold.

Exercise 3.2.12 Give the boundary of the square the structure of a smooth manifold.

Exercise 3.2.13 Let $\lambda : \mathbf{R} \to \mathbf{R}$ be defined by

$$\lambda(t) = \begin{cases} 0 & \text{for } t \leq 0 \\ e^{-1/t} & \text{for } t > 0 \end{cases}$$

This is a smooth function (note that all derivatives in zero are zero: the McLaurin series fails miserably and λ is definitely not analytic) with values between zero and one. Consequently, $t \mapsto \mathrm{sgn}(t)\lambda(|t|)$ gives a non-analytic diffeomorphism $\mathbf{R} \to (-1, 1)$.

3.3 Maximal atlases

We easily see that some manifolds can be equipped with many different smooth atlases. An example is the circle. Stereographic projection gives a different atlas than what you get if you for instance parametrize by means of the angle. But we do not want to distinguish between these two “smooth structures”, and in order to systematize this we introduce the concept of a *maximal atlas*.

Definition 3.3.1 Let M be a manifold and $\mathcal{A}$ a smooth atlas on M. Then we define $\mathcal{D}(\mathcal{A})$ as the following set of charts on M:

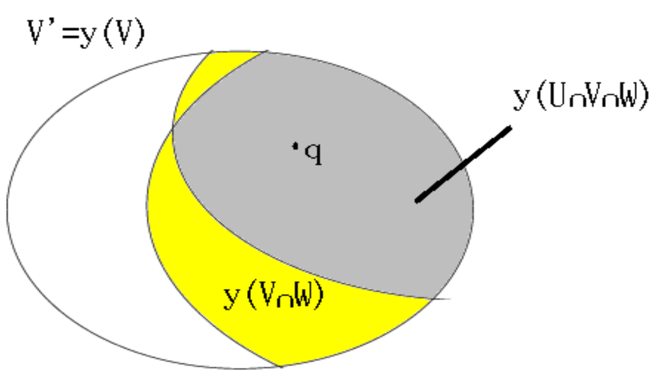
$$\mathcal{D}(\mathcal{A}) = \left\{ \text{charts } y\colon V \to V' \text{ on } M \;\middle|\; \begin{array}{c} \text{for all charts } (x, U) \text{ in } \mathcal{A}, \text{ the composite} \\ x|_W (y|_W)^{-1}\colon y(W) \to x(W) \\ \text{is a diffeomorphism, where } W = U \cap V \end{array} \right\}.$$

Lemma 3.3.2 *Let M be a manifold and $\mathcal{A}$ a smooth atlas on M. Then $\mathcal{D}(\mathcal{A})$ is a smooth atlas.*

Proof: Let $y\colon V \to V'$ and $z\colon W \to W'$ be two charts in $\mathcal{D}(\mathcal{A})$. We have to show that

$$z|_{V\cap W} \circ (y|_{V\cap W})^{-1}$$

is smooth. Let q be any point in $y(V \cap W)$. We prove that $z \circ y^{-1}$ is smooth in a neighborhood of q. Choose a chart $x\colon U \to U'$ in $\mathcal{A}$ with $y^{-1}(q) \in U$.



Letting $O = U \cap V \cap W$, we get that

$$\begin{aligned} z|_O \circ (y|_O)^{-1} &= z|_O \circ ((x|_O)^{-1} \circ x|_O) \circ (y|_O)^{-1} \\ &= \left(z|_O \circ (x|_O)^{-1}\right) \circ \left(x|_O \circ (y|_O)^{-1}\right) \end{aligned}$$

Since y and z are in $\mathcal{D}(\mathcal{A})$ and x is in $\mathcal{A}$ we have by definition that both the maps in the composite above are smooth, and we are done. $\square$

The crucial equation can be visualized by the following diagram

$$\begin{array}{ccccc} & & O & & \\ & \overset{y|_O}{\swarrow} & \downarrow{\scriptstyle x|_O} & \overset{z|_O}{\searrow} & \\ y(O) & & x(O) & & z(O) \end{array}$$

Going up and down with $x|_O$ in the middle leaves everything fixed so the two functions from $y(O)$ to $z(O)$ are equal.

Definition 3.3.3 A smooth atlas is *maximal* if there is no strictly bigger smooth atlas containing it.

Exercise 3.3.4 Given a smooth atlas $\mathcal{A}$, prove that $\mathcal{D}(\mathcal{A})$ is maximal. Hence any smooth atlas is a subset of a unique maximal smooth atlas.

Definition 3.3.5 A *smooth structure* on a topological manifold is a maximal smooth atlas. A *smooth manifold* $(M, \mathcal{A})$ is a topological manifold M equipped with a smooth structure $\mathcal{A}$. A *smooth manifold* is a topological manifold for which there exists a smooth structure.

Note 3.3.6 The following words are synonymous: smooth, differential and $\mathcal{C}^\infty$.

Many authors let the term "differentiable manifold" mean what we call a "smooth manifold", i.e., a topological manifold with a **chosen** smooth structure. The distinction between differentiable and smooth is not always relevant, but the reader may find pleasure in knowing that the topological manifold S^7 has 28 different smooth structures [7], and $\mathbf{R}^4$ has uncountably many.

As a side remark, one should notice that most physical situations involve differential equations of some sort, and so depend on the smooth structure, and not only on the underlying topological manifold. For instance, Baez remarks in This Week's Finds in Mathematical Physics *(Week 141)* that all of the 992 smooth structures on the 11-sphere are relevant to string-theory.

Note 3.3.7 We are only interested in the smooth (infinitely differentiable) case, but in some situation it is sensible to ask for less. For instance, that all chart transformations are $\mathcal{C}^1$ (all the single partial differentials exist and are continuous). However, the distinction is not really important since having an atlas with $\mathcal{C}^1$ chart transformations implies that there is a unique maximal smooth atlas such that the mixed chart transformations are $\mathcal{C}^1$ (see e.g., [**?**, Theorem 2.9]).

Note 3.3.8 In practice we do not give the maximal atlas, but only a small practical smooth atlas and apply $\mathcal{D}$ to it. Often we write just M instead of $(M, \mathcal{A})$ if $\mathcal{A}$ is clear from the context. To check that two smooth atlases $\mathcal{A}$ and $\mathcal{B}$ give the same smooth structure on M (i.e., that $\mathcal{D}(\mathcal{A}) = \mathcal{D}(\mathcal{B})$) it is enough to verify that for each $p \in M$ there are charts $(x, U) \in \mathcal{A}$ and $(y, V) \in \mathcal{B}$ with $p \in W = U \cap V$ such that $x|_W (y|_W)^{-1} \colon y(W) \to x(W)$ is a diffeomorphism.

Exercise 3.3.9 Show that the two smooth structures we have defined on S^n (the standard atlas in Example 3.1.4 and the stereographic projections of Example 3.2.7) are contained in a common maximal atlas. Hence they define the same smooth manifold, which we will simply call the *(standard smooth) sphere*.

Exercise 3.3.10 Choose your favorite diffeomorphism $x\colon \mathbf{R}^n \to \mathbf{R}^n$. Why is the smooth structure generated by x equal to the smooth structure generated by the identity? What does the maximal atlas for this smooth structure (the only one we'll ever consider) on $\mathbf{R}^n$ look like?

Exercise 3.3.11 Prove that any smooth manifold $(M, \mathcal{A})$ has a countable smooth atlas $\mathcal{V}$ (so that $\mathcal{D}(\mathcal{V}) = \mathcal{A}$).

Following up Example 3.1.10 we see that we can construct smooth manifolds from scratch, without worrying too much about the topology:

Lemma 3.3.12 *Given*

1. *a set M,*

2. *a collection $\mathcal{A}$ of subsets of M and*

3. *an injection $x_U\colon U \to \mathbf{R}^n$ for each $U \in \mathcal{A}$,*

such that

1. *there is a countable subcollection of $\mathcal{A}$ which covers M*

2. *for $p, q \in M$, either there is a $U \in \mathcal{A}$ such that $p, q \in U$ or there are $U, V \in \mathcal{A}$ such that U and V are disjoint with $p \in U$ and $q \in V$, and*

3. *if $U, V \in \mathcal{A}$ then the bijection $x_U(U \cap V) \to x_V(U \cap V)$ sending q to $x_V {x_U}^{-1}(q)$ is a smooth map between open subsets of $\mathbf{R}^n$.*

Then there is a unique topology on M such that $(M, \mathcal{D}(\{(x_U, U)\}_{U\in\mathcal{A}}))$ is a smooth manifold.

Proof: For the x_Us to be homeomorphisms we *must* have that a subset $W \subseteq M$ is open if and only if for all $U \in \mathcal{A}$ the set $x_U(U \cap W)$ is an open subset of $\mathbf{R}^n$. As before, M is a topological manifold, and by the last condition $\{(x_U, U)\}_{U\in\mathcal{A}}$ is a smooth atlas. ∎

Example 3.3.13 As an example of how to construct smooth manifolds using Lemma 3.3.12, we define a family of very important smooth manifolds called the Grassmann manifolds. These manifolds show up in a number of applications, and are important to the theory of vector bundles. The details in the construction below consists mostly of some rather tedious linear algebra, and may well be deferred to a second reading (at which time you should take the opportunity to check the assertions that are not immediate).

For $0 < n \leq k$, let $G_n^k = G_n(\mathbf{R}^k)$ be the set of all n-dimensional subspaces of $\mathbf{R}^k$. Note that G_1^{n+1} is nothing but the projective space $\mathbf{R}\mathrm{P}^n$. We will equip G_n^k with the structure of a $(k-n)n$-dimensional smooth manifold, the *Grassmann manifold*.

If $V, W \subseteq \mathbf{R}^k$ are subspaces, we let $pr^V\colon \mathbf{R}^k \to V$ be the orthogonal projection to V (with the usual inner product) and $pr^V_W\colon W \to V$ the restriction of pr^V to W. We let $\mathrm{Hom}(V, W)$ be the vector space of all linear maps from V to W. Concretely, and for the sake of the smoothness arguments below, using the standard basis for $\mathbf{R}^k$ we may identify $\mathrm{Hom}(V, W)$ with the $\dim(V)\cdot\dim(W)$-dimensional subspace of the space of $k \times k$-matrices A with the property that if $v \in V$ and $v' \in V^\perp$, then $Av \in W$ and $Av' = 0$.

If $V \in G_n^k$ let U_V be the set $\{W \in G_n^k | W \cap V^\perp = 0\}$, and let $\mathcal{A} = \{U_V\}_{V \in G_n^k}$. Another characterization of U_V is as the set of all $W \in G_n^k$ such that $pr^V_W\colon W \to V$ is an isomorphism. The vector space $\mathrm{Hom}(V, V^\perp)$ of all linear transformations $V \to V^\perp$ is isomorphic to the vector space of all $(k-n) \times n$-matrices (make a choice of bases for V and $V^\perp$), which again is isomorphic to $\mathbf{R}^{(k-n)n}$. Let $x_V : U_V \to \mathrm{Hom}(V, V^\perp)$ send $W \in U_V$ to the composite

$$x_V(W)\colon V \xrightarrow{(pr^V_W)^{-1}} W \xrightarrow{pr^{V^\perp}_W} V^\perp.$$

Notice that x_V is a bijection, with inverse sending $f \in \mathrm{Hom}(V, V^\perp)$ to the graph $\Gamma(f) = \{v + f(v) \in \mathbf{R}^k | v \in V\} \subseteq \mathbf{R}^k$.

If $V, W \in G_n^k$, then $x_V(U_V \cap U_W) = \{f \in \mathrm{Hom}(V, V^\perp) \,|\, \Gamma(f) \cap W^\perp = 0\}$. We must check that the chart transformation

$$x_V(U_V \cap U_W) \xrightarrow{{x_V}^{-1}} U_V \cap U_W \xrightarrow{x_W} x_W(U_V \cap U_W)$$

sending $f\colon V \to V^\perp$ to

$$W \xrightarrow{(pr^W_{\Gamma(f)})^{-1}} \Gamma(f) \xrightarrow{pr^W_{\Gamma(f)}} W^\perp$$

is smooth. For ease of notation we write $g_f = x_W {x_V}^{-1}(f) = pr^W_{\Gamma(f)}(pr^W_{\Gamma(f)})^{-1}$ for this map.

Now, if $x \in V$, then $(pr^V_{\Gamma(f)})^{-1}(x) = x + f(x)$, and so the composite isomorphism

$$A_f = pr^W_{\Gamma(f)}(pr^V_{\Gamma(f)})^{-1}\colon V \to W,$$

sending x to $A_f(x) = pr^W x + pr^W f(x)$ depends smoothly on f. By Cramer's rule, the inverse $B_f = A_f^{-1}$ also depends smoothly on f.

Finally, if $y \in W$, then $(pr^W_{\Gamma(f)})^{-1}(y) = y + g_f(y)$ is equal to $(pr^V_{\Gamma(f)})^{-1}(B_f(y)) = B_f(y) + f(B_f(y))$, and so

$$g_f = B_f + fB_f - 1$$

depends smoothly on f

The point-set conditions are satisfied by the following purely linear algebraic assertions. For a subset $S \subseteq \{1, \ldots, k\}$ of cardinality n, let $V_S \in G_n^k$ be the subspace of all vectors $v \in \mathbf{R}^k$ with $v_j = 0$ for all $j \in S$. The finite subcollection of $\mathcal{A}$ consisting of the U_{V_S} as S varies covers G_n^k. If $W_1, W_2 \in G_n^k$ there is a $V \in G_n^k$ such that $W_1, W_2 \in G_n^k$.

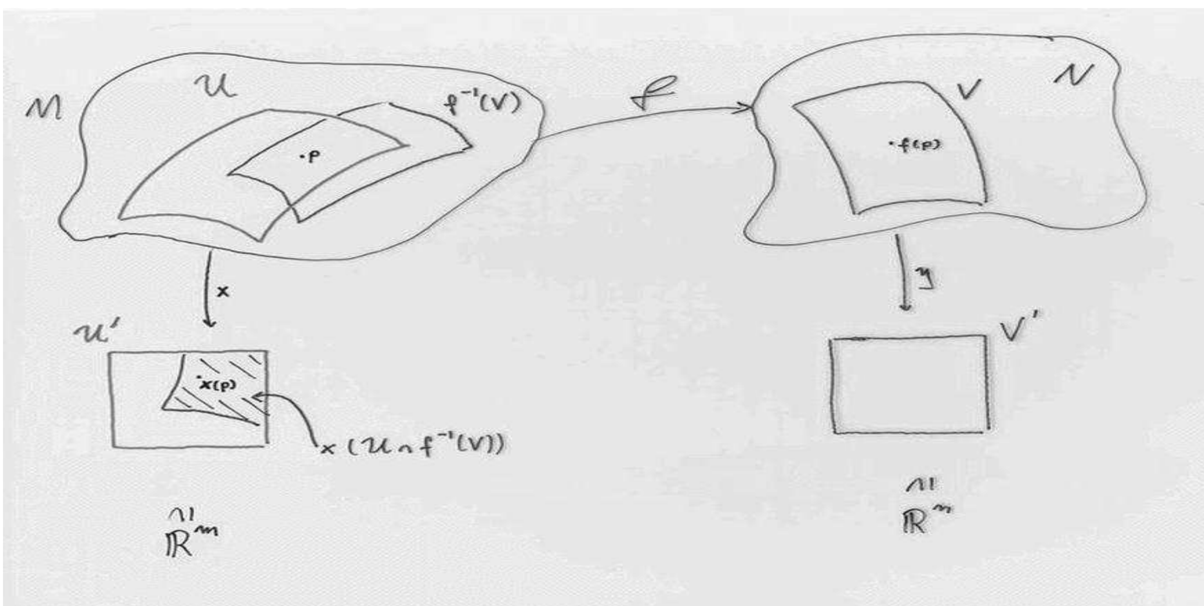
3.4 Smooth maps

Having defined smooth manifolds, we need to define smooth maps between them. No surprise: smoothness is a local question, so we may fetch the notion from Euclidean space by means of charts.

Definition 3.4.1 Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be smooth manifolds and $p \in M$. A continuous map $f\colon M \to N$ is *smooth at* p (or *differentiable at* p) if for any chart $x\colon U \to U' \in \mathcal{A}$ with $p \in U$ and any chart $y\colon V \to V' \in \mathcal{B}$ with $f(p) \in V$ the map

$$y \circ f|_{U \cap f^{-1}(V)} \circ (x|_{U \cap f^{-1}(V)})^{-1}\colon x(U \cap f^{-1}(V)) \to V'$$

is smooth at $x(p)$.



We say that f is a *smooth map* if it is smooth at all points of M.

The picture above will often find a less typographically challenging expression: "go up, over and down in the picture

$$\begin{array}{ccc} W & \xrightarrow{f|_W} & V \\ {\scriptstyle x|_W}\downarrow & & \downarrow{\scriptstyle y} \\ x(W) & & V' \end{array}$$

where $W = U \cap f^{-1}(V)$, and see whether you have a smooth map of open subsets of Euclidean spaces". Note that $x(W) = x(f^{-1}(V))$.

Note 3.4.2 To see whether f in the definition 3.4.1 above is smooth at $p \in M$ you do not actually have to check **all** charts! We formulate this as a lemma: its proof can be viewed as a worked exercise.

Lemma 3.4.3 *Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be smooth manifolds. A function $f\colon M \to N$ is smooth if (and only if) for all $p \in M$ there exist charts $(x, U) \in \mathcal{A}$ and $(y, V) \in \mathcal{B}$ with $p \in W = U \cap f^{-1}(V)$ such that the composite*

$$y \circ f|_W \circ (x|_W)^{-1}\colon x(W) \to y(V)$$

is smooth.

Proof: Given such charts we prove that f is smooth at p. This implies that f is smooth since p is arbitrary.

The function $f|_W$ is continuous since $y \circ f|_W \circ (x|_W)^{-1}$ is smooth (and so continuous), and x and y are homeomorphisms. We must show that given **any** charts $(\tilde{x}, \tilde{U}) \in \mathcal{A})$ and $(\tilde{y}, \tilde{V}) \in \mathcal{B}$ with $p \in \tilde{W} = \tilde{U} \cap f^{-1}(\tilde{V})$ we have that $\tilde{y} f|_{\tilde{W}} (\tilde{x}|_{\tilde{W}})^{-1}$ is smooth at p. Now, for $q \in W \cap \tilde{W}$ we can rewrite the function in question as a composition

$$\tilde{y} f \tilde{x}^{-1}(q) = (\tilde{y} y^{-1})(y f x^{-1})(x \tilde{x}^{-1})(q),$$

of smooth functions defined on Euclidean spaces: $x\tilde{x}^{-1}$ and $y\tilde{y}^{-1}$ are smooth since $\mathcal{A}$ and $\mathcal{B}$ are smooth atlases. □

Exercise 3.4.4 The map $\mathbf{R} \to S^1$ sending $p \in \mathbf{R}$ to $e^{ip} = (\cos p, \sin p) \in S^1$ is smooth.

Exercise 3.4.5 Show that the map $g\colon \mathbf{S}^2 \to \mathbf{R}^4$ given by

$$g(p_0, p_1, p_2) = (p_1 p_2, p_0 p_2, p_0 p_1, p_0^2 + 2p_1^2 + 3p_2^2)$$

defines a smooth injective map

$$\tilde{g}\colon \mathbf{R}\mathrm{P}^2 \to \mathbf{R}^4$$

via the formula $\tilde{g}([p]) = g(p)$.

Exercise 3.4.6 Show that a map $f\colon \mathbf{R}\mathrm{P}^n \to M$ is smooth iff the composite

$$S^n \stackrel{g}{\to} \mathbf{R}\mathrm{P}^n \stackrel{f}{\to} M$$

is smooth, where g is the projection.

Definition 3.4.7 A smooth map $f\colon M \to N$ is a *diffeomorphism* if it is a bijection, and the inverse is smooth too. Two smooth manifolds are *diffeomorphic* if there exists a diffeomorphism between them.

Note 3.4.8 Note that this use of the word diffeomorphism coincides with the one used earlier for open subsets of $\mathbf{R}^n$.

Example 3.4.9 The smooth map $\mathbf{R} \to \mathbf{R}$ sending $p \in \mathbf{R}$ to p^3 is a smooth homeomorphism, but it is not a diffeomorphism: the inverse is not smooth at $0 \in \mathbf{R}$. The problem is that the derivative is zero at $0 \in \mathbf{R}$: if a smooth map $f\colon \mathbf{R} \to \mathbf{R}$ has nowhere vanishing derivative, then it is a diffeomorphism. The inverse function theorem 5.2.1 gives the corresponding criterion for (local) smooth invertibility also in higher dimensions.

Example 3.4.10 If $a < b \in \mathbf{R}$, then the straight line $f(t) = (b-a)t + a$ gives a diffeomorphism $f\colon (0,1) \to (a,b)$ with inverse given by $f^{-1}(t) = (t-a)/(b-a)$. Note that

$$\tan\colon (-\pi/2, \pi/2) \to \mathbf{R}$$

is a diffeomorphism. Hence all open intervals are diffeomorphic to the entire real line.

Exercise 3.4.11 Show that $\mathbf{R}\mathrm{P}^1$ and S^1 are diffeomorphic.

Exercise 3.4.12 Show that $\mathbf{C}\mathrm{P}^1$ and S^2 are diffeomorphic.

Lemma 3.4.13 *If $f\colon (M, \mathcal{U}) \to (N, \mathcal{V})$ and $g\colon (N, \mathcal{V}) \to (P, \mathcal{W})$ are smooth, then the composite $gf\colon (M, \mathcal{U}) \to (P, \mathcal{W})$ is smooth too.*

Proof: This is true for maps between Euclidean spaces, and we lift this fact to smooth manifolds. Let $p \in M$ and choose appropriate charts

$x\colon U \to U' \in \mathcal{U}$, such that $p \in U$,

$y\colon V \to V' \in \mathcal{V}$, such that $f(p) \in V$,

$z\colon W \to W' \in \mathcal{W}$, such that $gf(p) \in W$.

Then $T = U \cap f^{-1}(V \cap g^{-1}(W))$ is an open set containing p, and we have that

$$zgfx^{-1}|_{x(T)} = (zgy^{-1})(yfx^{-1})|_{x(T)}$$

which is a composite of smooth maps of Euclidean spaces, and hence smooth. □

In a picture, if $S = V \cap g^{-1}(W)$ and $T = U \cap f^{-1}(S)$:

$$\begin{array}{ccccc} T & \xrightarrow{f|_T} & S & \xrightarrow{g|_S} & W \\ \downarrow{\scriptstyle x|_T} & & \downarrow{\scriptstyle y|_S} & & \downarrow{\scriptstyle z|_W} \\ x(T) & & y(S) & & z(W) \end{array}$$

Going up and down with y does not matter.

Exercise 3.4.14 Let $f\colon M \to N$ be a homeomorphism of topological spaces. If M is a smooth manifold then there is a unique smooth structure on N that makes f a diffeomorphism.

Definition 3.4.15 Let $(M,\mathcal{U})$ and $(N,\mathcal{V})$ be smooth manifolds. Then we let

$$\mathcal{C}^\infty(M,N) = \{\text{smooth maps } M \to N\}$$

and

$$\mathcal{C}^\infty(M) = \mathcal{C}^\infty(M,\mathbf{R}).$$

Note 3.4.16 A small digression, which may be disregarded by the categorically illiterate. The outcome of the discussion above is that we have a category $\mathcal{C}^\infty$ of smooth manifolds: the objects are the smooth manifolds, and if M and N are smooth, then

$$\mathcal{C}^\infty(M,N)$$

is the set of morphisms. The statement that $\mathcal{C}^\infty$ is a category uses that the identity map is smooth (check), and that the composition of smooth functions is smooth, giving the composition in $\mathcal{C}^\infty$:

$$\mathcal{C}^\infty(N,P) \times \mathcal{C}^\infty(M,N) \to \mathcal{C}^\infty(M,P)$$

The diffeomorphisms are the isomorphisms in this category.

Definition 3.4.17 A smooth map $f\colon M \to N$ is a *local diffeomorphism* if for each $p \in M$ there is an open set $U \subseteq M$ containing p such that $f(U)$ is an open subset of N and

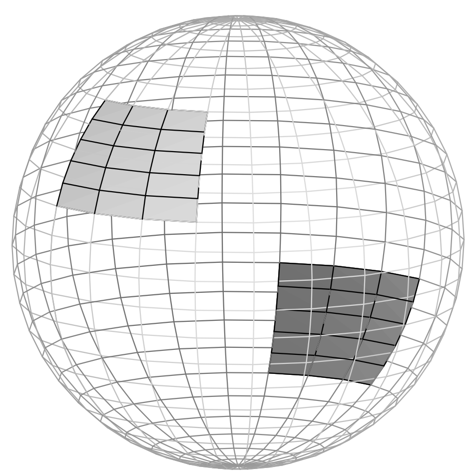
$$f|_U\colon U \to f(U)$$

is a diffeomorphism.

Example 3.4.18 The projection $S^n \to \mathbf{R}\mathrm{P}^n$ is a local diffeomorphism.
Here is a more general example: let M be a smooth manifold, and

$$i\colon M \to M$$

a diffeomorphism with the property that $i(p) \neq p$, but $i(i(p)) = p$ for all $p \in M$ (such an animal is called a *fixed point free involution*).
The quotient space M/i gotten by identifying p and $i(p)$ has a smooth structure, such that the projection $f\colon M \to M/i$ is a local diffeomorphism.
We leave the proof of this claim as an exercise:



Small open sets in $\mathbf{R}\mathrm{P}^2$ correspond to unions $U \cup (-U)$ where $U \subseteq S^2$ is an open set totally contained in one hemisphere.

Exercise 3.4.19 Show that M/i has a smooth structure such that the projection $f\colon M \to M/i$ is a local diffeomorphism.

Exercise 3.4.20 If $(M,\mathcal{U})$ is a smooth n-dimensional manifold and $p \in M$, then there is a chart $x\colon U \to \mathbf{R}^n$ such that $x(p) = 0$.

Note 3.4.21 In differential topology we consider two smooth manifolds to be the same if they are diffeomorphic, and all properties one studies are unaffected by diffeomorphisms.

Is it possible to give a classification of manifolds? That is, can we list all the smooth manifolds? On the face of it this is a totally over-ambitious question, but actually quite a lot is known.

The circle is the only compact (10.7.1) connected (10.9.1) smooth 1-manifold.

In dimension two it is only slightly more interesting. As we discussed in 2.4.3, you can obtain any compact (smooth) connected 2-manifold by punching g holes in the sphere S^2 and glue onto this either g handles or g Möbius bands.

In dimension four and up total chaos reigns (and so it is here all the interesting stuff is). Well, actually only the part within the parentheses is true in the last sentence: there is a lot of structure, much of it well understood. However all of it is beyond the scope of these notes. It involves quite a lot of manifold theory, but also algebraic topology and a subject called surgery which in spirit is not so distant from the cutting and pasting techniques we used on surfaces in 2.4.3. For dimension three, the reader may refer back to section 2.4.9.

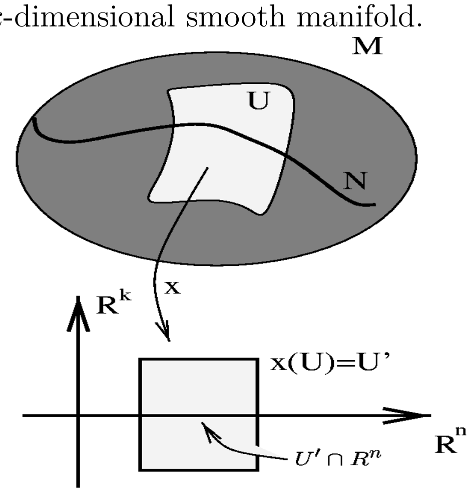
3.5 Submanifolds

We give a slightly unorthodox definition of submanifolds. The "real" definition will appear only very much later, and then in the form of a theorem! This approach makes it possible to discuss this important concept before we have developed the proper machinery to express the "real" definition. (This is really not all that unorthodox, since it is done in the same way in for instance both [3] and [5]).

Definition 3.5.1 Let $(M,\mathcal{U})$ be a smooth $n+k$-dimensional smooth manifold.

An n-dimensional *(smooth) submanifold* in M is a subset $N \subseteq M$ such that for each $p \in N$ there is a chart $x\colon U \to U'$ in $\mathcal{U}$ with $p \in U$ such that

$$x(U \cap N) = U' \cap (\mathbf{R}^n \times \{0\}) \subseteq \mathbf{R}^n \times \mathbf{R}^k.$$



In this definition we identify $\mathbf{R}^{n+k}$ with $\mathbf{R}^n \times \mathbf{R}^k$. We often write $\mathbf{R}^n \subseteq \mathbf{R}^n \times \mathbf{R}^k$ instead of $\mathbf{R}^n \times \{0\} \subseteq \mathbf{R}^n \times \mathbf{R}^k$ to signify the subset of all points with the k last coordinates equal to zero.

Note 3.5.2 The language of the definition really makes some sense: if $(M, \mathcal{U})$ is a smooth manifold and $N \subseteq M$ a submanifold, then we give N the smooth structure

$$\mathcal{U}|_N = \{(x|_{U \cap N}, U \cap N) | (x, U) \in \mathcal{U}\}$$

Note that the inclusion $N \to M$ is smooth.

Example 3.5.3 Let n be a natural number. Then $K_n = \{(p, p^n)\} \subseteq \mathbf{R}^2$ is a differential submanifold.

We define a smooth chart

$$x \colon \mathbf{R}^2 \to \mathbf{R}^2, \qquad (p, q) \mapsto (p, q - p^n)$$

Note that as required, x is smooth with smooth inverse given by

$$(p, q) \mapsto (p, q + p^n)$$

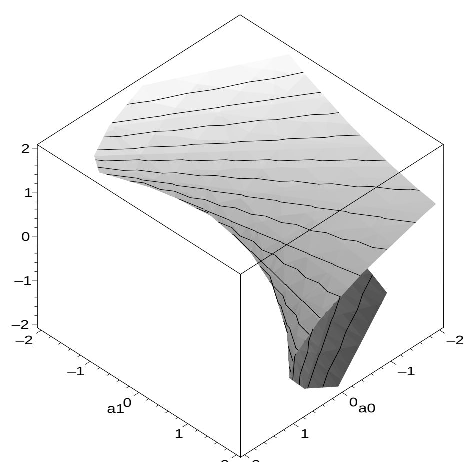
and that $x(K_n) = \mathbf{R}^1 \times \{0\}$.

Exercise 3.5.4 Prove that $S^1 \subset \mathbf{R}^2$ is a submanifold. More generally: prove that $S^n \subset \mathbf{R}^{n+1}$ is a submanifold.

Exercise 3.5.5 Show that the subset $C \subseteq \mathbf{R}^{n+1}$ given by

$$C = \{(a_0, \ldots, a_{n-1}, t) \in \mathbf{R}^{n+1} \mid t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0 = 0\},$$

a part of which is illustrated for $n = 2$ in the picture below, is a smooth submanifold.



Exercise 3.5.6 The subset $K = \{(p, |p|) \mid p \in \mathbf{R}\} \subseteq \mathbf{R}^2$ is **not** a smooth submanifold.

Note 3.5.7 If $\dim(M) = \dim(N)$ then $N \subset M$ is an open subset (called an *open submanifold*. Otherwise $\dim(M) > \dim(N)$.

Example 3.5.8 Let $M_n\mathbf{R}$ be the set of $n \times n$ matrices. This is a smooth manifold since it is homeomorphic to $\mathbf{R}^{n^2}$. The subset $\mathrm{GL}_n(\mathbf{R}) \subseteq M_n\mathbf{R}$ of invertible matrices is an open submanifold. (since the determinant function is continuous, so the inverse image of the open set $\mathbf{R} \setminus \{0\}$ is open)

Exercise 3.5.9 If V is an n-dimensional vector space, let $\mathrm{GL}(V)$ be the set of linear isomorphisms $\alpha\colon V \cong V$. By representing any linear isomorphism of $\mathbf{R}^n$ in terms of the standard basis, we may identify $\mathrm{GL}(\mathbf{R}^n)$ and $\mathrm{GL}_n(\mathbf{R})$. Any linear isomorphism $f\colon V \cong W$ gives a bijection $\mathrm{GL}(f)\colon \mathrm{GL}(V) \cong \mathrm{GL}(W)$ sending $\alpha\colon V \cong V$ to $f\alpha f^{-1}\colon W \cong W$. Hence, any linear isomorphism $f\colon V \cong \mathbf{R}^n$ (i.e., a choice of basis) gives a bijection $GL(f)\colon \mathrm{GL}(V) \cong \mathrm{GL}_n\mathbf{R}$, and so a smooth manifold structure on $\mathrm{GL}(V)$ (with a diffeomorphism to the open subset $\mathrm{GL}_n\mathbf{R}$ of Euclidean n^2-space).

Prove that the smooth structure on $\mathrm{GL}(V)$ does not depend on the choice of $f\colon V \cong \mathbf{R}^n$.

If $h\colon V \cong W$ is a linear isomorphism, prove that $GL(h)\colon \mathrm{GL}(V) \cong \mathrm{GL}(W)$ is a diffeomorphism respecting composition and the identity element.

Example 3.5.10 Let $M_{m\times n}\mathbf{R}$ be the set of $m \times n$ matrices (if $m = n$ we write $M_n(\mathbf{R})$ instead of $M_{n\times n}(\mathbf{R})$. This is a smooth manifold since it is homeomorphic to $\mathbf{R}^{mn}$. Let $0 \le r \le \min(m, n)$. That a matrix has rank r means that it has an $r \times r$ invertible submatrix, but no larger invertible submatrices.

The subset $M^r_{m\times n}(\mathbf{R}) \subseteq M_{m\times n}\mathbf{R}$ of matrices of rank r is a submanifold of codimension $(n - r)(m - r)$. Since some of the ideas will be valuable later on, we spell out a proof.

For the sake of simplicity, we treat the case where our matrices have an invertible $r \times r$ submatrices in the upper left-hand corner. The other cases are covered in a similar manner, taking care of indices (or by composing the chart we give below with a diffeomorphism on $M_{m\times n}\mathbf{R}$ given by multiplying with permutation matrices so that the invertible submatrix is moved to the upper left-hand corner).

So, consider the open set U of matrices

$$X = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

with $A \in M_r(\mathbf{R})$, $B \in M_{r\times(n-r)}(\mathbf{R})$, $C \in M_{(m-r)\times r}(\mathbf{R})$ and $D \in M_{(m-r)\times(n-r)}(\mathbf{R})$ such that $\det(A) \neq 0$ (i.e., such that $A \in \mathrm{GL}_r(\mathbf{R})$). The matrix X has rank exactly r if and only if the last $n - r$ columns are in the span of the first r. Writing this out, this means that X is of rank r if and only if there is an $r \times (n - r)$-matrix T such that

$$\begin{bmatrix} B \\ D \end{bmatrix} = \begin{bmatrix} A \\ C \end{bmatrix} T,$$

which is equivalent to $T = A^{-1}B$ and $D = CA^{-1}B$. Hence

$$U \cap M^r_{m\times n}(\mathbf{R}) = \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in U \,\middle|\, D - CA^{-1}B = 0 \right\}.$$

The map

$$\begin{aligned} U \to & \mathrm{GL}_r(\mathbf{R}) \times M_{r\times(n-r)}(\mathbf{R}) \times M_{(m-r)\times r}(\mathbf{R}) \times M_{(m-r)\times(n-r)}(\mathbf{R}) \\ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \mapsto & (A, B, C, D - CA^{-1}B) \end{aligned}$$

is a diffeomorphism onto an open subset of $M_r(\mathbf{R}) \times M_{r\times(n-r)}(\mathbf{R}) \times M_{(m-r)\times r}(\mathbf{R}) \times M_{(m-r)\times(n-r)}(\mathbf{R}) \cong \mathbf{R}^{mn}$, and therefore gives a chart having the desired property that $U \cap M^r_{m\times n}(\mathbf{R})$ is the set of points such that the last $(m-r)(n-r)$ coordinates vanish.

Definition 3.5.11 A smooth map $f\colon N \to M$ is an *imbedding* if

the image $f(N) \subseteq M$ is a submanifold, and

the induced map

$$N \to f(N)$$

is a diffeomorphism.

Exercise 3.5.12 The map

$$\begin{aligned} f\colon \mathbf{R}\mathrm{P}^n \to & \mathbf{R}\mathrm{P}^{n+1} \\ [p] = [p_0, \dots, p_n] \mapsto & [p, 0] = [p_0, \dots, p_n, 0] \end{aligned}$$

is an imbedding.

Note 3.5.13 Later we will give a very efficient way of creating smooth submanifolds, getting rid of all the troubles of finding actual charts that make the subset look like $\mathbf{R}^n$ in $\mathbf{R}^{n+k}$. We shall see that if $f\colon M \to N$ is a smooth map and $q \in N$ then more often than not the inverse image

$$f^{-1}(q) = \{p \in M \mid f(p) = q\}$$

is a submanifold of M. Examples of such submanifolds are the sphere and the space of orthogonal matrices (the inverse image of the identity matrix under the map sending a matrix A to A^tA).

Example 3.5.14 An example where we have the opportunity to use a bit of topology. Let $f\colon M \to N$ be an imbedding, where M is a (non-empty) compact n-dimensional smooth manifold and N is a connected n-dimensional smooth manifold. Then f is a diffeomorphism. This is so because $f(M)$ is compact, and hence closed, and open since it is a codimension zero submanifold. Hence $f(M) = N$ since N is connected. But since f is an imbedding, the map $M \to f(M) = N$ is – by definition – a diffeomorphism.

Exercise 3.5.15 (important exercise. Do it: you will need the result several times). Let $i_1\colon N_1 \to M_1$ and $i_2\colon N_2 \to M_2$ be smooth imbeddings and let $f\colon N_1 \to N_2$ and $g\colon M_1 \to M_2$ be continuous maps such that $i_2 f = g i_1$ (i.e., the diagram

$$\begin{array}{ccc} N_1 & \xrightarrow{f} & N_2 \\ {\scriptstyle i_1}\downarrow & & {\scriptstyle i_2}\downarrow \\ M_1 & \xrightarrow{g} & M_2 \end{array}$$

commutes). Show that if g is smooth, then f is smooth.

Exercise 3.5.16 Show that the composite of imbeddings is an imbedding.

3.6 Products and sums

Definition 3.6.1 Let $(M,\mathcal{U})$ and $(N,\mathcal{V})$ be smooth manifolds. The *(smooth) product* is the smooth manifold you get by giving the product $M \times N$ the smooth structure given by the charts

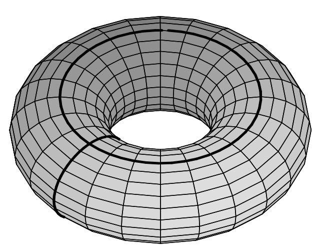
$$\begin{aligned} x \times y\colon U \times V &\to U' \times V' \\ (p,q) &\mapsto (x(p), y(q)) \end{aligned}$$

where $(x,U) \in \mathcal{U}$ and $(y,V) \in \mathcal{V}$.

Exercise 3.6.2 Check that this definition makes sense.

Note 3.6.3 Even if the atlases we start with are maximal, the charts of the form $x \times y$ do not form a maximal atlas on the product, but as always we can consider the associated maximal atlas.

Example 3.6.4 We know a product manifold already: the *torus* $S^1 \times S^1$.



The torus is a product. The bolder curves in the illustration try to indicate the submanifolds $\{1\} \times S^1$ and $S^1 \times \{1\}$.

Exercise 3.6.5 Show that the projection

$$\begin{aligned} \mathrm{pr}_1 \colon M \times N &\to M \\ (p, q) &\mapsto p \end{aligned}$$

is a smooth map. Choose a point $p \in M$. Show that the map

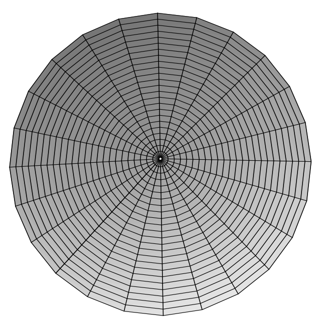
$$\begin{aligned} i_p \colon N &\to M \times N \\ q &\mapsto (p, q) \end{aligned}$$

is an imbedding.

Exercise 3.6.6 Show that giving a smooth map $Z \to M \times N$ is the same as giving a pair of smooth maps $Z \to M$ and $Z \to N$. Hence we have a bijection

$$\mathcal{C}^\infty(Z, M \times N) \cong \mathcal{C}^\infty(Z, M) \times \mathcal{C}^\infty(Z, N).$$

Exercise 3.6.7 Show that the infinite cylinder $\mathbf{R}^1 \times S^1$ is diffeomorphic to $\mathbf{R}^2 \setminus \{0\}$.



Looking down into the infinite cylinder.

More generally: $\mathbf{R}^1 \times S^n$ is diffeomorphic to $\mathbf{R}^{n+1} \setminus \{0\}$.

Exercise 3.6.8 Let $f \colon M \to M'$ and $g \colon N \to N'$ be imbeddings. Then

$$f \times g \colon M \times N \to M' \times N'$$

is an imbedding.

Exercise 3.6.9 Show that there exists an imbedding $S^{n_1} \times \cdots \times S^{n_k} \to \mathbf{R}^{1+\sum_{i=1}^k n_i}$.

Exercise 3.6.10 Why is the multiplication of matrices

$$\mathrm{GL}_n(\mathbf{R}) \times \mathrm{GL}_n(\mathbf{R}) \to \mathrm{GL}_n(\mathbf{R}), \qquad (A, B) \mapsto A \cdot B$$

a smooth map? This, together with the existence of inverses, makes $\mathrm{GL}_n(\mathbf{R})$ a "*Lie group*".

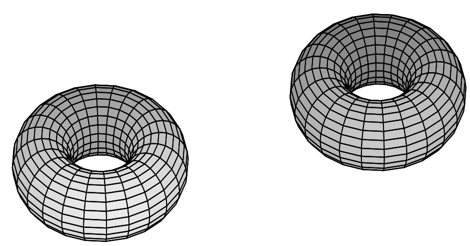
For the record: a *Lie group* is a smooth manifold M with a smooth "multiplication" $M \times M \to M$ that is associative, has a neutral element and all inverses (in $\mathrm{GL}_n(\mathbf{R})$ the neutral element is the identity matrix).

Exercise 3.6.11 Why is the multiplication

$$S^1 \times S^1 \to S^1, \qquad (e^{i\theta}, e^{i\tau}) \mapsto e^{i\theta} \cdot e^{i\tau} = e^{i(\theta+\tau)}$$

a smooth map? This is our second example of a Lie Group.

Definition 3.6.12 Let $(M,\mathcal{U})$ and $(N,\mathcal{V})$ be smooth manifolds. The *(smooth) disjoint union* (or *sum*) is the smooth manifold you get by giving the disjoint union $M \coprod N$ the smooth structure given by $\mathcal{U} \cup \mathcal{V}$.

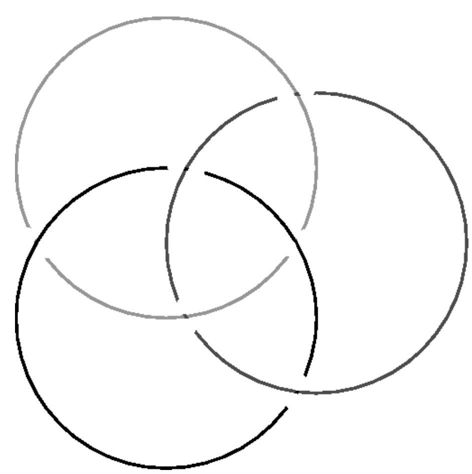


The disjoint union of two tori (imbedded in $\mathbf{R}^3$).

Exercise 3.6.13 Check that this definition makes sense.

Note 3.6.14 As for the product, the atlas we give the sum is not maximal (a chart may have disconnected source and target). There is nothing a priori wrong with taking the disjoint union of an m-dimensional manifold with an n-dimensional manifold. The result will of course neither be m nor n-dimensional. Such examples will not be important to us, and you will find that we in arguments may talk about a smooth manifold, and without hesitation later on start talking about its dimension. This is justified since we can consider one component at a time, and each component will have a well defined dimension.

Example 3.6.15 The Borromean rings gives an interesting example showing that the imbedding in Euclidean space is irrelevant to the manifold: the Borromean rings is the disjoint union of three circles $S^1 \coprod S^1 \coprod S^1$. Don't get confused: it is the imbedding in $\mathbf{R}^3$ that makes your mind spin: the manifold itself is just three copies of the circle! Moral: an imbedded manifold is something more than just a manifold that *can be* imbedded.



Exercise 3.6.16 Prove that the inclusion

$$inc_1 \colon M \subset M \coprod N$$

is an imbedding.

Exercise 3.6.17 Show that giving a smooth map $M \coprod N \to Z$ is the same as giving a pair of smooth maps $M \to Z$ and $N \to Z$. Hence we have a bijection

$$\mathcal{C}^\infty(M \coprod N, Z) \cong \mathcal{C}^\infty(M, Z) \times \mathcal{C}^\infty(N, Z).$$

Chapter 4

The tangent space

In this chapter we will study **linearizations**. You have seen this many times before as *tangent lines* and *tangent planes* (for curves and surfaces in euclidean space), and the main difficulty you will encounter is that the linearizations must be defined intrinsically – i.e., in terms of the manifold at hand – and not with reference to some big ambient space. We will shortly (in 4.0.6) give a simple and perfectly fine technical definition of the tangent space, but for future convenience we will use the concept of *germs* in our final definition. This concept makes notation and bookkeeping easy and is good for all things local (in the end it will turn out that due to the existence of so-called smooth bump functions 4.1.13 we could have stayed global in our definitions).

An important feature of the tangent space is that it is a vector space, and a smooth map of manifolds gives a linear map of vector spaces. Eventually, the chain rule expresses the fact that the tangent space is a "natural" construction (which actually is a very precise statement that will reappear several times in different contexts. It is the hope of the author that the reader, through the many examples, in the end will appreciate the importance of being natural – as well as earnest).

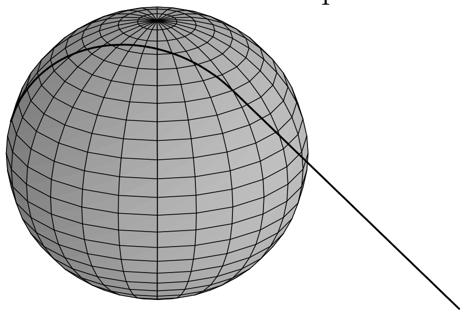
Beside the tangent space, we will also briefly discuss its sibling, the *cotangent space*, which is concerned with linearizing the space of real valued functions, and which is the relevant linearization for many applications.

Another interpretation of the tangent space is as the space of *derivations*, and we will discuss these briefly since they figure prominently in many expositions. They are more abstract and less geometric than the path we have chosen – as a matter of fact, in our presentation derivations are viewed as a "double dualization" of the tangent space.

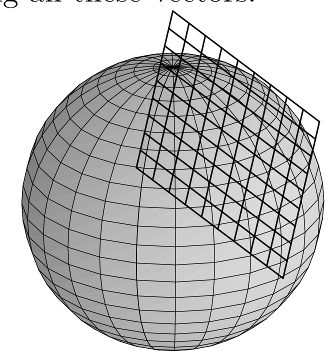
4.0.1 The idea of the tangent space of a submanifold of euclidean space

Given a submanifold M of euclidean space $\mathbf{R}^n$, it is fairly obvious what we should mean by the "tangent space" of M at a point $p \in M$.

In purely physical terms, the tangent space should be the following subspace of $\mathbf{R}^n$: If a particle moves on some curve in M and at p suddenly "loses the grip on M" it will continue out in the ambient space along a straight line (its “tangent”). This straight line is determined by its velocity vector at the point where it flies out into space. The tangent space should be the linear subspace of $\mathbf{R}^n$ containing all these vectors.



A particle loses its grip on M and flies out on a tangent



A part of the space of all tangents

When talking about manifolds it is important to remember that there **is no** ambient space to fly out into, but we still may talk about a tangent space.

4.0.2 Partial derivatives

The tangent space is all about the linearization in Euclidean space. To fix notation we repeat some multivariable calculus.

Definition 4.0.3 Let $f\colon U \to \mathbf{R}$ be a function where U is an open subset of $\mathbf{R}^n$ containing $p = (p_1, \ldots p_n)$. The *ith partial derivative of f at p* is the number (if it exists)

$$D_i f(p) = D_i|_p f = \lim_{h \to 0} \frac{1}{h} \left(f(p + he_i) - f(p) \right),$$

where e_i is the ith unit vector $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ (with a 1 in the ith coordinate). We collect the partial derivatives in an $1 \times n$-matrix

$$Df(p) = D|_p f = (D_1 f(p), \ldots, D_n f(p)).$$

Definition 4.0.4 If $f = (f_1, \ldots, f_m)\colon U \to \mathbf{R}^m$ is a function where U is an open subset of $\mathbf{R}^n$ containing $p = (p_1, \ldots p_n)$, then the *Jacobian matrix* is the $m \times n$-matrix

$$Df(p) = D|_p(f) = \begin{bmatrix} Df_1(p) \\ \vdots \\ Df_m(p) \end{bmatrix}.$$

In particular, if $g = (g_1, \ldots g_n)\colon (a, b) \to \mathbf{R}^m$ the Jacobian is an $n \times 1$-matrix, or element in $\mathbf{R}^n$, which we write as

$$g'(c) = Dg(c) = \begin{bmatrix} g_1'(c) \\ \vdots \\ g_n'(c) \end{bmatrix} \in \mathbf{R}^n.$$

Note 4.0.5 When considered as a vector space, we insist that the elements in $\mathbf{R}^n$ are standing vectors (so that linear maps can be represented by multiplication by matrices from the left), when considered as a manifold the distinction between lying and standing vectors is not important, and we use either convention as may be typographically convenient.

It is a standard fact from multivariable calculus (see e.g., [11, 2-8]) that if $f\colon U \to \mathbf{R}^m$ is continuously differentiable at p (all the partial derivatives exist and are continuous at p), where U is an open subset of $\mathbf{R}^n$, then the Jacobian is the matrix associated (in the standard bases) with the unique linear transformation $L\colon \mathbf{R}^n \to \mathbf{R}^m$ such that

$$\lim_{h\to 0} \frac{1}{h}\left(f(p+h) - f(p) - L(h)\right) = 0.$$

4.0.6 Predefinition of the tangent space

Let M be a smooth manifold, and let $p \in M$. Consider the set of all curves $\gamma\colon \mathbf{R} \to M$ with $\gamma(0) = p$. On this set we define the following equivalence relation: given two curves $\gamma\colon \mathbf{R} \to M$ and $\gamma_1\colon \mathbf{R} \to M$ with $\gamma(0) = \gamma_1(0) = p$ we say that γ and γ_1 are equivalent if for all charts $x\colon U \to U'$ with $p \in U$ we have an equality of vectors

$$(x\gamma)'(0) = (x\gamma_1)'(0).$$

Then the tangent space of M at p is the set of all equivalence classes.

There is nothing wrong with this definition, in the sense that it is naturally isomorphic to the one we are going to give in a short while (see 4.2.1). However, in order to work efficiently with our tangent space, it is fruitful to introduce some language. It is really not necessary for our curves to be defined on all of $\mathbf{R}$, but on the other hand it is not important to know the domain of definition as long as it contains a neighborhood around the origin.

4.1 Germs

Whatever ones point of view on tangent vectors is, it is a local concept. The tangent of a curve passing through a given point p is only dependent upon the behavior of the curve close to the point. Hence it makes sense to divide out by the equivalence relation which says that all curves that are equal on some neighborhood of the point are equivalent. This is the concept of *germs*.

Definition 4.1.1 Let M and N be smooth manifolds, and let $p \in M$. On the set

$$\{f | f\colon U_f \to N \text{ is smooth, and } U_f \text{ an open neighborhood of } p\}$$

we define an equivalence relation where f is equivalent to g, written $f \sim g$, if there is an open neighborhood $V_{fg} \subseteq U_f \cap U_g$ of p such that

$$f(q) = g(q), \text{ for all } q \in V_{fg}$$

Such an equivalence class is called a *germ*, and we write

$$\bar{f}\colon (M,p) \to (N, f(p))$$

for the germ associated to $f\colon U_f \to N$. We also say that f *represents* $\bar{f}$.

Definition 4.1.2 Let M be a smooth manifold and p a point in M. A *function germ* at p is a germ $\bar{\phi}\colon (M,p) \to (\mathbf{R}, \phi(p))$. Let

$$\mathcal{O}_{M,p} = \mathcal{O}_p$$

be the set of function germs at p.

Example 4.1.3 In $\mathcal{O}_{\mathbf{R}^n,0}$ there are some very special function germs, namely those associated to the *standard coordinate functions* pr_i sending $p = (p_1, \dots, p_n)$ to $\mathrm{pr}_i(p) = p_i$ for $i = 1, \dots, n$.

Note 4.1.4 Germs are quite natural things. Most of the properties we need about germs are "obvious" if you do not think too hard about them, so it is a good idea to skip the rest of the section which spells out these details before you know what they are good for. Come back later if you need anything precise.

Exercise 4.1.5 Show that the relation $\sim$ actually is an equivalence relation as claimed in Definition 4.1.1.

The only thing that is slightly ticklish with the definition of germs is the transitivity of the equivalence relation: assume

$$f\colon U_f \to N, \qquad g\colon U_g \to N, \text{ and } h\colon U_h \to N$$

and $f \sim g$ and $g \sim h$. Writing out the definitions, we see that $f = g = h$ on the open set $V_{fg} \cap V_{gh}$, which contains p.

Let

$$\bar{f}\colon (M,p) \to (N, f(p))$$

and

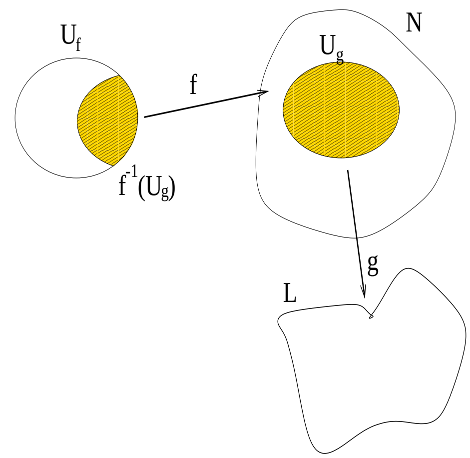
$$\bar{g}\colon (N, f(p)) \to (L, g(f(p)))$$

be two germs represented by the functions $f\colon U_f \to N$ and $g\colon U_g \to L$. Then we define the *composite*

$$\bar{g}\,\bar{f}\colon (M,p) \to (L, g(f(p)))$$

as the germ associated to the composite

$$f^{-1}(U_g) \xrightarrow{f|_{f^{-1}(U_g)}} U_g \xrightarrow{\ g\ } L$$



The composite of two germs: just remember to restrict the domain of the representatives.

(which makes sense since $f^{-1}(U_g) \subseteq M$ is an open set containing p).

Exercise 4.1.6 Show that the composition $\bar{g}\bar{f}$ of germs is well defined in the sense that it does not depend on the chosen representatives g and f. Also, show "associativity": $\bar{h}(\bar{g}\bar{f}) = (\bar{h}\bar{g})\bar{f}$, and that if $\bar{h}$ and $\bar{f}$ are represented by identity functions, then $\bar{h}\bar{g} = \bar{g} = \bar{g}\bar{f}$.

We occasionally write $\overline{gf}$ instead of $\bar{g}\bar{f}$ for the composite, even though the pedants will point out that we have to adjust the domains before composing representatives.

Also, we will be cavalier about the range of germs, in the sense that if $q \in V \subseteq N$ we will sometimes not distinguish notationally between a germ $(M,p) \to (V,q)$ and the germ $(M,p) \to (N,q)$ given by composition with the inclusion.

A germ $\bar{f}\colon (M,p) \to (N,q)$ is *invertible* if (and only if) there is a germ $\bar{g}\colon (N,q) \to (M,p)$ such that the composites $\bar{f}\bar{g}$ and $\bar{g}\bar{f}$ are represented by identity maps.

Lemma 4.1.7 *A germ $\bar{f}\colon (M,p) \to (N,q)$ represented by $f\colon U_f \to N$ is invertible if and only if there is a diffeomorphism $\phi\colon U \to V$ with $U \subseteq U_f$ a neighborhood of p and V a neighborhood of q such that $f(t) = \phi(t)$ for all $t \in U$.*

Proof: If $\phi\colon U \to V$ is a diffeomorphism such that $f(t) = \phi(t)$ for all $t \in U$, then ϕ^{-1} represents an inverse to $\bar{f}$. Conversely, let $g\colon V_g \to M$ represent an inverse to $\bar{f}$. Then there is a neighborhood $p \in U_{gf}$ such that $u = gf(u)$ for all $u \in U_{gf} \subseteq U_f \cap f^{-1}(V_g)$ and a neighborhood $q \in V_{fg} \subseteq g^{-1}(U_f) \cap V_g$ such that $v = fg(v)$ for all $v \in V_{fg}$. Letting $U = U_{gf} \cap f^{-1}(V_{gf})$ and $V = g^{-1}(U_{gf}) \cap V_{gf}$, the restriction of f to U defines the desired diffeomorphism $\phi\colon U \to V$. ∎

Note 4.1.8 The set $\mathcal{O}_{M,p}$ of function germs forms a vector space by pointwise addition and multiplication by real numbers:

$$\begin{array}{rlll} \bar\phi+\bar\psi=\overline{\phi+\psi} & \text{where } (\phi+\psi)(q)=\phi(q)+\psi(q) & \text{for } q\in U_\phi\cap U_\psi \\ k\cdot\bar\phi=\overline{k\cdot\phi} & \text{where } (k\cdot\phi)(q)=k\cdot\phi(q) & \text{for } q\in U_\phi \\ \bar 0 & \text{where } 0(q)=0 & \text{for } q\in M \end{array}$$

It furthermore has the pointwise multiplication, making it what is called a "commutative **R**-algebra":

$$\begin{array}{rlll} \bar\phi\cdot\bar\psi=\overline{\phi\cdot\psi} & \text{where } (\phi\cdot\psi)(q)=\phi(q)\cdot\psi(q) & \text{for } q\in U_\phi\cap U_\psi \\ \bar 1 & \text{where } 1(q)=1 & \text{for } q\in M \end{array}$$

That these structures obey the usual rules follows by the same rules on $\mathbf{R}$.

Since we both multiply and compose germs, we should perhaps be careful in distinguishing the two operations by remembering to write $\circ$ whenever we compose, and $\cdot$ when we multiply. We will be sloppy about this, and the $\circ$ will mostly be invisible. We try to remember to write the $\cdot$, though.

Definition 4.1.9 A germ $\bar f\colon (M,p)\to(N,f(p))$ defines a function

$$f^*\colon \mathcal{O}_{f(p)}\to\mathcal{O}_p$$

by sending a function germ $\bar\phi\colon (N,f(p))\to(\mathbf{R},\phi f(p))$ to

$$\overline{\phi f}\colon (M,p)\to(\mathbf{R},\phi f(p))$$

("precomposition").

Note that f^* preserves addition and multiplication.

Lemma 4.1.10 *If* $\bar f\colon (M,p)\to(N,f(p))$ *and* $\bar g\colon (N,f(p))\to(L,g(f(p)))$ *then*

$$f^*g^*=(gf)^*\colon \mathcal{O}_{L,g(f(p))}\to\mathcal{O}_{M,p}$$

Proof: Both sides send $\bar\psi\colon (L,g(f(p)))\to(\mathbf{R},\psi(g(f(p))))$ to the composite

$$\begin{array}{ccccc} (M,p) & \xrightarrow{\bar f} & (N,f(p)) & \xrightarrow{\bar g} & (L,g(f(p))) \\ & & & & \downarrow{\scriptstyle\bar\psi} \\ & & & & (\mathbf{R},\psi(g(f(p)))), \end{array}$$

i.e., $f^*g^*(\bar\psi)=f^*(\overline{\psi g})=(\overline{\psi g})\bar f=\bar\psi(\overline{gf})=(gf)^*(\bar\psi)$. ∎

The superscript $*$ may help you remember that this construction reverses the order, since it may remind you of transposition of matrices.

Since manifolds are locally Euclidean spaces, it is hardly surprising that on the level of function germs, there is no difference between $(\mathbf{R}^n,0)$ and (M,p).

Lemma 4.1.11 *There are isomorphisms $\mathcal{O}_{M,p} \cong \mathcal{O}_{\mathbf{R}^n,0}$ preserving all algebraic structure.*

Proof: Pick a chart $x\colon U \to U'$ with $p \in U$ and $x(p) = 0$ (if $x(p) \neq 0$, just translate the chart). Then

$$x^*\colon \mathcal{O}_{\mathbf{R}^n,0} \to \mathcal{O}_{M,p}$$

is invertible with inverse $(x^{-1})^*$ (note that $\overline{id_U} = \overline{id_M}$ since they agree on an open subset (namely U) containing p). ∎

Note 4.1.12 So is this the end of the subject? Could we just as well study $\mathbf{R}^n$? **No**! these isomorphisms depend on **a choice of charts**. This is OK if you just look at one point at a time, but as soon as things get a bit messier, this is every bit as bad as choosing particular coordinates in vector spaces.

4.1.13 Smooth bump functions

Germs allow us to talk easily about local phenomena. There is another way of focusing our attention on neighborhoods of a point p in a smooth manifold M, namely by using bump functions. Their importance lies in the fact that they focus the attention on a neighborhood of p, ignoring everything “far away”. The existence of smooth bump functions is a true luxury about smooth manifolds, which makes the smooth case much more flexible than the analytic case. We will return to this topic when we define partitions of unity.

Definition 4.1.14 Let X be a space and p a point in X. A *bump function* around p is a map $\phi\colon X \to \mathbf{R}$, which takes values in the closed interval $[0, 1]$ only, which takes the constant value 1 in (the closure of) a neighborhood of p, and takes the constant value 0 outside some bigger neighborhood.

We will only be interested in *smooth* bump functions.

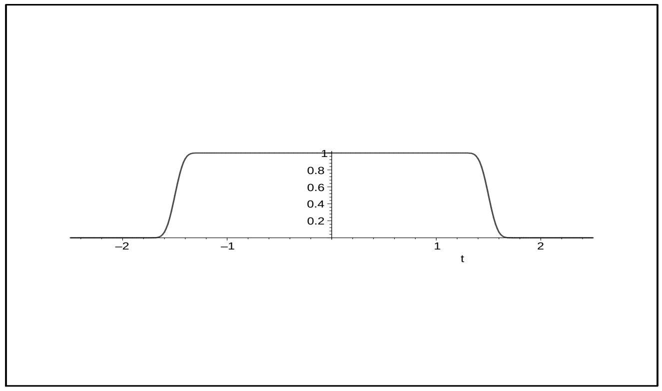
Definition 4.1.15 Let X be a space. The *support* of a function $f : X \to \mathbf{R}$ is the closure of the subset of X with nonzero values, i.e.,

$$\operatorname{supp}(f) = \overline{\{x \in X | f(x) \neq 0\}}$$

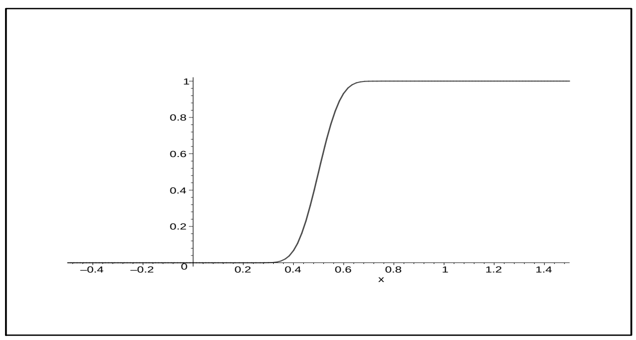
Lemma 4.1.16 *Given $r, \epsilon > 0$, there is a smooth bump function*

$$\gamma_{r,\epsilon}\colon \mathbf{R}^n \to \mathbf{R}$$

with $\gamma_{r,\epsilon}(t) = 1$ for $|t| \le r$ and $\gamma_{r,\epsilon}(t) = 0$ for $|t| \ge r + \epsilon$.
More generally, if M is a manifold and $p \in M$, then there exist smooth bump functions around p.



Proof: Let $\beta_\epsilon\colon \mathbf{R} \to \mathbf{R}$ be any smooth function with non-negative values and support $[0, \epsilon]$ (for instance, you may use the $\beta_\epsilon(t) = \lambda(t) \cdot \lambda(t - \epsilon)$ where λ is the function of Exercise 3.2.13).
Since β_ϵ is smooth, it is integrable with $\int_0^\epsilon \beta_\epsilon(x)\ dx > 0$, and we may define the smooth step function α_ϵ : $\mathbf{R} \to \mathbf{R}$ which ascends from zero to one smoothly between zero and ϵ by means of



$$\alpha_\epsilon(t) = \frac{\int_0^t \beta_\epsilon(x)\ dx}{\int_0^\epsilon \beta_\epsilon(x)\ dx}.$$

Finally, $\gamma_{(r,\epsilon)} : \mathbf{R}^n \to \mathbf{R}$ is given by

$$\gamma_{(r,\epsilon)}(x) = 1 - \alpha_\epsilon(|x| - r).$$

As to the more general case, choose a chart (x, U) for the smooth manifold M with $p \in U$. By translating, we may assume that $x(p) = 0$. Since $x(U) \subseteq \mathbf{R}^n$ is open, there are $r, \epsilon > 0$ such that the open ball of radius $r + 2\epsilon$ is contained in $x(U)$. The function given by sending $q \in M$ to $\gamma_{(r,\epsilon)}x(q)$ if $q \in U$ and to 0 if $q \neq U$ is a smooth bump function around p. ∎

Example 4.1.17 Smooth bump functions are very handy, for instance if you want to join curves in a smooth fashion (for instance if you want to design smooth highways!) They also allow you to drive smoothly on a road with corners: the curve $\gamma\colon \mathbf{R} \to \mathbf{R}^2$ given by $\gamma(t) = (te^{-1/t^2}, |te^{-1/t^2}|)$ is smooth, although its image is not.

Exercise 4.1.18 Given $\epsilon > 0$, prove that there is a diffeomorphism $f\colon (-\epsilon, \epsilon) \to \mathbf{R}$ such that $f(t) = t$ for $|t|$ small. Conclude that any germ $\bar{\gamma}\colon (\mathbf{R}, 0) \to (M, p)$ is represented by a "globally defined" curve $\gamma\colon \mathbf{R} \to M$.

Exercise 4.1.19 Show that any function germ $\bar{\phi}\colon (M,p) \to (\mathbf{R}, \phi(p))$ has a smooth representative $\phi\colon M \to \mathbf{R}$.

Exercise 4.1.20 Let M and N be smooth manifolds and $f\colon M \to N$ a continuous map. Show that f is smooth if for all smooth $\phi\colon N \to \mathbf{R}$ the composite $\phi f\colon M \to \mathbf{R}$ is smooth.

4.2 The tangent space

Note that if $\bar{\gamma}\colon (\mathbf{R}, 0) \to (\mathbf{R}^n, \gamma(0))$ is some germ into Euclidean space, the derivative at zero does not depend on a choice of representative (i.e., if γ and γ_1 are two representatives for $\bar{\gamma}$, then $\gamma'(0) = \gamma_1'(0)$), and we write $\gamma'(0)$ without ambiguity.

Definition 4.2.1 Let $(M, \mathcal{A})$ be a smooth n-dimensional manifold. Let $p \in M$ and let

$$W_p = \{\text{germs } \bar{\gamma} : (\mathbf{R}, 0) \to (M, p)\}.$$

Two germs $\bar{\gamma}, \bar{\gamma}_1 \in W_p$ are said to be equivalent, written $\bar{\gamma} \approx \bar{\gamma}_1$, if for all function germs $\bar{\phi} : (M,p) \to (\mathbf{R}, \phi(p))$ we have that $(\phi\gamma)'(0) = (\phi\gamma_1)'(0)$. We define the *tangent space* of M at p to be the set of equivalence classes

$$T_pM = W_p/\approx .$$

We write $[\bar{\gamma}]$ (or simply $[\gamma]$) for the $\approx$-equivalence class of $\bar{\gamma}$. This definition is essentially the same as the one we gave in section 4.0.6 (see Lemma 4.2.11 below). So for the definition of the tangent space, it is not necessary to involve the definition of germs, but it is convenient when working with the definition since we are freed from specifying domains of definition all the time.

As always, it is not the objects, but the maps comparing them that are important, and so we need to address how the tangent space construction is to act on smooth maps and germs

Definition 4.2.2 Let $\bar{f}\colon (M,p) \to (N, f(p))$ be a germ. Then we define

$$T_pf\colon T_pM \to T_{f(p)}N$$

by

$$T_pf([\gamma]) = [f\gamma].$$

Exercise 4.2.3 This is well defined.

Anybody recognize the next lemma? It is the chain rule!

Lemma 4.2.4 *If $\bar{f}\colon (M,p) \to (N, f(p))$ and $\bar{g}\colon (N, f(p)) \to (L, g(f(p)))$ are germs, then*

$$T_{f(p)}g\, T_p f = T_p(gf).$$

Proof: Let $\bar{\gamma}\colon (\mathbf{R}, 0) \to (M,p)$, then

$$T_{f(p)}g(T_p f([\gamma])) = T_{f(p)}g([f\gamma]) = [gf\gamma] = T_p(gf)([\gamma])$$

∎

That's the ultimate proof of the chain rule! The ultimate way to remember it is: the two ways around the triangle

$$\begin{array}{ccc} T_pM & \xrightarrow{T_pf} & T_{f(p)}N \\ & \searrow^{T_p(gf)} & \downarrow{\scriptstyle T_{f(p)}g} \\ & & T_{gf(p)}L \end{array} \quad .$$

are the same ("the diagram commutes").

Note 4.2.5 For the categorists: the tangent space is an assignment from pointed manifolds to vector spaces, and the chain rule states that it is a "functor".

Exercise 4.2.6 Show that if the germ $\bar{f}\colon (M,p) \to (N,(f(p))$ is invertible (i.e., there is a germ $\bar{g}\colon (N,(f(p)) \to (M,p)$ such that $\bar{g}\bar{f}$ is the identity germ on (M,p) and $\bar{f}\bar{g}$ is the identity germ on $(N, f(p)))$, then T_pf is a bijection with inverse $T_{f(p)}g$. In particular, the tangent space construction sends diffeomorphisms to bijections.

4.2.7 The vector space structure

The "flat chain rule" 4.2.8 from multivariable calculus will be used to show that the tangent spaces are vector spaces and that T_pf is a linear map, but if we were content with working with sets only, the one line proof of the chain rule in 4.2.4 would be all we'd ever need. For convenience, we cite the flat chain rule below. For a proof, see e.g., [11, 2-9], or any decent book on multi-variable calculus.

Lemma 4.2.8 *(The flat chain rule) Let $g\colon (a,b) \to U$ and $f\colon U \to \mathbf{R}$ be smooth functions where U is an open subset of $\mathbf{R}^n$ and $c \in (a,b)$. Then*

$$\begin{aligned}(fg)'(c) =& D(f)(g(c)) \cdot g'(c) \\ =& \sum_{j=1}^{n} D_j f(g(c)) \cdot g_j'(c)\end{aligned}$$

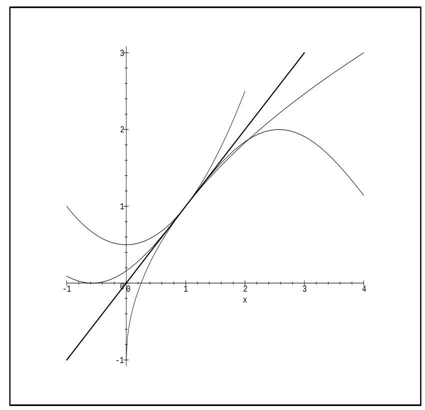
Exercise 4.2.9 Show that the equivalence relation on W_p in Definition 4.2.1 could equally well be described as follows: Two germs $\bar{\gamma}, \bar{\gamma}_1 \in W_p$ are said to be equivalent, if for all charts $(x,U) \in \mathcal{A}$ with $p \in U$ we have that $(x\gamma)'(0) = (x\gamma_1)'(0)$.

Exercise 4.2.10 Show that for two germs $\bar{\gamma}, \bar{\gamma}_1 \colon (\mathbf{R}, 0) \to (M, p)$ to define the same tangent vector, it is enough that $(x\gamma)'(0) = (x\gamma_1)'(0)$ for *some* chart (x, U).

Summing up, the predefinition of the tangent space given in section 4.0.6 agrees with the official definition (we allow ourselves to make the conclusion of exercises official when full solutions are provided):

Proposition 4.2.11 *The tangent space at a point p is the set of all (germs of) curves sending 0 to p, modulo the identification of all curves having equal derivatives at 0 in some chart.*

Proof: This is the contents of the Exercises 4.2.9 and 4.2.10, and since by Exercise 4.1.18 all germs of curves have representatives defined on all of $\mathbf{R}$, the parenthesis could really be removed. ∎



Many curves give rise to the same tangent.

In particular if $M = \mathbf{R}^n$, then two curves $\gamma_1, \gamma_2 \colon (\mathbf{R}, 0) \to (\mathbf{R}^n, p)$ define the same tangent vector if and only if the derivatives are equal:

$$\gamma_1'(0) = \gamma_2'(0)$$

(using the identity chart). Hence, a tangent vector in $\mathbf{R}$ is uniquely determined by (p and) its derivative at 0, and so $T_p\mathbf{R}^n$ may be identified with $\mathbf{R}^n$:

Lemma 4.2.12 *A germ $\bar{\gamma} \colon (\mathbf{R}, 0) \to (\mathbf{R}^n, p)$ is $\approx$-equivalent to the germ represented by*

$$t \mapsto p + \gamma'(0)t.$$

That is, all elements in $T_p\mathbf{R}^n$ are represented by linear curves, giving a bijection

$$T_p\mathbf{R}^n \cong \mathbf{R}^n, \qquad [\gamma] \mapsto \gamma'(0).$$

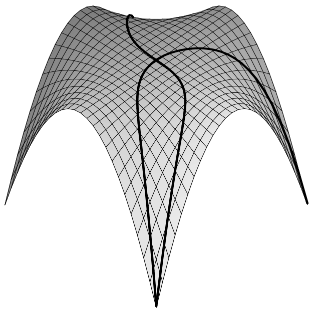
More generally, if M is an n-dimensional smooth manifold, p a point in M and (x, U) a chart with $p \in U$, then the map

$$A_x \colon T_pM \to \mathbf{R}^n, \qquad A_x([\gamma]) = (x\gamma)'(0)$$

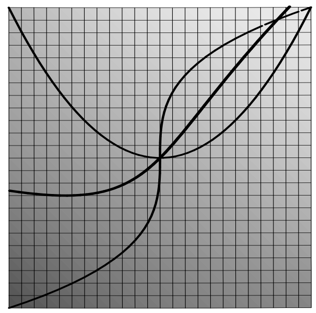
is a bijection with inverse ${A_x}^{-1}(v) = [B_x^v]$ where $B_x^v(t) = x^{-1}(x(p) + tv)$.

Proof: It is enough to check that the purported formula for the inverse actually works. We check both composites, using that $xB_x^v(t) = x(p) + tv$, and so $(xB_x^v)'(0) = v$: ${A_x}^{-1}A_x([\gamma]) = [B_x^{(x\gamma)'(0)}] = [\gamma]$ and $A_x{A_x}^{-1}(v) = (xB_x^v)'(0) = v$. ∎

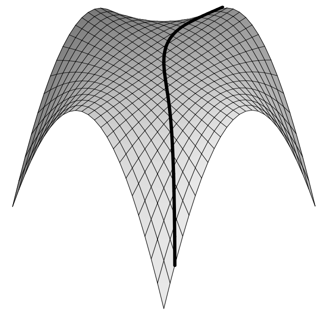
Note 4.2.13 The tangent space is a vector space, and like always we fetch the structure locally by means of charts. Visually it goes like this:



Two curves on M is sent by a chart x to



$\mathbf{R}^n$, where they are added, and the sum



is sent back to M with x^{-1}.

Explicitly, if $[\gamma_1], [\gamma_2] \in T_pM$ and $a, b \in \mathbf{R}$ we define

$$a[\gamma_1] + b[\gamma_2] = {A_x}^{-1}\left(aA_x[\gamma_1] + bA_x[\gamma_2]\right).$$

This is all well and fine, but would have been quite worthless if the vector space structure depended on a choice of chart. Of course, it does not.

Lemma 4.2.14 *The above formula for a vector space structure on T_pM is independent of the choice of chart.*

Proof: If (y, V) is another chart on M with $p \in V$, then we must show that

$${A_x}^{-1}\left(aA_x[\gamma_1] + bA_x[\gamma_2]\right) = {A_y}^{-1}\left(aA_y[\gamma_1] + bA_y[\gamma_2]\right),$$

or alternatively, that

$$aA_x[\gamma_1] + bA_x[\gamma_2] = A_x{A_y}^{-1}\left(aA_y[\gamma_1] + bA_y[\gamma_2]\right).$$

Spelling this out, we see that the question is whether the vectors $a(x\gamma_1)'(0) + b(x\gamma_2)'(0)$ and $\frac{d}{dt}\Big|_{t=0}\left(xy^{-1}\left(y(p) + t(a(y\gamma_1)'(0) + b(y\gamma_2)'(0)\right)\right)$ are equal. The flat chain rule gives that the last expression is equal to $D(xy^{-1})(y(p)) \cdot \left(a(y\gamma_1)'(0) + b(y\gamma_2)'(0)\right)$, which, by linearity of matrix multiplication, is equal to

$$aD(xy^{-1})(y(p)) \cdot (y\gamma_1)'(0) + bD(xy^{-1})(y(p)) \cdot (y\gamma_2)'(0).$$

A final application of the flat chain rule on each of the summands ends the proof. ∎

Proposition 4.2.15 *Let $\bar{f}\colon (M,p) \to (N, f(p))$ be a germ, then the tangent map $T_p f\colon T_pM \to T_{f(p)}N$ is linear. If (x, U) is a chart in M with $p \in U$ and (y, V) a chart in N with $f(p) \in V$, then the diagram*

$$\begin{array}{ccc} T_pM & \xrightarrow{T_pf} & T_{f(p)}N \\ \cong\big\downarrow A_x & & \cong\big\downarrow A_y \\ \mathbf{R}^m & \xrightarrow{D(yfx^{-1})(x(p))\cdot} & \mathbf{R}^n \end{array}$$

commutes, where the bottom horizontal map is the linear map given by multiplication with the Jacobi matrix $D(yfx^{-1})(x(p))$.

Proof: That T_pf is linear follows from the commutativity of the diagram: T_pf is the result of composing three linear maps (the last is ${A_y}^{-1}$). To show that the diagram commutes, start with a $[\gamma] \in T_pM$. Going down and right we get $D(yfx^{-1})(x(p)) \cdot (x\gamma)'(0)$ and going right and down we get $(yf\gamma)'(0)$. That these two expressions agree is the chain rule: $(yf\gamma)'(0) = (yfx^{-1}x\gamma)'(0) = D(yfx^{-1})(x(p)) \cdot (x\gamma)'(0)$. ∎

Proposition 4.2.15 is extremely useful, not only because it proves that T_pf is linear, but also because it gives us a concrete way of calculating the tangent map. Many questions can be traced back to a question of whether T_pf is onto ("p is a regular point"), and we see that Proposition 4.2.15 translates this to the question of whether the Jacobi matrix $D(yfx^{-1})(x(p))$ has rank equal to the dimension of N.

Example 4.2.16 Consider the map $\det\colon M_2(\mathbf{R}) \to \mathbf{R}$ sending the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

to its determinant $\det(A) = a_{11}a_{22} - a_{12}a_{21}$. Using the chart $x\colon M_2(\mathbf{R}) \to \mathbf{R}^4$ with

$$x(A) = \begin{bmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{bmatrix}$$

(and the identity chart on $\mathbf{R}$) we have that the Jacobi matrix is the 1×4-matrix

$$D(\det x^{-1})(x(A)) = [a_{22}, -a_{21}, -a_{12}, a_{11}]$$

(check this!). Thus we see that the rank of $D(\det x^{-1})(x(A))$ is 0 if $A = 0$ and 1 if $A \neq 0$. Hence $T_A \det\colon T_AM_2(\mathbf{R}) \to T_{\det A}\mathbf{R}$ is onto if and only if $A \neq 0$ (and $T_0 \det = 0$).

Exercise 4.2.17 Consider the determinant map $\det\colon M_n(\mathbf{R}) \to \mathbf{R}$ for $n > 1$. Show that $T_A \det$ is onto if the rank of the $n \times n$-matrix A is greater than $n-2$ and $T_A \det = 0$ if $rkA < n-1$.

Exercise 4.2.18 Let $L\colon \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation. Show that $DL(p)$ is the matrix associated with L in the standard basis (and so independent of the point p).

4.3 The cotangent space

Although the tangent space has a clear physical interpretation as the space of all possible velocities at a certain point of a manifold, it turns out that for many applications – including mechanics – the *cotangent space* is even more fundamental.

As opposed to the tangent space, which is defined in terms of maps **from** the real line to the manifold, the cotangent space is defined in turn of maps **to** the real line. We are really having a glimpse of a standard mathematical technique: if you want to understand an object, a good way is to understand the maps to or from something you think you understand (in this case the real line). The real line is the "yardstick" for spaces.

Recall from 4.1.2 that $\mathcal{O}_{M,p}$ denotes the algebra of function germs $\bar{\phi}\colon (M,p) \to (\mathbf{R}, \phi(p))$. If W is a subspace of a vector space V, then the quotient space V/W is the vector space you get from V by dividing out by the equivalence relation $v \sim v + w$ for $v \in V$ and $w \in W$. The vector space structure on V/W is defined by demanding that the map $V \to V/W$ sending a vector to its equivalence class is linear.

Definition 4.3.1 Let M be a smooth manifold, and let $p \in M$. Let $J = J_pM \subseteq \mathcal{O}_{M,p}$ be the vector space of all smooth function germs $\bar{\phi}\colon (M,p) \to (\mathbf{R}, 0)$ (i.e., such that $\phi(p) = 0$), and let J^2 be the sub-vector space spanned by all products $\bar{\phi} \cdot \bar{\psi}$ where $\bar{\phi}$ and $\bar{\psi}$ are in J. The *cotangent space, T_p^*M, of M at p* is the quotient space J/J^2. The elements of T_p^*M are referred to as *cotangent vectors*.

Let $\bar{f}\colon (M,p) \to (N, f(p))$ be a smooth germ. Then $T^*f = T_p^*f\colon T^*_{f(p)}N \to T_p^*M$ is the linear transformation given by sending the sending the cotangent vector represented by the function germ $\bar{\psi}\colon (N, f(p)) \to (\mathbf{R}, 0)$ to the cotangent vector represented by $\bar{\psi}\bar{f}\colon (M,p) \to (N, f(p)) \to (\mathbf{R}, 0)$.

Lemma 4.3.2 *If $\bar{f}\colon (M, g(p)) \to (N, fg(p))$ and $\bar{g}\colon (L,p) \to (M, g(p))$ are smooth germs, then $T^*(fg) = T^*gT^*f$, i.e.,*

$$\begin{array}{ccc} T^*_{fg(p)}N & \xrightarrow{T^*f} & T^*_{g(p)}M \\ & \searrow^{T^*(fg)} & \downarrow T^*g \\ & & T^*_pL \end{array}$$

commutes.

Proof: There is only one way to compose the ingredients, and the lemma follows since composition is associative: $\psi(fg) = (\psi f)g$. ∎

Exercise 4.3.3 Prove that if $\bar{f}$ is an invertible germ, then f^* is an isomorphism.

Note 4.3.4 In the classical literature there is frequently some magic about "contravariant and covariant tensors" transforming this or that way. To some of us this is impossible to remember, but it *is* possible to remember whether our construction turns arrows around or not.

The tangent space keeps the direction: a germ $\bar{f}\colon (M,p) \to (N,q)$ gives a map $Tf\colon T_pM \to T_qN$, and the chain rule tells us that composition is OK – $T(fg) = TfTg$. The cotangent construction turns around the arrows: we get a map $T^*f\colon T_q^*N \to T_p^*M$ and the "cochain rule" 4.3.2 says that composition follows suit – $T^*(fg) = T^*gT^*f$.

Definition 4.3.5 The linear map $d\colon \mathcal{O}_{M,p} \to T_p^*(M)$ given by sending $\bar{\phi}\colon M \to \mathbf{R}$ to the class $d\phi \in J_p/J_p^2$ represented by $\bar{\phi} - \phi(p) = [q \mapsto \phi(q) - \phi(p)] \in J_p$ is called the *differential*.

The differential is obviously a surjection, and when we pick an arbitrary element from the cotangent space, it is often convenient to let it be on the form $d\phi$. We note that $T^*f(d\phi) = d(\phi f)$.

Exercise 4.3.6 The differential $d\colon \mathcal{O}_{M,p} \to T_p^*(M)$ is natural, i.e., if $\bar{f}\colon (M,p) \to (N,q)$ is a smooth germ, then

$$\begin{array}{ccc} \mathcal{O}_{N,q} & \xrightarrow{f^*} & \mathcal{O}_{M,p} \\ {\scriptstyle d}\big\downarrow & & {\scriptstyle d}\big\downarrow \\ T_q^*N & \xrightarrow{T^*f} & T_p^*M \end{array},$$

where $f^*(\bar{\phi}) = \bar{\phi}\bar{f}$, commutes.

Lemma 4.3.7 *The differential $d\colon \mathcal{O}_{M,p} \to T_p^*(M)$ is a* derivation*, i.e., is is a linear map of real vector spaces satisfying the* Leibniz condition*:*

$$d(\phi \cdot \psi) = d\phi \cdot \psi + \phi \cdot d\psi,$$

where $\phi \cdot d\psi = d\psi \cdot \phi$ is the cotangent vector represented by $q \mapsto \phi(q) \cdot (\psi(q) - \psi(p))$.

Proof: We want to show that $d\phi \cdot \psi(p) + \phi(p) \cdot d\psi - d(\phi \cdot \psi)$ vanishes. It is represented by $(\bar{\phi} - \phi(p)) \cdot \bar{\psi} + \bar{\phi} \cdot (\bar{\psi} - \psi(p)) - (\bar{\phi} \cdot \bar{\psi} - \phi(p) \cdot \psi(p)) \in J_p$, which, upon collecting terms, is equal to $(\bar{\phi} - \phi(p)) \cdot (\bar{\psi} - \psi(p)) \in J_p^2$, and hence represents zero in $T_p^*M = J_p/J_p^2$. ∎

In order to relate the tangent and cotangent spaces, we need to understand the situation $(\mathbf{R}^n, 0)$. The corollary of the following lemma pins down the rôle of $J^2_{\mathbf{R}^n,0}$.

Lemma 4.3.8 *Let $\phi\colon U \to \mathbf{R}$ be a smooth map where U is an open ball in $\mathbf{R}^n$ containing the origin. Then*

$$\phi(p) = \phi(0) + \sum_{i=1}^n p_i \cdot \phi_i(p), \qquad \textit{where} \qquad \phi_i(p) = \int_0^1 D_i\phi(t \cdot p)\, dt.$$

Note that $\phi_i(0) = D_i\phi(0)$.

Proof: For $p \in U$ and $t \in [0,1]$, let $F(t) = \phi(t \cdot p)$. Then $\phi(p) - \phi(0) = F(1) - F(0) = \int_0^1 F'(t)\, dt$ by the fundamental theorem of calculus, and $F'(t) = \sum_{i=1}^n p_i D_i\phi(t \cdot p)$ by the chain rule. ∎

Corollary 4.3.9 *The map $J_{\mathbf{R}^n,0} \to M_{1\times n}(\mathbf{R})$ sending $\bar\phi$ to $D\phi(0)$ has kernel $J^2_{\mathbf{R}^n,0}$.*

Proof: The Leibniz rule implies that $J^2_{\mathbf{R}^n,0}$ is in the kernel $\{\bar\phi \in J_{\mathbf{R}^n,0} | D\phi(0) = 0\}$: If $\phi(p) = \psi(p) = 0$, then $D(\phi\cdot\psi)(0) = \phi(0)\cdot D\psi(0) + D\phi(0)\cdot\psi(0) = 0$. Conversely, assuming that $\phi(0) = 0$ and $D\phi(0) = 0$, the decomposition $\phi = 0 + \sum_{j=1}^n \mathrm{pr}_j \phi_j$ of Lemma 4.3.8 (where $\mathrm{pr}_j \colon \mathbf{R}^n \to \mathbf{R}$ is the jth projection, which obviously gives an element in $J_{\mathbf{R}^n,0}$) expresses $\bar\phi$ as an element of $J^2_{\mathbf{R}^n,0}$, since $\phi_j(0) = D_j\phi(0) = 0$. ∎

Definition 4.3.10 Let V be a real vector space. The *dual* of V, written V^*, is the vector space $\mathrm{Hom}_{\mathbf{R}}(V, \mathbf{R})$ of all linear maps $V \to \mathbf{R}$. Addition and multiplication by scalars are performed pointwise, in the sense that if $a, b \in \mathbf{R}$ and $f, g \in V^*$, then $af + bg$ is the linear map sending $v \in V$ to $af(v) + bg(v) \in \mathbf{R}$.

If $f\colon V \to W$ is linear, then the *dual* linear map $f^*\colon W^* \to V^*$indexf@$f^*\colon W^* \to V^*$ is defined by sending $h\colon W \to \mathbf{R}$ to the composite $hf\colon V \to W \to \mathbf{R}$.

Notice that $(gf)^* = f^*g^*$.

Example 4.3.11 If $V = \mathbf{R}^n$, then any linear transformation $V \to \mathbf{R}$ is uniquely represented by a $1 \times n$-matrix, and we get an isomorphism

$$(\mathbf{R}^n)^* \cong M_{1\times n}(\mathbf{R}) = \{v^t | v \in \mathbf{R}^n\}.$$

If $f\colon \mathbf{R}^n \to \mathbf{R}^m$ is represented by the $m \times n$-matrix A, then $f^*\colon (\mathbf{R}^m)^* \to (\mathbf{R})^*$ is represented by the transpose A^t of A in the sense that if $h \in (\mathbf{R}^m)^*$ corresponds to v^t, then $f^*(h) = hf \in (\mathbf{R}^n)^*$ corresponds to $v^tA = (A^tv)^t$.

This means that if V is a finite dimensional vector space, then V and V^* are isomorphic (they have the same dimension), but there is no *preferred* choice of isomorphism.

The promised natural isomorphism between the cotangent space and the dual of the tangent space is given by the following proposition.

Proposition 4.3.12 *Consider the assignment*

$$\alpha = \alpha_{M,p}\colon T_p^*M \to (T_pM)^*, \qquad d\phi \mapsto \{[\gamma] \mapsto (\phi\gamma)'(0)\}.$$

1. *$\alpha_{M,p}$ is a well defined linear map.*
2. *$\alpha_{M,p}$ is natural in (M,p), in the sense that if $\bar f\colon (M,p) \to (N,q)$ is a germ, then the diagram*

$$\begin{array}{ccc} T_q^*N & \xrightarrow{T^*f} & T_p^*M \\ {\scriptstyle\alpha_{N,q}}\downarrow & & \downarrow{\scriptstyle\alpha_{M,p}} \\ (T_qN)^* & \xrightarrow{(Tf)^*} & (T_pM)^* \end{array}$$

commutes.

3. *Let (x, U) be a chart for M with $p \in U$, and $A_x\colon T_pM \to \mathbf{R}^m$ the isomorphism of Lemma 4.2.12 given by $A_x[\gamma] = (x\gamma)'(0)$. Then the composite*

$$T_p^*M \xrightarrow{\alpha_{M,p}} (T_pM)^* \xrightarrow[\cong]{(A_x{}^{-1})^*} (\mathbf{R}^m)^*$$

sends the cotangent vector $d\phi$ to (the linear transformation $\mathbf{R}^m \to \mathbf{R}$ given by multiplication with) the Jacobi matrix $D(\phi x^{-1})(x(p))$.

4. *$\alpha_{M,p}$ is an isomorphism.*

Proof:

1. We show that J_p^2 is in the kernel of the (obviously well defined) linear transformation $J_p(M) \to (T_pM)^*$ sending $\bar{\phi}$ to the linear transformation $[\gamma] \mapsto (\phi\gamma)'(0)$. If $\phi(p) = \psi(p) = 0$, then the Leibniz rule gives

$$((\phi \cdot \psi)\gamma)'(0) = ((\phi\gamma) \cdot (\psi\gamma))'(0) = (\phi\gamma)(0) \cdot (\psi\gamma)'(0) + (\phi\gamma)'(0) \cdot (\psi\gamma)(0) = 0,$$

regardless of γ.

2. Write out the definitions and conclude that both ways around the square send a cotangent vector $d\phi$ to the linear map $\{[\gamma] \mapsto (\phi \circ f \circ \gamma)'(0)\}$.

3. Recalling that $A_x{}^{-1}(v) = [t \mapsto x^{-1}(x(p) + tv)]$ we get that the composite sends the cotangent vector $d\phi$ to the element in $(\mathbf{R}^m)^*$ given by sending $v \in \mathbf{R}^m$ to the derivative at 0 of $t \mapsto \phi x^{-1}(x(p)+tv)$, which, by the chain rule is exactly $D(\phi x^{-1})(x(p)) \cdot v$.

4. By naturality, we just have to consider the case $(M, p) = (\mathbf{R}^m, 0)$ (use naturality with $\bar{f}$ the germ of a chart). Hence we are reduced to showing that the composite $(A_x{}^{-1})^*\alpha_{\mathbf{R}^m,0}$ is an isomorphism when x is the identity chart. But this is exactly Corollary 4.3.9: the kernel of $J_{\mathbf{R}^m,0} \to (\mathbf{R}^m)^* \cong M_{1\times m}(\mathbf{R})$ sending $\bar{\phi}$ to $D\phi(0)$ is precisely $J^2_{\mathbf{R}^m,0}$ and so the induces map from $T_0^*\mathbf{R}^m = J_{\mathbf{R}^m,0}/J^2_{\mathbf{R}^m,0}$ is an isomorphism.

∎

In order to get a concrete grip on the cotangent space, we should understand the linear algebra of dual vector spaces a bit better.

Definition 4.3.13 If $\{v_1, \dots, v_n\}$ is a basis for the vector space V, then the *dual basis* $\{v_1^*, \dots, v_n^*\}$ for V^* is given by $v_j^*\left(\sum_{i=1}^n a_i v_i\right) = a_j$.

Exercise 4.3.14 Check that the dual basis **is** a basis and that f^* is a linear map with associated matrix the transpose of the matrix of f.

Note 4.3.15 If (x, U) is a chart for M around $p \in M$ and let $x_i = \mathrm{pr}_i x$ be the "ith coordinate". The proof of proposition 4.3.12 shows that $\{dx_i\}_{i=1,\dots,n}$ is a basis for the cotangent space T_p^*M. The isomorphism $\alpha_{M,p}$ sends this basis to the dual basis of the basis $\{A_x{}^{-1}(e_i)\}_{i=1,\dots,n}$ for T_pM (where $\{e_i\}_{i=1,\dots,n}$ is the standard basis for $\mathbf{R}^n$).

Exercise 4.3.16 Verify the claim in the note. Also show that

$$d\phi = \sum_{i=1}^{n} D_i(\phi x^{-1})(x(p)) \cdot dx_i.$$

To get notation as close as possible to the classical, one often writes $\partial\phi/\partial x_i(p)$ instead of $D_i(\phi x^{-1})(x(p))$, and gets the more familiar expression

$$d\phi = \sum_{i=1}^{n} \frac{\partial \phi}{\partial x_i}(p) \cdot dx_i.$$

One good thing about understanding manifolds is that we finally can answer the question "what is the x in that formula. What does actually 'variables' mean, and what is the mysterious symbol 'dx_i'?" The x is the name of a particular chart. In the special case where $x = id\colon \mathbf{R}^n = \mathbf{R}^n$ we see that x_i is just a name for the projection onto the ith coordinate and $D_i(\phi)(p) = \partial\phi/\partial x_i(p)$.

Note 4.3.17 Via the correspondence between a basis and its dual in terms of transposition we can explain the classical language of "transforming this or that way". If $x\colon \mathbf{R}^n \cong \mathbf{R}^n$ is a diffeomorphism (and so is a chart in the standard smooth structure of $\mathbf{R}^n$, or a "change of coordinates") and $p \in \mathbf{R}^n$, then the diagram

$$\begin{array}{ccc} T_p\mathbf{R}^n & \xrightarrow{T_p x} & T_{x(p)}\mathbf{R}^n \\ {\scriptstyle [\gamma]\mapsto\gamma'(0)}\downarrow\cong & & {\scriptstyle [\gamma]\mapsto\gamma'(0)}\downarrow\cong \\ \mathbf{R}^n & \xrightarrow{Dx(p)\cdot} & \mathbf{R}^n \end{array}$$

commutes, that is, the change of coordinates x transforms tangent vectors by multiplication by the Jacobi matrix $Dx(p)$. For cotangent vectors the situation is that,

$$\begin{array}{ccc} T^*_{x(p)}\mathbf{R}^n & \xrightarrow{T^* x} & T^*_p\mathbf{R}^n \\ \cong\downarrow{\scriptstyle d\phi\mapsto[D\phi(p)]^t} & & \cong\downarrow{\scriptstyle d\phi\mapsto[D\phi(p)]^t} \\ \mathbf{R}^n & \xrightarrow{[Dx(p)]^t\cdot} & \mathbf{R}^n \end{array}$$

commutes.

Exercise 4.3.18 Let $0 \neq p \in M = \mathbf{R}^2 = \mathbf{C}$, let $x\colon \mathbf{R}^2 = \mathbf{R}^2$ be the identity chart and $y\colon V \cong V'$ be polar coordinates: $y^{-1}(r,\theta) = re^{i\theta}$, where V is $\mathbf{C}$ minus some ray from the origin not containing p, and V' the corresponding strip of radii and angles. Show that the upper horizontal arrow in

$$\begin{array}{ccccc} & & T^*_p M & & \\ & \overset{T^*x}{\nearrow} & & \overset{T^*y}{\nwarrow} & \\ T^*_{x(p)}\mathbf{R}^2 & & \longrightarrow & & T^*_{y(p)}\mathbf{R}^2 \\ \downarrow{\scriptstyle d\phi\mapsto[D\phi(x(p))]^t} & & & & \downarrow{\scriptstyle d\phi\mapsto[D\phi(y(p))]^t} \\ \mathbf{R}^2 & & \longrightarrow & & \mathbf{R}^2 \end{array}$$

is $T^*(xy^{-1})$ and the lower horizontal map is given by multiplication by the transposed Jacobi matrix $D(xy^{-1})(y(p))^t$, and calculate this explicitly in terms of p_1 and p_2.

Conversely, in the same diagram with tangent spaces instead of cotangent spaces (remove the superscript *, reverse the diagonal maps, and let the vertical maps be given by $[\gamma] \mapsto (x\gamma)'(0)$ and $[\gamma] \mapsto (y\gamma)'(0)$ respectively), the upper horizontal map is $T_{x(p)}(yx^{-1})$ and the lower is given by multiplication with the Jacobi matrix $D(yx^{-1})(x(p))$, and calculate this explicitly in terms of p_1 and p_2.

Example 4.3.19 If this example makes no sense to you, don't worry, it's for the physicists among us! Classical mechanics is all about the relationship between the tangent and cotangent space. More precisely, the kinetic energy E should be thought of as (half) a *inner product* g on the tangent space i.e., as a symmetric bilinear and positive definite map

$$g = 2E\colon T_pM \times T_pM \to \mathbf{R}.$$

This is the equation $E = \frac{1}{2}m|v|^2$ you know from high school, giving the kinetic energy as something proportional to the norm applied to the velocity v. The usual – mass independent – inner product in euclidean space gives $g(v,v) = v^t \cdot v = |v|^2$, in mechanics the mass is incorporated into the inner product.

The assignment $[\gamma] \mapsto g([\gamma], -)$ where $g([\gamma], -)\colon T_pM \to \mathbf{R}$ is the linear map $[\gamma_1] \mapsto g([\gamma], [\gamma_1])$ defines an isomorphism $T_pM \cong \mathrm{Hom}_{\mathbf{R}}(T_pM, \mathbf{R})$ (isomorphism since g is positive definite). The *momentum* of a particle with mass m moving along the curve γ is, at time $t = 0$, exactly the cotangent vector $g([\gamma], -)$ (this is again the old formula $p = mv$: the mass is intrinsic to the inner product, and the v should really be transposed ($p = g(v, -) = mv^t$) so as to be ready to be multiplied with with another v to give $E = \frac{1}{2}m|v|^2 = \frac{1}{2}p \cdot v$).

4.4 Derivations[1]

Although the definition of the tangent space by means of curves is very intuitive and geometric, the alternative point of view of the tangent space as the space of "derivations" can be very convenient. A derivation is a linear transformation satisfying the Leibniz rule:

Definition 4.4.1 Let M be a smooth manifold and $p \in M$. A *derivation* (on M at p) is a linear transformation

$$X\colon \mathcal{O}_{M,p} \to \mathbf{R}$$

satisfying the *Leibniz rule*

$$X(\bar\phi \cdot \bar\psi) = X(\bar\phi) \cdot \psi(p) + \phi(p) \cdot X(\bar\psi)$$

for all function germs $\bar\phi, \bar\psi \in \mathcal{O}_{M,p}$.

We let $D|_pM$ be the set of all derivations.

[1]This material is not used in an essential way in the rest of the book. It is included for completeness, and for comparison with other sources.

Example 4.4.2 Let $M = \mathbf{R}$. Then $\phi \mapsto \phi'(p)$ is a derivation. More generally, if $M = \mathbf{R}^n$ then all the partial derivatives $\phi \mapsto D_j(\phi)(p)$ are derivations.

Note 4.4.3 Note that the set $D|_pM$ of derivations is a vector space: adding two derivations or multiplying one by a real number gives a new derivation. We shall later see that the partial derivatives form a basis for the vector space $D|_p\mathbf{R}^n$.

Definition 4.4.4 Let $\bar{f}\colon (M,p) \to (N, f(p))$ be a germ. Then we have a linear transformation

$$D|_pf\colon D|_pM \to D|_{f(p)}N$$

given by

$$D|_pf(X) = Xf^*$$

(i.e. $D|_pf(X)(\bar{\phi}) = X(\phi f)$.).

Lemma 4.4.5 *If $\bar{f}\colon (M,p) \to (N, f(p))$ and $\bar{g}\colon (N, f(p)) \to (L, g(f(p)))$ are germs, then*

$$\begin{array}{ccc} D|_pM & \xrightarrow{D|_pf} & D|_{f(p)}N \\ & {\scriptstyle D|_p(gf)} \searrow & \downarrow {\scriptstyle D|_{f(p)}g} \\ & & D|_{gf(p)}L \end{array}$$

commutes.

Proof: Let $X\colon \mathcal{O}_{M,p} \to \mathbf{R}$ be a derivation, then

$$D|_{f(p)}g(D|_pf(X)) = D|_{f(p)}g(Xf^*) = (Xf^*)g^* = X(gf)^* = D|_pgf(X).$$

∎

4.4.6 The space of derivations is the dual of the cotangent space

Given our discussion of the cotangent space $T_p^*M = J_p/J_p^2$ in the previous section, it is easy to identify the space of derivations as the dual of the cotangent space (and so the *double dual* of the tangent space).[2]

However, it is instructive to see how naturally the derivations fall out of our discussion of the cotangent space (this is of course a reflection of a deeper theory of derivations you may meet later if you study algebra).

Proposition 4.4.7 *Let M be a smooth manifold and $p \in M$. Then*

$$\beta_{M,p}\colon \left(T_p^*M\right)^* \longrightarrow D|_pM, \qquad \beta_{M,p}(g) = \{\mathcal{O}_{M,p} \xrightarrow{\bar{\phi} \mapsto d\phi} T_p^*M \xrightarrow{g} \mathbf{R}\}$$

[2]For the benefit of those who did not study the cotangent space, we give an independent proof of this fact in the next subsection, along with some further details about the structure of the space of derivations.

is a natural isomorphism; if $f\colon M \to N$ is a smooth map, then

$$\begin{array}{ccc}\left(T_p^*M\right)^* & \xrightarrow{\beta_{M,p}} & D|_pM \\ {\scriptstyle (T^*f)^*}\downarrow & & \downarrow{\scriptstyle D|_pf} \\ \left(T_{f(p)}^*N\right)^* & \xrightarrow{\beta_{N,f(p)}} & D|_{f(p)}N\end{array}$$

commutes.

Proof: Recall that $T_p^*M = J_p/J_p^2$ where $J_p \subseteq \mathcal{O}_{M,p}$ consists of the germs vanishing at p. That $\beta_{M,p}(g)$ is a derivation follows since g is linear and since d satisfies the Leibniz rule by Lemma 4.3.7: $\beta_{M,p}(g)$ applied to $\bar{\phi}\cdot\bar{\psi}$ gives $g(d(\phi\cdot\psi)) = \phi(p)\cdot g(d\psi) + g(d\phi)\cdot\psi(p)$. The inverse of $\beta_{M,p}$ is given as follows. Given a derivation $h\colon \mathcal{O}_{M,p} \to \mathbf{R}$, notice that the Leibniz rule gives that $J_p^2 \subseteq \ker\{h\}$, and so h defines a map $\beta_{M,p}^{-1}(h)\colon J_p/J_p^2 \to \mathbf{R}$.

Showing that the diagram commutes boils down to following an element $g \in (T_p^*M)^*$ both ways and observing that the result either way is the derivation sending $\bar{\phi} \in \mathcal{O}_{M,p}$ to $g(d(\phi f)) \in \mathbf{R}$. ∎

For a vector space V, there is a canonical map to the double dualization $V \to (V^*)^*$ sending $v \in V$ to $v^{**}\colon V^* \to \mathbf{R}$ given by $v^{**}(f) = f(v)$. This map is always injective, and if V is finite dimensional it is an isomorphism. This is also natural: if $f\colon V \to W$ is linear, then

$$\begin{array}{ccc}V & \longrightarrow & (V^*)^* \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle (f^*)^*} \\ W & \longrightarrow & (W^*)^*\end{array}$$

commutes.

Together with the above result, this gives the promised natural isomorphism between the double dual of the tangent space and the space of derivations:

Corollary 4.4.8 *There is a chain of natural isomorphism*

$$T_pM \xrightarrow{\cong} ((T_pM)^*)^* \xrightarrow{(\alpha_{M,p})^*} (T_p^*M)^* \xrightarrow{\beta_{M,p}} D|_pM.$$

The composite sends $[\gamma] \in T_pM$ to $X_\gamma \in D|_pM$ whose value at $\bar{\phi} \in \mathcal{O}_{M,p}$ is $X_\gamma(\bar{\phi}) = (\phi\gamma)'(0)$.

Note 4.4.9 In the end, this all sums up to say that T_pM and $D|_pM$ are one and the same thing (the categorists would say that "the functors are naturally isomorphic"), and so we will let the notation $D|_pM$ slip quietly into oblivion.

Notice that in the proof of Corollary 4.4.8 it is crucial that the tangent spaces are finite dimensional. However, the proof of Proposition 4.4.7 is totally algebraic, and does not depend on finite dimensionality.

4.4.10 The space of derivations is spanned by partial derivatives

Even if we know that the space of derivations is just another name for the tangent space, a bit of hands-on knowledge about derivations can often be useful. This subsection does not depend on the previous, and as a side effect gives a direct proof of $T_pM \cong D|_pM$ without talking about the cotangent space.

The chain rule gives as before, that we may use charts and transport all calculations to $\mathbf{R}^n$.

Proposition 4.4.11 *The partial derivatives* $\{D_i|_0\}i = 1, \ldots, n$ *form a basis for* $D|_0\mathbf{R}^n$.

Exercise 4.4.12 Prove Proposition 4.4.11

Thus, given a chart $\bar{x}\colon (M,p) \to (\mathbf{R}^n, 0)$ we have a basis for $D|_pM$, and we give this basis the old-fashioned notation to please everybody:

Definition 4.4.13 Consider a chart $\bar{x}\colon (M,p) \to (\mathbf{R}^n, x(p))$. Define the derivation in T_pM

$$\left.\frac{\partial}{\partial x_i}\right|_p = (D|_px)^{-1}\left(D_i|_{x(p)}\right),$$

or in more concrete language: if $\bar{\phi}\colon (M,p) \to (\mathbf{R}, \phi(p))$ is a function germ, then

$$\left.\frac{\partial}{\partial x_i}\right|_p (\bar{\phi}) = D_i(\phi x^{-1})(x(p))$$

Note 4.4.14 Note that if $\bar{f}\colon (M,p) \to (N, f(p))$ is a germ, then the matrix associated with the linear transformation $D|_pf\colon D|_pM \to D|_{f(p)}N$ in the basis given by the partial derivatives of x and y is nothing but the Jacobi matrix $D(yfx^{-1})(x(p))$. In the current notation the i,j-entry is

$$\left.\frac{\partial(y_if)}{\partial x_j}\right|_p.$$

Definition 4.4.15 Let M be a smooth manifold and $p \in M$. To every germ $\bar{\gamma}\colon (\mathbf{R}, 0) \to (M,p)$ we may associate a derivation $X_\gamma\colon \mathcal{O}_{M,p} \to \mathbf{R}$ by setting

$$X_\gamma(\bar{\phi}) = (\phi\gamma)'(0)$$

for every function germ $\bar{\phi}\colon (M,p) \to (\mathbf{R}, \phi(p))$.

Note that $X_\gamma(\bar{\phi})$ is the derivative at zero of the composite

$$(\mathbf{R}, 0) \xrightarrow{\bar{\gamma}} (M,p) \xrightarrow{\bar{\phi}} (\mathbf{R}, \phi(p))$$

Exercise 4.4.16 Check that the map $T_pM \to D|_pM$ sending $[\gamma]$ to X_γ is well defined.

Using the definitions we get the following lemma, which says that the map $T_0\mathbf{R}^n \to D|_0\mathbf{R}^n$ is surjective.

Lemma 4.4.17 *If $v \in \mathbf{R}^n$ and $\bar{\gamma}$ the germ associated to the curve $\gamma(t) = v \cdot t$, then $[\gamma]$ sent to*

$$X_\gamma(\bar{\phi}) = D(\phi)(0) \cdot v = \sum_{i=0}^{n} v_i D_i(\phi)(0)$$

and so if $v = e_j$ is the jth unit vector, then X_γ is the jth partial derivative at zero.

Lemma 4.4.18 *Let $\bar{f}\colon (M,p) \to (N, f(p))$ be a germ. Then*

$$\begin{array}{ccc} T_pM & \xrightarrow{T_pf} & T_{f(p)}N \\ \downarrow & & \downarrow \\ D|_pM & \xrightarrow{D|_pf} & D|_{f(p)}N \end{array}$$

commutes.

Exercise 4.4.19 Prove Lemma 4.4.18.

Proposition 4.4.20 *Let M be a smooth manifold and p a point in M. The assignment $[\gamma] \mapsto X_\gamma$ defines a natural isomorphism*

$$T_pM \cong D|_pM$$

between the tangent space at p and the vector space of derivations $\mathcal{O}_{M,p} \to \mathbf{R}$.

Proof: The term "natural" in the proposition refers to the statement in Lemma 4.4.18. In fact, we can use this to prove the rest of the proposition.

Choose a germ chart $\bar{x}\colon (M,p) \to (\mathbf{R}^n, 0)$. Then Lemma 4.4.18 proves that

$$\begin{array}{ccc} T_pM & \underset{\cong}{\xrightarrow{T_px}} & T_0\mathbf{R}^n \\ \downarrow & & \downarrow \\ D|_pM & \underset{\cong}{\xrightarrow{D|_px}} & D|_0\mathbf{R}^n \end{array}$$

commutes, and the proposition follows if we know that the right hand map is a linear isomorphism.

But we have seen in Proposition 4.4.11 that $D|_0\mathbf{R}^n$ has a basis consisting of partial derivatives, and we noted in Lemma 4.4.17 that the map $T_0\mathbf{R}^n \to D|_0\mathbf{R}^n$ hits all the basis elements, and now the proposition follows since the dimension of $T_0\mathbf{R}^n$ is n (a surjective linear map between vector spaces of the same (finite) dimension is an isomorphism). ∎

Chapter 5

Regular values

In this chapter we will acquire a powerful tool for constructing new manifolds as inverse images of smooth functions. This result is a consequence of the *rank theorem*, which says roughly that smooth maps are – locally around "most" points – like linear projections or inclusions of Euclidean spaces.

5.1 The rank

Remember that the rank of a linear transformation is the dimension of its image. In terms of matrices, this can be captured by saying that a matrix has rank at least r if it contains an $r \times r$ invertible submatrix.

Definition 5.1.1 Let $\bar{f}\colon (M,p) \to (N, f(p))$ be a smooth germ. The *rank* $rk_p f$ *of* f *at* p is the rank of the linear map $T_p f$. We say that a germ $\bar{f}$ has *constant rank* r if it has a representative $f\colon U_f \to N$ whose rank $rkT_q f = r$ for all $q \in U_f$. We say that a germ $\bar{f}$ has rank $\geq r$ if it has a representative $f\colon U_f \to N$ whose rank $rkT_q f \geq r$ for all $q \in U_f$.

In view of Proposition 4.2.15, the rank of f at p is the same as the rank of the Jacobi matrix $D(yfx^{-1})(x(p))$, where (x,U) is a chart around p and (y,V) a chart around $f(p)$.

Lemma 5.1.2 *Let $\bar{f}\colon (M,p) \to (N, f(p))$ be a smooth germ. If $rk_p f = r$ then there exists a neighborhood of p such that $rk_q f \geq r$ for all $q \in U$.*

Proof: Note that the subspace $M^{\geq r}_{n\times m}(\mathbf{R}) \subseteq M_{n\times m}(\mathbf{R})$ of $n \times m$-matrices of rank at least r is open: the determinant function is continuous, so the set of matrices such that a given $r\times r$-submatrix is invertible is open (in fact, if for $S \subseteq \{1,\dots n\}$ and $T \subseteq \{1,\dots,m\}$ are two sets with r elements each we let $\det_{S,T}\colon M_{n\times m}(\mathbf{R}) \to \mathbf{R}$ be the continuous function sending the $n \times m$-matrix (a_{ij}) to $\det((a_{ij})_{i\in S, j\in T})$ we see that $M^{\geq r}_{n\times m}(\mathbf{R})$ is the finite intersection $\bigcap_{S,T} \det^{-1}_{S,T}(\mathbf{R} \setminus \{0\})$ of open sets).

Choose a representative $f\colon U_f \to N$ and charts (x,U) and (y,V) with $p \in U$ and $f(p) \in V$. Let $W = U_f \cap U \cap f^{-1}(V)$, and consider the continuous function $J\colon W \to$

$M_{n\times m}(\mathbf{R})$ sending $q \in W$ to $J(q) = D_j(\mathrm{pr}_i y f x^{-1})(x(q))$. The desired neighborhood of p is then $J^{-1}(M^r_{n\times m}(\mathbf{R}))$. ∎

Note 5.1.3 In the previous proof we used the useful fact that the subspace $M^{\geq r}_{n\times m}(\mathbf{R}) \subseteq M_{n\times m}(\mathbf{R})$ of $n \times m$-matrices of rank at least r is open. As a matter of fact, we showed in Example 3.5.10 that the subspace $M^r_{n\times m}(\mathbf{R}) \subseteq M_{n\times m}(\mathbf{R})$ of $n\times m$-matrices of rank (exactly equal to) r is a submanifold of codimension $(m-r)(n-r)$. Perturbing a rank r matrix may kick you out of this manifold and into one of higher rank (but if the perturbation is small enough you can avoid the matrices of smaller rank).

To remember what way the inequality in Lemma 5.1.2 goes, it may help to recall that the zero matrix is the only matrix of rank 0 (and so all the neighboring matrices are of higher rank), and likewise that the subset $M^{\min(m,n)}_{n\times m}(\mathbf{R}) \subseteq M_{n\times m}(\mathbf{R})$ of matrices of maximal rank is open. The rank "does not decrease locally".

Example 5.1.4 The map $f\colon \mathbf{R} \to \mathbf{R}$ given by $f(p) = p^2$ has $Df(p) = 2p$, and so

$$rk_p f = \begin{cases} 0 & p = 0 \\ 1 & p \neq 0 \end{cases}$$

Example 5.1.5 Consider the determinant $\det\colon M_2(\mathbf{R}) \to \mathbf{R}$ with

$$\det(A) = a_{11}a_{22} - a_{12}a_{21}, \text{ for } A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

By the calculation in Example 4.2.16 we see that

$$rk_A \det = \begin{cases} 0 & A = 0 \\ 1 & A \neq 0 \end{cases}$$

For dimension $n \geq 2$, the analogous statement is that $rk_A \det = 0$ if and only if $rkA < n-1$.

Example 5.1.6 Consider the map $f\colon S^1 \subseteq \mathbf{C} \to \mathbf{R}$ given by $f(x+iy) = x$. Cover S^1 by the "angle" charts $(x, (0, 2\pi))$ and $(y, (-\pi, \pi))$ with $x(t) = y(t) = e^{it}$. Then $fx^{-1}(t) = fy^{-1}(t) = \cos(t)$, and so we see that the rank of f at z is 1 if $z \neq \pm 1$ and 0 if $z = \pm 1$.

Definition 5.1.7 Let $f\colon M \to N$ be a smooth map where N is n-dimensional. A point $p \in M$ is *regular* if $T_p f$ is surjective (i.e., if $rk_p f = n$). A point $q \in N$ is a *regular value* if all $p \in f^{-1}(q)$ are regular points. Synonyms for "non-regular" are *critical* or *singular*.

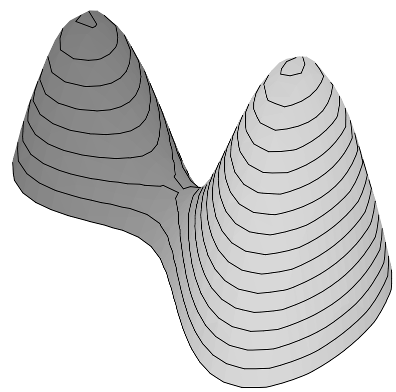
Note that a point q which is not in the image of f is a regular value since $f^{-1}(q) = \emptyset$.

Note 5.1.8 These names are well chosen: the critical values are critical in the sense that they exhibit bad behavior. The inverse image $f^{-1}(q) \subseteq M$ of a regular value q will turn out to be a submanifold, whereas inverse images of critical points usually are not.

On the other hand, according to Sard's theorem 5.6.1 the regular values are the common state of affairs (in technical language: critical values have "measure zero" while regular values are "dense").

Example 5.1.9 The names correspond to the normal usage in multi variable calculus. For instance, if you consider the function

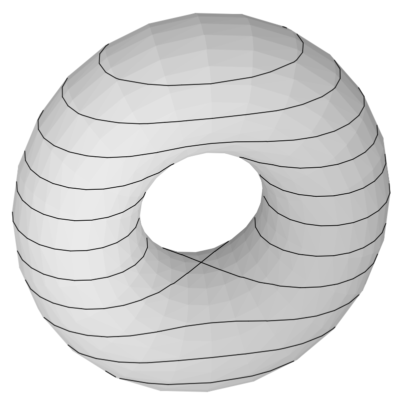
$$f\colon \mathbf{R}^2 \to \mathbf{R}$$



whose graph is depicted to the right, the critical points – i.e., the points $p \in \mathbf{R}^2$ such that

$$D_1 f(p) = D_2 f(p) = 0$$

– will correspond to the two local maxima and the saddle point. We note that the contour lines at all other values are nice 1-dimensional submanifolds of $\mathbf{R}^2$ (circles, or disjoint unions of circles).



In the picture to the right, we have considered a standing torus, and looked at its height function. The contour lines are then inverse images of various height values. If we had written out the formulas we could have calculated the rank of the height function at every point of the torus, and we would have found four critical points: one on the top, one on "the top of the hole", one on "the bottom of the hole" (the point on the figure where you see two contour lines cross) and one on the bottom. The contours at these heights look like points or figure eights, whereas contour lines at other values are one or two circles.

The robot example 2.1, was also an example of this type of phenomenon.

Example 5.1.10 The robot example is another example. In that example we considered a function

$$f\colon S^1 \times S^1 \to \mathbf{R}^1$$

and found three critical values.

To be more precise:

$$f(e^{i\theta}, e^{i\phi}) = |3 - e^{i\theta} - e^{i\phi}| = \sqrt{11 - 6\cos\theta - 6\cos\phi + 2\cos(\theta - \phi)},$$

and so (using charts corresponding to the angles: conveniently all charts give the same formulas in this example) the Jacobi matrix at $(e^{i\theta}, e^{i\phi})$ equals

$$\frac{1}{f(e^{i\theta}, e^{i\phi})}[3\sin\theta - \cos\phi\sin\theta + \sin\phi\cos\theta, 3\sin\phi - \cos\theta\sin\phi + \sin\theta\cos\phi].$$

The rank is one, unless both coordinates are zero, in which case we get that we must have $\sin\theta = \sin\phi = 0$, which leaves the points

$$(1,1), \qquad (-1,-1), \qquad (1,-1), \qquad \text{and } (-1,1)$$

giving the critical values 1, 5 and (twice) 3: exactly the points we noticed as troublesome.

Exercise 5.1.11 Fill out the details in the robot example.

5.2 The inverse function theorem

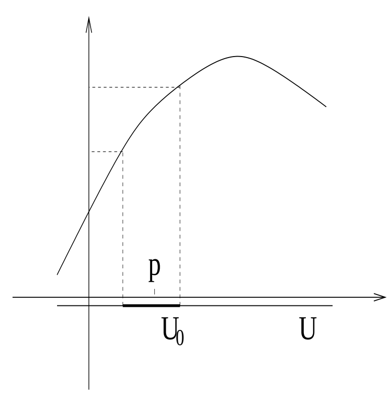
The technical foundation for the theorems to come is the inverse function theorem from multivariable calculus which we cite below. A proof can be found in [11, Theorem 2.11] or any other decent book on multi-variable calculus.

Theorem 5.2.1 *If $f : U_1 \to U_2$ is a smooth function where $U_1, U_2 \subseteq \mathbf{R}^n$. Let $p \in U_1$ and assume the the Jacobi matrix $[Df(p)]$ is invertible in the point p. Then there exists a neighborhood around p on which f is smoothly invertible, i.e., there exists an open subset $U_0 \subseteq U_1$ containing p such that*

$$f|_{U_0} : U_0 \to f(U_0)$$

is a diffeomorphism. The inverse has Jacobi matrix

$$[D(f^{-1})(f(x))] = [Df(x)]^{-1}$$



Recall from Lemma 4.1.7 that an invertible germ $(M,p) \to (N,q)$ is exactly a germ induced by a diffeomorphism $\phi\colon U \to V$ between neighborhoods of p and q.

Theorem 5.2.2 (The inverse function theorem) *A germ*

$$\bar{f} : (M,p) \to (N, f(p))$$

is invertible if and only if

$$T_pf : T_pM \to T_{f(p)}N$$

is invertible, in which case $T_{f(p)}(f^{-1}) = (T_pf)^{-1}$.

Proof: Choose charts (x, U) and (y, V) with $p \in W = U \cap f^{-1}(V)$. By Proposition 4.2.15, $T_p f$ is an isomorphism if and only if the Jacobi matrix $D(yfx^{-1})(x(p))$ is invertible (which incidentally implies that $\dim(M) = \dim(N)$).

By the inverse function theorem 5.2.1 in the flat case, this is the case iff yfx^{-1} is a diffeomorphism when restricted to a neighborhood $U_0 \subseteq x(U)$ of $x(p)$. As x and y are diffeomorphisms, this is the same as saying that $f|_{x^{-1}(U_0)}$ is a diffeomorphism. ∎

Corollary 5.2.3 *Let $f: M \to N$ be a smooth map between smooth n-dimensional manifolds. Then f is a diffeomorphism if and only if it is bijective and $T_p f$ is of rank n for all $p \in M$.*

Proof: One way is obvious. For the other implication assume that f is bijective and $T_p f$ is of rank n for all $p \in M$. Since f is bijective it has an inverse function. A function has at most one inverse function (!) so the smooth inverse functions existing locally by the inverse function theorem, must be equal to the globally defined inverse function which hence is smooth. ∎

Exercise 5.2.4 Let G be a Lie group (a smooth manifold with a smooth associative multiplication, with a unit and all inverses). Show that the map $G \to G$ given by sending an element g to its inverse g^{-1} is smooth (some authors have this as a part of the definition of a Lie group, which is totally redundant. However, if G is only a topological space with a continuous associative multiplication, with a unit and all inverses, it does not automatically follow that inverting elements gives a continuous function).

5.3 The rank theorem

The rank theorem says that if the rank of a smooth map $f: M \to N$ is constant in a neighborhood of a point, then there are charts so that f looks like a a composite $\mathbf{R}^m \to \mathbf{R}^r \subseteq \mathbf{R}^n$, where the first map is the projection onto the first $r \leq m$ coordinate directions, and the last one is the inclusion of the first $r \leq n$ coordinates. So for instance, a map of rank 1 between 2-manifolds looks locally like

$$\mathbf{R}^2 \to \mathbf{R}^2, \qquad (q_1, q_2) \mapsto (q_1, 0).$$

If $\sigma: \{1, \dots, n\} \to \{1, \dots, n\}$ is a bijection (a *permutation*), we refer to the diffeomorphism sending $(t_1, \dots, t_n)$ to $(t_{\sigma^{-1}(1)}, \dots, t_{\sigma^{-1}(n)})$ as a *permutation of the coordinates* corresponding to σ and denoted $\sigma: \mathbf{R}^n \to \mathbf{R}^n$.

In the formulation of the rank theorem we give below, the two last cases are the extreme situations where the rank is maximal (and hence constant).

Lemma 5.3.1 *(The rank theorem)Let M and N be smooth manifolds of dimension* $\dim(M) = m$ *and* $\dim(N) = n$*, and let* $\bar{f}\colon (M,p) \to (N, f(p))$ *be a germ.*

1. *If $\bar{f}$ is of rank $\geq r$. Then for any chart (z, V) for N with $q \in V$ there exists a chart (x, U) for M with $p \in U$ and permutation $\sigma\colon \mathbf{R}^n \to \mathbf{R}^n$ such that the diagram of germs*
$$\begin{array}{ccc} (\mathbf{R}^m, x(p)) & \xrightarrow{\overline{\sigma z f x^{-1}}} & (\mathbf{R}^n, \sigma z(p)) \\ {\scriptstyle \overline{pr}}\downarrow & & {\scriptstyle \overline{pr}}\downarrow \\ (\mathbf{R}^r, prx(p)) & = & (\mathbf{R}^r, prx(p)) \end{array}$$
commutes, where pr is the projection onto the first r coordinates, $pr(t_1, \ldots, t_m) = (t_1, \ldots, t_r)$.

2. *If $\bar{f}$ has constant rank r, then there exists a charts (x, U) for M and (y, V) for N with $p \in U$ and $q \in V$ such that $\overline{yfx^{-1}} = \overline{ipr}$, where $ipr(t_1, \ldots, t_m) = (t_1, \ldots, t_r, 0, \ldots, 0)$.*

3. *If $\bar{f}$ is of rank n (and so $m \geq n$), then for any chart (y, V) for N with $f(p) \in V$, there exists a chart (x, U) for M with $p \in U$ such that $\overline{yfx^{-1}} = \overline{pr}$, where $pr(t_1, \ldots, t_m) = (t_1, \ldots, t_n)$.*

4. *If $\bar{f}$ is of rank m (and so $m \leq n$), then for any chart (x, U) for M with $p \in U$ there exists a chart (y, V) for N with $f(p) \in V$ such that $\overline{yfx^{-1}} = \bar{i}$, where $i(t_1, \ldots, t_m) = (t_1, \ldots, t_m, 0, \ldots, 0)$.*

Proof: This is a local question: if we start with arbitrary charts, we will fix them up so that we have the theorem. Hence we may just as well assume that $(M, p) = (\mathbf{R}^m, 0)$ and $(N, f(p)) = (\mathbf{R}^n, 0)$, that f is a representative of the germ, and that the Jacobian $Df(0)$ has the form
$$Df(0) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$
where A is an invertible $r \times r$ matrix. This is where we use that we may permute the coordinates: at the outset there was no guarantee that the upper left $r \times r$-matrix A was invertible: we could permute the columns by choosing x wisely (except in the fourth part where x is fixed, but where this is unnecessary since $r = m$), but the best we could guarantee without introducing the σ was that there would be an invertible $r \times r$-matrix somewhere in the first r columns. For the third part of the theorem, this is unnecessary since $r = n$.

Let $f_i = \mathrm{pr}_i f$, and for the first, second and third parts, define $x\colon (\mathbf{R}^m, 0) \to (\mathbf{R}^m, 0)$ by
$$x(t) = (f_1(t), \ldots, f_r(t), t_{r+1}, \ldots, t_m)$$
(where $t_j = \mathrm{pr}_j(t)$). Then
$$Dx(0) = \begin{bmatrix} A & B \\ 0 & I \end{bmatrix}$$

and so $\det Dx(0) = \det(A) \neq 0$. By the inverse function theorem 5.2.2, $\bar{x}$ is an invertible germ with inverse $\bar{x}^{-1}$. Choose a representative for $\bar{x}^{-1}$ which we by a slight abuse of notation will call x^{-1}. Since for sufficiently small $t \in M = \mathbf{R}^m$ we have

$$(f_1(t), \dots, f_n(t)) = f(t) = fx^{-1}x(t) = fx^{-1}(f_1(t), \dots, f_r(t), t_{r+1}, \dots, t_m)$$

we see that

$$fx^{-1}(t) = (t_1, \dots, t_r, f_{r+1}x^{-1}(t), \dots, f_n x^{-1}(t))$$

and we have proven the first and third parts of the rank theorem.

For the second part, assume $rkDf(t) = r$ for all t. Since $\bar{x}$ is invertible

$$D(fx^{-1})(t) = Df(x^{-1}(t))D(x^{-1})(t)$$

also has rank r for all t in the domain of definition. Note that

$$D(fx^{-1})(t) = \begin{bmatrix} I \qquad 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ [D_j(f_i x^{-1})(t)]_{\substack{i=r+1,\dots,n \\ j=1,\dots m}} \end{bmatrix}$$

so since the rank is exactly r we must have that the lower right hand $(n-r)\times(m-r)$-matrix

$$\left[D_j(f_i x^{-1})(t)\right]_{\substack{r+1 \,\leq\, i \,\leq\, n \\ r+1 \,\leq\, j \,\leq\, m}}$$

is the zero matrix (which says that "$f_i x^{-1}$ does not depend on the last $m-r$ coordinates for $i > r$"). Define $\bar{y}\colon (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ by setting

$$y(t) = \Big(t_1, \dots, t_r, t_{r+1} - f_{r+1}x^{-1}(\bar{t}), \dots, t_n - f_n x^{-1}(\bar{t})\Big)$$

where $\bar{t} = (t_1, \dots, t_r, 0, \dots, 0)$. Then

$$Dy(t) = \begin{bmatrix} I & 0 \\ ? & I \end{bmatrix}$$

so $\bar{y}$ is invertible and $\overline{yfx^{-1}}$ is represented by

$$\begin{aligned} t = (t_1, \dots, t_m) \mapsto & \Big(t_1, \dots, t_r, f_{r+1}x^{-1}(t) - f_{r+1}x^{-1}(\bar{t}), \dots, f_n x^{-1}(t) - f_n x^{-1}(\bar{t})\Big) \\ =&(t_1, \dots, t_r, 0, \dots, 0) \end{aligned}$$

where the last equation holds since $D_j(f_i x^{-1})(t) = 0$ for $r < i \leq n$ and $r < j \leq m$ so $\dots, f_n x^{-1}(t) - f_n x^{-1}(\bar{t}) = 0$ for $r < i \leq n$ for t close to the origin.

For the fourth part, we need to shift the entire burden to the chart on $N = \mathbf{R}^n$. Consider the germ $\bar{\eta}\colon (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ represented by $\eta(t) = (0, \dots, 0, t_{m+1}, \dots, t_n) + f(t_1, \dots, t_m)$. Since

$$D\eta(0) = \begin{bmatrix} A & 0 \\ C & I \end{bmatrix}$$

is invertible, $\bar{\eta}$ is invertible. Let $\bar{y} = \bar{\eta}^{-1}$ and let y be the corresponding diffeomorphism. Since $\bar{f}$ is represented by $(t_1, \dots, t_m) \mapsto \eta(t_1, \dots, t_m, 0, \dots, 0)$, we get that $\bar{y}\bar{f}$ is represented by $(t_1, \dots, t_m) \mapsto (t_1, \dots, t_m, 0, \dots, 0)$, as required. ∎

Exercise 5.3.2 Let $f\colon M \to N$ be a smooth map between n-dimensional smooth manifolds. Assume that M is compact and that $q \in N$ a regular value. Prove that $f^{-1}(q)$ is a finite set and that there is a neighborhood V around q such that for each $q' \in V$ we have that $f^{-1}(q') \cong f^{-1}(q)$.

Exercise 5.3.3 Prove the fundamental theorem of algebra: any non-constant complex polynomial P has a zero.

Exercise 5.3.4 Let $f\colon M \to M$ be smooth such that $f \circ f = f$ and M connected. Prove that $f(M) \subseteq M$ is a submanifold. If you like point-set topology, prove that $f(M) \subseteq M$ is closed.

Note 5.3.5 It is a remarkable fact that any smooth manifold can be obtained by the method of Exercise 5.3.4 with M an open subset of Euclidean space and f some suitable smooth map. If you like algebra, then you might like to think that smooth manifolds are to open subsets of Euclidean spaces what projective modules are to free modules.

We will not be in a position to prove this, but the idea is as follows. Given a manifold T, choose a smooth imbedding $i\colon T \subseteq \mathbf{R}^N$ for some N (this is possible by the Whitney imbedding theorem, which we prove in 9.2.6 for T compact). Thicken $i(T)$ slightly to a "tubular neighborhood" U, which is an open subset of $\mathbf{R}^N$ together with a lot of structure (it is isomorphic to what we will later refer to as the "total space of the normal bundle" of the inclusion $i(T) \subseteq \mathbf{R}^N$), and in particular comes equipped with a smooth map $f\colon U \to U$ (namely the "projection $U \to i(T)$ of the bundle" composed with the "zero section $i(T) \to U$" – you'll recognize these words once we have talked about vector bundles) such that $f \circ f = f$ and $f(U) = i(T)$.

5.4 Regular values

Since by Lemma 5.1.2 the rank can not decrease locally, there are certain situations where constant rank is guaranteed, namely when the rank is maximal.

Definition 5.4.1 A smooth map $f\colon M \to N$ is

a *submersion* if $rkT_pf = dim\, N$ (that is T_pf is surjective)

an *immersion* if $rkT_pf = dim\, M$ (T_pf is injective)

for all $p \in M$.

In these situation the third and/or fourth versions in the rank theorem 5.3.1 applies.

Note 5.4.2 To say that a map $f\colon M \to N$ is a submersion is equivalent to claiming that all points $p \in M$ are regular (T_pf is surjective), which again is equivalent to claiming that all $q \in N$ are regular values (values that are not hit are regular by definition).

Theorem 5.4.3 *Let*

$$f \colon M \to N$$

be a smooth map where M is $n+k$-dimensional and N is n-dimensional. If $q = f(p)$ is a regular value, then

$$f^{-1}(q) \subseteq M$$

is a k-dimensional smooth submanifold.

Proof: We must display a chart (x, W) such that $x(W \cap f^{-1}(q)) = x(W) \cap (\mathbf{R}^k \times \{0\})$.

Since p is regular, the rank of f must be n in a neighborhood of p, so by the rank theorem 5.3.1, there are charts (x, U) and (y, V) around p and q such that $x(p) = 0$, $y(q) = 0$ and

$$yfx^{-1}(t_1, \dots, t_{n+k}) = (t_1, \dots, t_n), \text{ for } t \in x(U \cap f^{-1}(V))$$

Let $W = U \cap f^{-1}(V)$, and note that $f^{-1}(q) = (yf)^{-1}(0)$. Then

$$\begin{aligned} x(W \cap f^{-1}(q)) =& x(W) \cap \left(yfx^{-1}\right)^{-1}(0) \\ =& \{(0, \dots, 0, t_{n+1}, \dots, t_{n+k}) \in x(W)\} \\ =& (\{0\} \times \mathbf{R}^k) \cap x(W) \end{aligned}$$

and so (permuting the coordinates) $f^{-1}(q) \subseteq M$ is a k-dimensional submanifold as claimed.

∎

Exercise 5.4.4 Give a new proof which shows that $S^n \subset \mathbf{R}^{n+1}$ is a smooth submanifold.

Note 5.4.5 Not all submanifolds can be realized as the inverse image of a regular value of some map (e.g., the zero section in the tautological line bundle $\eta_1 \to S^1$ can not, see 6.1.3), but the theorem still gives a rich source of important examples of submanifolds.

Example 5.4.6 Consider the *special linear group*

$$SL_n(\mathbf{R}) = \{A \in \mathrm{GL}_n(\mathbf{R}) \mid \det(A) = 1\}$$

We show that $SL_2(\mathbf{R})$ is a 3-dimensional manifold. The determinant function is given by

$$\det \colon M_2(\mathbf{R}) \to \mathbf{R}$$
$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \mapsto \det(A) = a_{11}a_{22} - a_{12}a_{21}$$

and so with the obvious coordinates $M_2(\mathbf{R}) \cong \mathbf{R}^4$ (sending A to $[a_{11}\, a_{12}\, a_{21}\, a_{22}]^t$) we have that

$$D(\det)(A) = \begin{bmatrix} a_{22} & -a_{21} & -a_{12} & a_{11} \end{bmatrix}$$

Hence the determinant function has rank 1 at all matrices, except the zero matrix, and in particular 1 is a regular value.

Exercise 5.4.7 Show that $SL_2(\mathbf{R})$ is diffeomorphic to $S^1 \times \mathbf{R}^2$.

Exercise 5.4.8 If you have the energy, you may prove that $SL_n(\mathbf{R})$ is an (n^2-1)-dimensional manifold.

Example 5.4.9 The subgroup $O(n) \subseteq \mathrm{GL}_n(\mathbf{R})$ of *orthogonal matrices* is a submanifold of dimension $\frac{n(n-1)}{2}$.

To see this, recall that $A \in \mathrm{GL}_n(\mathbf{R})$ is orthogonal iff $A^tA = I$. Note that A^tA is always symmetric. The space $Sym(n)$ of all symmetric matrices is diffeomorphic to $\mathbf{R}^{n(n+1)/2}$ (the entries on and above the diagonal can be chosen arbitrarily, and will then determine the remaining entries uniquely). We define a map

$$\begin{aligned} f\colon \mathrm{GL}_n(\mathbf{R}) &\to Sym(n) \\ A &\mapsto A^tA \end{aligned}$$

which is smooth (since matrix multiplication and transposition is smooth), and such that

$$O(n) = f^{-1}(I)$$

We must show that I is a regular value, and we offer two proofs, one computational using the Jacobi matrix, and one showing more directly that T_Af is surjective for all $A \in O(n)$. We present both proofs, the first one since it is very concrete, and the second one since it is short and easy to follow.

First the Jacobian argument. We use the usual chart on $\mathrm{GL}_n(\mathbf{R}) \subseteq M_n(\mathbf{R}) \cong \mathbf{R}^{n^2}$ by listing the entries in lexicographical order, and the chart

$$pr\colon Sym(n) \cong \mathbf{R}^{n(n+1)/2}$$

with $\mathrm{pr}_{ij}[A] = a_{ij}$ if $A = [a_{ij}]$ (also in lexicographical order) only defined for $1 \le i \le j \le n$. Then $\mathrm{pr}_{ij} f([A]) = \sum_{k=1}^n a_{ki}a_{kj}$, and a straight forward calculation yields that if $A = [a_{ij}]$ then

$$D_{kl}\mathrm{pr}_{ij} f(A) = \begin{cases} a_{ki} & i < j = l \\ a_{kj} & l = i < j \\ 2a_{kl} & i = j = l \\ 0 & \text{otherwise} \end{cases}$$

In particular

$$D_{kl}\mathrm{pr}_{ij} f(I) = \begin{cases} 1 & k = i < j = l \\ 1 & l = i < j = k \\ 2 & i = j = k = l \\ 0 & \text{otherwise} \end{cases}$$

and $rkDf(I) = n(n+1)/2$ since $Df(I)$ is on echelon form, with no vanishing rows (example: for $n = 2$ and $n = 3$ the Jacobi matrices are

$$\begin{bmatrix} 2 & & & \\ & 1 & 1 & \\ & & & 2 \end{bmatrix}, \text{ and } \begin{bmatrix} 2 & & & & & & & & \\ & 1 & & 1 & & & & & \\ & & 1 & & & & 1 & & \\ & & & & 2 & & & & \\ & & & & & 1 & & 1 & \\ & & & & & & & & 2 \end{bmatrix}$$

(in the first matrix the columns are the partial derivatives in the 11, 12, 21 and 22-variable, and the rows are the projection on the 11 12 and 22-factor. Likewise in the second one)).

For any $A \in \mathrm{GL}_n(\mathbf{R})$ we define the diffeomorphism

$$L_A : \mathrm{GL}_n(\mathbf{R}) \to \mathrm{GL}_n(\mathbf{R})$$

by $L_A(B) = A \cdot B$. Note that if $A \in O(n)$ then

$$f(L_A(B)) = f(AB) = (AB)^t AB = B^t A^t AB = B^t B = f(B)$$

and so by the chain rule and the fact that $D(L_A)(B) = A$ we get that

$$Df(I) = D(fL_A)(I) = D(f)(L_A I)D(L_A)(I) = D(f)(A)A$$

implying that $rkD(f)(A) = n(n+1)/2$ for all $A \in O(n)$. This means that A is a regular point for all $A \in O(n) = f^{-1}(I)$, and so I is a regular value, and $O(n)$ is an

$$n^2 - n(n+1)/2 = n(n-1)/2$$

dimensional submanifold.

For the other proof of the fact that I is a regular value, notice that all tangent vectors in $T_A\mathrm{GL}_n(\mathbf{R}) = T_A M_n(\mathbf{R})$ are in the equivalence class of a linear curve

$$\nu_B(s) = A + sB, \qquad B \in M_n(\mathbf{R}), \quad s \in \mathbf{R}$$

We have that

$$f\nu_B(s) = (A+sB)^t(A+sB) = A^tA + s(A^tB + B^tA) + s^2B^tB$$

and so

$$T_Af[\nu_B] = [f\nu_B] = [\gamma_B]$$

where $\gamma_B(s) = A^tA + s(A^tB + B^tA)$. Similarly, all tangent vectors in $T_I Sym(n)$ are in the equivalence class of a linear curve

$$\alpha_C(s) = I + sC$$

for C a symmetric matrix. If A is orthogonal, we see that $\gamma_{\frac{1}{2}AC} = \alpha_C$, and so $T_Af[\nu_{\frac{1}{2}AC}] = [\alpha_C]$, and T_Af is surjective. Since this is true for all $A \in O(n)$ we get that I is a regular value.

Exercise 5.4.10 Consider the inclusion $O(n) \subseteq M_n(\mathbf{R})$, giving a description of the tangent bundle of $O(n)$ along the lines of corollary 6.5.12. Show that under the isomorphism

$$TM_n(\mathbf{R}) \cong M_n(\mathbf{R}) \times M_n(\mathbf{R}), \qquad [\gamma] \leftrightarrows (\gamma(0), \gamma'(0))$$

the tangent bundle of $O(n)$ corresponds to the projection on the first factor

$$E = \{(g, A) \in O(n) \times M_n(\mathbf{R}) | A^t = -g^t A g^t\} \to O(n).$$

This also shows that $O(n)$ is parallelizable, since we get an obvious bundle isomorphism induced by

$$E \to O(n) \times \{B \in M_n(\mathbf{R}) | B^t = -B\}, \qquad (g, A) \mapsto (g, g^{-1}A)$$

(a matrix B satisfying $B^t = -B$ is called a *skew matrix*).

Note 5.4.11 The multiplication

$$O(n) \times O(n) \to O(n)$$

is smooth (since multiplication of matrices is smooth in $M_n(\mathbf{R}) \cong \mathbf{R}^{n^2}$, and 3.5.15), and so $O(n)$ is a Lie group. The same of course applies to $SL_n(\mathbf{R})$.

Exercise 5.4.12 Prove that

$$\begin{aligned} \mathbf{C} &\to M_2(\mathbf{R}) \\ x + iy &\mapsto \begin{bmatrix} x & -y \\ y & x \end{bmatrix} \end{aligned}$$

defines an imbedding. More generally it defines an imbedding

$$M_n(\mathbf{C}) \to M_n(M_2(\mathbf{R})) \cong M_{2n}(\mathbf{R})$$

Show also that this imbedding sends "conjugate transpose" to "transpose" and "multiplication" to "multiplication".

Exercise 5.4.13 Prove that the *unitary group*

$$U(n) = \{A \in \mathrm{GL}_n(\mathbf{C}) | \bar{A}^t A = I\}$$

is a Lie group of dimension n^2.

Exercise 5.4.14 Prove that $O(n)$ is compact and has two connected components. The component consisting of matrices of determinant 1 is called $SO(n)$, the *special orthogonal group*.

Note 5.4.15 $SO(2)$ is diffeomorphic to S^1 (prove this), and $SO(3)$ is diffeomorphic to the real projective 3-space (don't prove that).

Note 5.4.16 It is a beautiful fact that if G is a Lie group (e.g., $\mathrm{GL}_n(\mathbf{R})$) and $H \subseteq G$ is a closed subgroup (i.e., a closed subset which is closed under multiplication and such that if $h \in H$ then $h^{-1} \in H$), then $H \subseteq G$ is a "Lie subgroup". We will not prove this fact, (see e.g., Spivak's book I, theorem 10.15), but note that it implies that **all** closed matrix groups such as $O(n)$ are Lie groups since $\mathrm{GL}_n(\mathbf{R})$ is.

Example 5.4.17 Consider the map $f\colon S^1 \times S^1 \times S^1 \to \mathrm{SO}(3)$ uniquely defined by the composite $g\colon \mathbf{R}^3 \to S^1 \times S^1 \times S^1 \to \mathrm{SO}(3) \subseteq M_3(\mathbf{R})$ sending (α, β, γ) to

$$\begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & -\sin\beta \\ 0 & \sin\beta & \cos\beta \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

A quick calculation shows that the rank of this map is 3, unless $\sin\beta = 0$, in which case the rank is 2 (do this!). Hence all points on $S^1 \times S^1 \times S^1$ are regular except those in the sub-torus $S^1 \times \{1\} \times S^1 \subseteq S^1 \times S^1 \times S^1$ with middle coordinate 1 (why? Explain why the rank of the composite g gives the rank of f). On this sub-torus, the rotation is simply $\alpha + \gamma$ around the z-axis.

Hence, to most rotations there is a unique set of angles (α, β, γ), called the *Euler angles*, representing this rotation. Euler angles are used e.g., in computer graphics and in flight control and represent rotations uniquely except on the sub-torus $S^1 \times \{1\} \times S^1$ where the rotation has inverse image consisting of an entire circle. This situation is often referred to as the *gimbal lock* and is considered highly undesirable. This name derives from navigation where one uses a device called an inertial measurement unit (IMU) to keep a reference frame to steer by (it consists of three gimbals mounted inside each other at right angles to provide free rotation in all directions with gyroscopes in the middle to provide inertia fixing the reference frame). The map f above gives the correspondence between the rotation in question and the angles in the gimbals. However, at the critical value of f – the gimbal lock – the IMU fails to work causing a loss of reference frame. Hence a plane has to avoid maneuvering too close to the gimbal lock.

For a good treatment of Euler angles, check the Wikipedia or a NASA page giving some background on the worries the gimbal lock caused NASA's Apollo mission.

Exercise 5.4.18 A *k-frame* in $\mathbf{R}^n$ is a k-tuple of orthonormal vectors in $\mathbf{R}^n$. Define a *Stiefel manifold* V_n^k as the subset

$$V_n^k = \{k\text{-frames in } \mathbf{R}^n\}$$

of $\mathbf{R}^{nk}$. Show that V_n^k is a compact smooth $nk - \frac{k(k+1)}{2}$-dimensional manifold. Note that V_n^1 may be identified with S^{n-1}.

Note 5.4.19 In the literature you will often find a different definition, where a k-frame is just a k-tuple of linearly independent vectors. Then the Stiefel manifold is an open subset of the $M_{n\times k}(\mathbf{R})$, and so is clearly a smooth manifold – but this time of dimension nk.

A k-frame defines a k-dimensional linear subspace of $\mathbf{R}^n$. The *Grassmann manifold* G_n^k of Example 3.3.13 have as underlying set the set of k-dimensional linear subspaces of $\mathbf{R}^n$, and is topologized as the quotient space of the Stiefel manifold.

Exercise 5.4.20 Let P_n be the space of degree n polynomials. Show that the space of solutions in P_3 of the equation

$$(y'')^2 - y' + y(0) + xy'(0) = 0$$

is a 1-dimensional submanifold of P_3.

Exercise 5.4.21 Make a more interesting exercise along the lines of the previous, and solve it.

Exercise 5.4.22 Let $A \in M_n(\mathbf{R})$ be a symmetric matrix. For what values of $a \in \mathbf{R}$ is the *quadric*

$$M_a^A = \{p \in \mathbf{R}^n \,|\, p^t A p = a\}$$

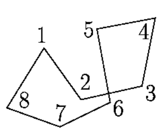
an $n-1$-dimensional smooth manifold?

Exercise 5.4.23 In a chemistry book I found van der Waal's equation, which gives a relationship between the temperature T, the pressure p and the volume V, which supposedly is somewhat more accurate than the ideal gas law $pV = nRT$ (n is the number of moles of gas, R is a constant). Given the relevant positive constants a and b, prove that the set of points $(p, V, T) \in (0,\infty) \times (nb, \infty) \times (0, \infty)$ satisfying the equation

$$\left(p - \frac{n^2 a}{V^2}\right)(V - nb) = nRT$$

is a smooth submanifold of $\mathbf{R}^3$.

Exercise 5.4.24 Consider the set $LF_{n,k}$ of *labeled flexible n-gons* in $\mathbf{R}^k$. A labeled flexible n-gon is what you get if you join $n > 2$ straight lines of unit length to a closed curve and label the vertices from 1 to n.



A labeled flexible 8-gon in $\mathbf{R}^2$.

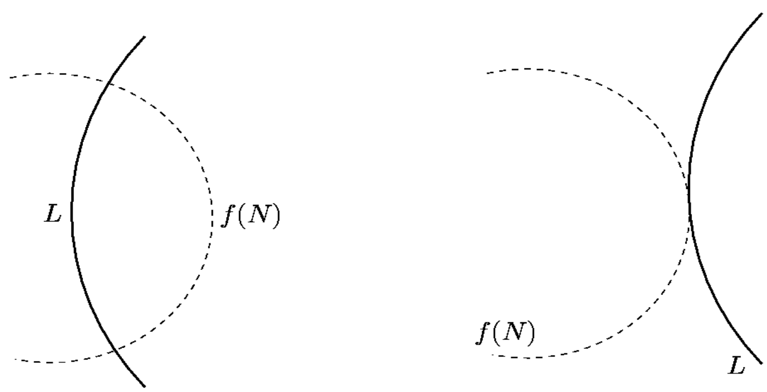
Let n be odd and $k = 2$. Show that $LF_{n,2}$ is a smooth submanifold of $\mathbf{R}^2 \times (S^1)^{n-1}$ of dimension n.

Exercise 5.4.25 Prove that the set of *non-self-intersecting* flexible n-gons in $\mathbf{R}^2$ is a manifold.

5.5 Transversality

In theorem 5.4.3 we learned about regular values, and inverse images of these. Often interesting submanifolds naturally occur not as inverse images of points, but as inverse images of submanifolds. How is one to guarantee that the inverse image of a submanifold is a submanifold? The relevant term is *transversality*.

Definition 5.5.1 Let $f\colon N \to M$ be a smooth map and $L \subset M$ a smooth submanifold. We say that f is *transverse* to $L \subset M$ if for all $p \in f^{-1}(L)$ the image of $T_p f$ and $T_{f(p)}L$ together span $T_{f(p)}M$.



The picture to the left is a typical transverse situation, whereas the situation to the right definitely can't be transverse since $\mathrm{Im}\{T_p f\}$ and $T_{f(p)}L$ only spans a one-dimensional space. Beware that pictures like this can be misleading, since the situation to the left fails to be transverse if f slows down at the intersection so that $\mathrm{Im}\{T_p f\} = 0$.

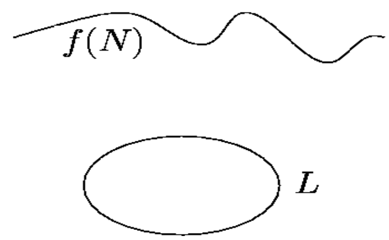
Note 5.5.2 If $L = \{q\}$ in the definition above, we recover the definition of a regular point.

Another common way of expressing transversality is to say that for all $p \in f^{-1}(L)$ the induced map

$$T_p N \xrightarrow{T_p f} T_{f(p)}M \longrightarrow T_{f(p)}M/T_{f(p)}L$$

is surjective. Here $T_{f(p)}M/T_{f(p)}L$ is the quotient space: If W is a subspace of a vector space V, then the quotient space V/W is the vector space you get from V by dividing out by the equivalence relation $v \sim v + w$ for $v \in V$ and $w \in W$. The vector space structure on V/W is defined by demanding that the map $V \to V/W$ sending a vector to its equivalence class is linear.

Note that the definition of transversality only refers to points in $f^{-1}(L)$, and so if $f(N) \cap L = \emptyset$ the condition is vacuous and f and L are transverse.



A map is always transverse to a submanifold its image does not intersect.

Furthermore if f is a submersion (i.e., $T_p f$ is always surjective), then f is transverse to all submanifolds.

Theorem 5.5.3 *Assume that the smooth map $f\colon N \to M$ is transverse to a k-codimensional submanifold $L \subseteq M$ and that $f(N) \cap L \neq \emptyset$. Then $f^{-1}(L) \subseteq N$ is a k-codimensional manifold.*

Proof: Let $q \in L$ and $p \in f^{-1}(q)$, and choose a chart (y, V) around q such that $y(q) = 0$ and

$$y(L \cap V) = y(V) \cap (\mathbf{R}^{n-k} \times \{0\})$$

Let $\pi\colon \mathbf{R}^n \to \mathbf{R}^k$ be the projection $\pi(t_1, \dots, t_n) = (t_{n-k+1}, \dots, t_n)$. Consider the diagram

$$\begin{array}{ccccc} T_pN & \xrightarrow{T_pf} & T_qM & \longrightarrow & T_qM/T_pL & & \\ & & {\scriptstyle T_qy}\downarrow\cong & & \downarrow\cong & & \\ & & T_0\mathbf{R}^n & \longrightarrow & T_0\mathbf{R}^n/T_0\mathbf{R}^{n-k} & \xrightarrow{\cong} & T_0\mathbf{R}^k \end{array}$$

The top horizontal composition is surjective by the transversality hypothesis, and the lower horizontal composite is defined as $T_0\pi$. Then we get that p is a regular point to the composite

$$U = f^{-1}(V) \xrightarrow{f|_U} V \xrightarrow{y} y(V) \xrightarrow{\pi|_{y(V)}} \mathbf{R}^k$$

and varying p in $f^{-1}(q)$ we get that $0 \in \mathbf{R}^k$ is a regular value. Hence

$$(\pi y f|_U)^{-1}(0) = f^{-1}y^{-1}\pi^{-1}(0) \cap U = f^{-1}(L) \cap U$$

is a submanifold of codimension k in U, and therefore $f^{-1}(L) \subseteq N$ is a k-codimensional submanifold. ∎

5.6 Sard's theorem[1]

As commented earlier, the regular points are dense. Although this is good to know and important for many applications, we will not need this fact, and are content to cite the

[1] This material is not used in an essential way in the rest of the book. It is included for completeness, and for comparison with other sources.

precise statement and let the proof be a guided sequence of exercises. Proofs can be found in many references, for instance in [8, Chapter 3].

Theorem 5.6.1 (Sard) *Let $f\colon M \to N$ be a smooth map. The set of critical values has measure zero.*

Recall that a subset $C \subseteq \mathbf{R}^n$ has measure zero if for every $\epsilon > 0$ there is a sequence of closed cubes $\{C_i\}_{i\in\mathbf{N}}$ with $C \subseteq \bigcup_{i\in\mathbf{N}} C_i$ and $\sum_{i\in\mathbf{N}} \mathrm{volume}(C_i) < \epsilon$.

In this definition it makes no essential difference if one uses open or closed cubes, rectangles or balls instead of closed cubes.

Exercise 5.6.2 Any open subset U of $\mathbf{R}^n$ is a countable union of closed balls.

Exercise 5.6.3 Prove that a countable union of measure zero subsets of $\mathbf{R}^n$ has measure zero.

Exercise 5.6.4 Let $f\colon U \to \mathbf{R}^m$ be a smooth map, where $U \subseteq \mathbf{R}^n$ is an open subset. Prove that if $C \subseteq U$ has measure zero, then so has the image $f(C)$. Conclude that a diffeomorphism $f\colon U \to U'$ between open subsets of Euclidean spaces provide a one-to-one correspondence between the subsets of measure zero in U and in U'.

Definition 5.6.5 Let $(M, \mathcal{A})$ be a smooth n-dimensional manifold and $C \subseteq M$ a subset. We say that C has measure zero if for each $(x, U) \in \mathcal{A}$ the subset $x(C \cap U) \subseteq \mathbf{R}^n$ has measure zero.

Given a subatlas $\mathcal{B} \subseteq \mathcal{A}$ we see that by Exercise 5.6.4 it is enough to check that $x(C \cap U) \subseteq \mathbf{R}^n$ has measure zero for all $(x, U) \in \mathcal{B}$.

Exercise 5.6.6 An open cover of the closed interval $[0, 1]$ by subintervals contains a finite open subcover whose sum of diameters is less than or equal to 2.

Exercise 5.6.7 Prove Fubini's theorem: Let $C \subseteq \mathbf{R}^n$ be a countable union of compact subsets. Assume that for each $t \in \mathbf{R}$ the set $\{(t_1, \dots, t_{n-1}) \in \mathbf{R}^{n-1} | (t_1, \dots, t_{n-1}, t) \in C\} \subseteq \mathbf{R}^{n-1}$ has measure zero. Then C has measure zero.

Exercise 5.6.8 Show that Sard's theorem follows if you show the following statement: Let $f\colon U \to \mathbf{R}^n$ be smooth where $U \subseteq \mathbf{R}^m$ is open, and let C be the set of critical points. Then $f(C) \subseteq \mathbf{R}^n$ has measure zero.

Let $f\colon U \to \mathbf{R}^n$ be smooth where $U \subseteq \mathbf{R}^m$ is open, and let C be the set of critical points. For $i > 0$, let C_i be the set of points $p \in U$ such that all partial derivatives of order less than or equal to i vanish, and let $C_0 = C$.

Exercise 5.6.9 Assume Sard's theorem is proven for manifolds of dimension less than m. Prove that $f(C_0 - C_1)$ has measure zero.

Exercise 5.6.10 Assume Sard's theorem is proven for manifolds of dimension less than m. Prove that $f(C_i - C_{i+1})$ has measure zero for all $i > 0$.

Exercise 5.6.11 Assume Sard's theorem is proven for manifolds of dimension less than m. Prove that $f(C_k)$ has measure zero for $nk \geq m$.

Exercise 5.6.12 Prove Sard's theorem.

5.7 Immersions and imbeddings

We are finally closing in on the promised "real" definition of submanifolds, or rather, of imbeddings. The condition of being an immersion is a readily checked property, since we only have to check the derivatives in every point. The rank theorem states that in some sense "locally" immersions are imbeddings. But how much more do we need? Obviously, an imbedding is injective.

Something more is needed, as we see from the following example

Example 5.7.1 Consider the injective smooth map

$$f\colon (0, 3\pi/4) \to \mathbf{R}^2$$

given by

$$f(t) = \sin(2t)(\cos t, \sin t)$$

Then

$$Df(t) = 2[(1 - 3\sin^2 t)\cos t, (3\cos^2 t - 1)\sin t]$$

is never zero and f is an immersion.

However,

$$(0, 3\pi/4) \to \mathrm{Im}\{f\}$$

is not a homeomorphism where

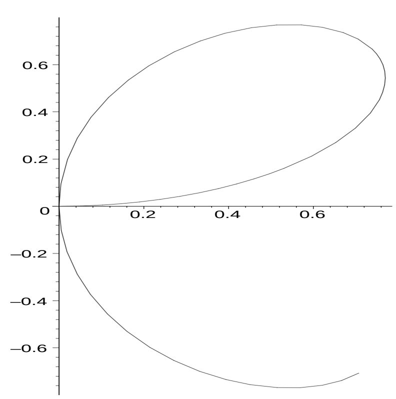
$$\mathrm{Im}\{f\} = f((0, 3\pi/4)) \subseteq \mathbf{R}^2$$

has the subspace topology. For, if it were a homeomorphism, then

$$f((\pi/4, 3\pi/4)) \subseteq \mathrm{Im}\{f\}$$

would be open (for the inverse to be continuous). But any open ball around $(0,0) = f(\pi/2)$ in $\mathbf{R}^2$ must contain a piece of $f((0, \pi/4))$, so $f((\pi/4, 3\pi/4)) \subseteq \mathrm{Im}\{f\}$ is not open.

Hence f is not an imbedding.



The image of f is a subspace of $\mathbf{R}^2$.

Exercise 5.7.2 Let

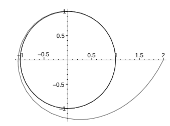
$$\mathbf{R}\coprod\mathbf{R}\to\mathbf{R}^2$$

be defined by sending x in the first summand to $(x,0)$ and y in the second summand to $(0,e^y)$. This is an injective immersion, but not an imbedding.

Exercise 5.7.3 Let

$$\mathbf{R}\coprod S^1\to\mathbf{C}$$

be defined by sending x in the first summand to $(1+e^x)e^{ix}$ and being the inclusion $S^1\subseteq\mathbf{C}$ on the second summand. This is an injective immersion, but not an imbedding.



The image is not a submanifold of $\mathbf{C}$.

But, strangely enough these examples exhibit the only thing that can go wrong: if an injective immersion is to be an imbedding, the map to the image has got to be a homeomorphism.

Theorem 5.7.4 *Let $f\colon M\to N$ be an immersion such that the induced map*

$$M\to\operatorname{Im}\{f\}$$

is a homeomorphism where $\operatorname{Im}\{f\}=f(M)\subseteq N$ has the subspace topology, then f is an imbedding.

Proof: Let $p\in M$. The rank theorem says that there are charts

$$x_1\colon U_1\to U_1'\subseteq\mathbf{R}^n$$

and

$$y_1\colon V_1\to V_1'\subseteq\mathbf{R}^{n+k}$$

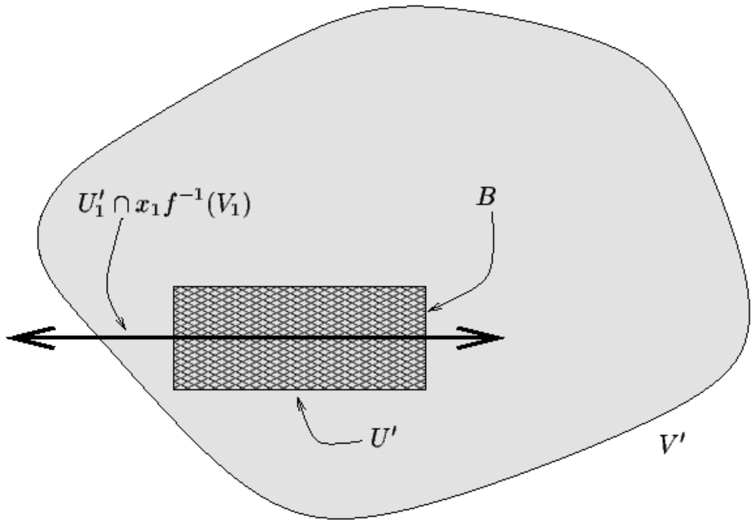
with $x_1(p)=0$ and $y_1(f(p))=0$ such that

$$y_1 f x_1^{-1}(t)=(t,0)\in\mathbf{R}^n\times\mathbf{R}^k=\mathbf{R}^{n+k}$$

for all $t\in x_1(U_1\cap f^{-1}(V_1))$.

Since V_1' is open, it contains open rectangles around the origin. Choose one such rectangle $V_2'=U'\times B$ so that $U'\subseteq x_1(U_1\cap f^{-1}(V_1))$ (see the picture below)

$$V_2'=U'\times B\subseteq(x_1(U_1\cap f^{-1}(V_1))\times\mathbf{R}^k)\cap V_1'\subseteq\mathbf{R}^n\times\mathbf{R}^k$$



Let $U = x_1^{-1}(U')$, $x = x_1|_U$ and $V_2 = y_1^{-1}(V_2')$.

Since $M \to f(M)$ is a homeomorphism, $f(U)$ is an open subset of $f(M)$, and since $f(M)$ has the subspace topology, $f(U) = W \cap f(M)$ where W is an open subset of N (here is the crucial point where complications as in example 5.7.1 are excluded: there are no other "branches" of $f(M)$ showing up in W).

Let $V = V_2 \cap W$, $V' = V_2' \cap y_1(W)$ and $y = y_1|_V$.

Then we see that $f(M) \subseteq N$ is a submanifold ($y(f(M) \cap V) = yf(U) = (\mathbf{R}^n \times \{0\}) \cap V'$), and $M \to f(M)$ is a bijective local diffeomorphism (the constructed charts show that both f and its inverse are smooth around every point), and hence a diffeomorphism. ∎

We note the following useful corollary:

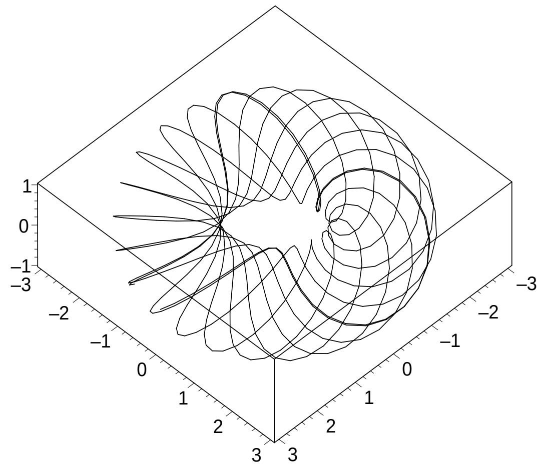
Corollary 5.7.5 *Let $f\colon M \to N$ be an injective immersion from a compact manifold M. Then f is an imbedding.*

Proof: We only need to show that the continuous map $M \to f(M)$ is a homeomorphism. It is injective since f is, and clearly surjective. But from point set topology (theorem 10.7.8) we know that it must be a homeomorphism since M is compact and $f(M)$ is Hausdorff ($f(M)$ is Hausdorff since it is a subspace of the Hausdorff space N). ∎

Exercise 5.7.6 Let $a, b \in \mathbf{R}$, and consider the map

$$\begin{aligned} f_{a,b}\colon \mathbf{R} &\to S^1 \times S^1 \\ t &\mapsto (e^{iat}, e^{ibt}) \end{aligned}$$

Show that $f_{a,b}$ is an immersion if either a or b is different from zero. Show that $f_{a,b}$ factors through an imbedding $S^1 \to S^1 \times S^1$ iff either $b = 0$ or a/b is rational.



Part of the picture if $a/b = \pi$ (this goes on forever)

Exercise 5.7.7 Consider smooth maps

$$M \xrightarrow{i} N \xrightarrow{j} L$$

Show that if the composite ji is an imbedding, then i is an imbedding.

Example 5.7.8 As a last example of Corollary 5.7.5 we can redo Exercise 3.5.12 and see that

$$\begin{aligned} f\colon \mathbf{R}\mathrm{P}^n &\to \mathbf{R}\mathrm{P}^{n+1} \\ [p] = [p_0, \dots, p_n] &\mapsto [p, 0] = [p_0, \dots, p_n, 0] \end{aligned}$$

is an imbedding: $\mathbf{R}\mathrm{P}^n$ is compact, f is injective and an immersion (since it is a local diffeomorphism, and so induces an isomorphism on tangent spaces).

Exercise 5.7.9 Let M be a smooth manifold, and consider M as a subset by imbedding it as the diagonal in $M \times M$, i.e., as the set $\{(p, p) \in M \times M\}$: show that it is a smooth submanifold.

Exercise 5.7.10 Consider two smooth maps

$$M \xrightarrow{f} N \xleftarrow{g} L$$

Define the *fiber product*

$$M \times_N L = \{(p, q) \in M \times L | f(p) = g(q)\}$$

(topologized as a subspace of the product $M \times L$: notice that if f and g are inclusions of subspaces, then $M \times_N L = M \cap L$). Assume that for all $(p, q) \in M \times_N L$ the subspaces spanned by the images of T_pM and T_qL equals all of $T_{f(p)}N$. Show that the fiber product $M \times_N L$ may be given a smooth structure such that the projections $M \times_N L \to M$ and $M \times_N L \to L$ are smooth.

Exercise 5.7.11 Let $\pi\colon E \to M$ be a submersion and $f\colon N \to M$ smooth. Let $E \times_M N$ be the fiber product of Exercise 5.7.10. Show that the projection $E \times_M N \to N$ is a submersion.

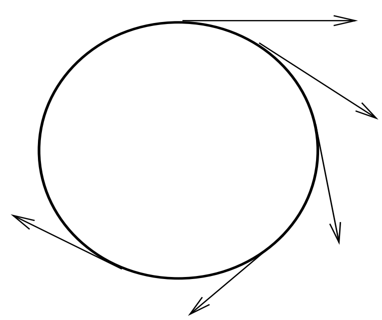
Chapter 6

Vector bundles

In this chapter we are going to collect all the tangent spaces of a manifold into one single object, the so-called *tangent bundle*.

6.0.12 The idea

We defined the tangent space at a point in a smooth manifold by considering curves passing through the point. In physical terms, the tangent vectors are the velocity vectors of particles passing through our given point. But the particle will have velocities and positions at other times than the one in which it passes through our given point, and the position and velocity may depend continuously upon the time. Such a broader view demands that we are able to keep track of the points on the manifold and their tangent space, and understand how they change from point to point.



A particle moving on S^1: some of the velocity vectors are drawn. The collection of all possible combinations of position and velocity ought to assemble into a "tangent bundle". In this case we see that $S^1 \times \mathbf{R}^1$ would do, but in most instances it won't be as easy as this.

As a *set* the tangent bundle ought to be given by pairs (p, v), where $p \in M$ and $v \in T_pM$, i.e.,

$$TM = \{(p, v) \mid p \in M, v \in T_pM\} = \coprod_{p \in M} T_pM.$$

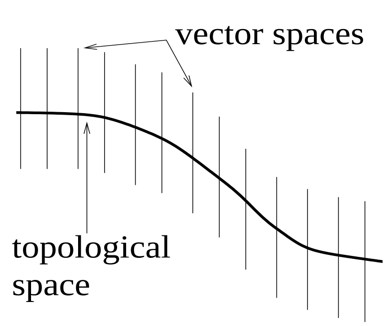
In the special case $M = \mathbf{R}^m$ we have a global chart (e.g., the identity chart), and so we have a global (not depending on the point p) identification of $T_p\mathbf{R}$ with the vector space $\mathbf{R}^m$ through the correspondence $[\gamma] \leftrightarrow \gamma'(0)$. Hence it is reasonable to say that $T\mathbf{R}^n$ can be identified with the product $\mathbf{R}^n \times \mathbf{R}^n$. Now, in general a manifold M is locally like $\mathbf{R}^m$, but the question is how this local information should be patched together to a global picture.

The tangent bundle is an example of an important class of objects called *vector bundles*. We start the discussion of vector bundles in general in this chapter, although our immediate applications will focus on the tangent bundle. We will pick up the glove in chapter 7 where we discuss the algebraic properties of vector bundles, giving tools that eventually could have brought the reader to fascinating topics like the topological K-theory of Atiyah and Hirzebruch [1] which is an important tool in algebraic topology.

We first introduce topological vector bundles, and then see how transition functions, very similar to the chart transformations, allow us to coin what it means for a bundle to be smooth. An observation shows that the work of checking that something actually **is** a vector bundle can be significantly simplified, paving the way for a sleek definition of the tangent bundle in 6.5.1. Equally simple, we get the cotangent bundle.

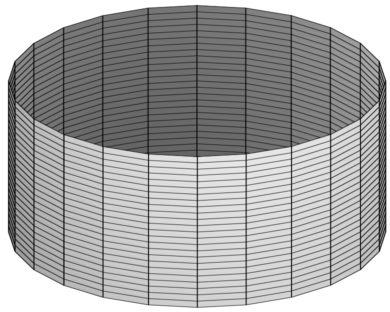
6.1 Topological vector bundles

Loosely speaking, a vector bundle is a collection of vector spaces parametrized in a locally controllable fashion by some space.



A vector bundle is a topological space to which a vector space is stuck at each point, and everything fitted continuously together.

The easiest example is simply the product $X \times R^n$, and we will have this as our local model.



The product of a space and an euclidean space is the local model for vector bundles. The cylinder $S^1 \times \mathbf{R}$ is an example.

Definition 6.1.1 An *n-dimensional (real topological) vector bundle* is a surjective continuous map

$$\begin{array}{c} E \\ \downarrow{\scriptstyle \pi} \\ X \end{array}$$

such that for every $p \in X$

— the *fiber*

$$\pi^{-1}(p)$$

has the structure of a real n-dimensional vector space

— there is an open set $U \subseteq X$ containing p

— a homeomorphism

$$h\colon \pi^{-1}(U) \to U \times \mathbf{R}^n$$

such that

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{h} & U \times \mathbf{R}^n \\ & {\scriptstyle \pi|_{\pi^{-1}(U)}} \searrow \quad \swarrow {\scriptstyle \mathrm{pr}_U} & \\ & U & \end{array}$$

commutes, and such that for every $q \in U$ the composite

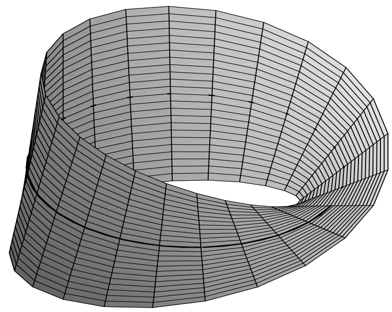
$$h_q\colon \pi^{-1}(q) \xrightarrow{h|_{\pi^{-1}(q)}} \{q\} \times \mathbf{R}^n \xrightarrow{(q,t)\mapsto t} \mathbf{R}^n$$

is a vector space isomorphism.

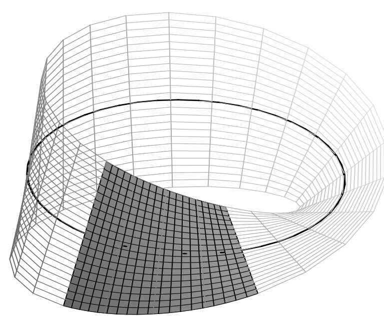
Example 6.1.2 The "unbounded Möbius band" given by

$$E = (\mathbf{R} \times [0,1])/((p,0) \sim (-p,1))$$

defines a 1-dimensional vector bundle by projecting onto the central circle $E \to [0,1]/(0 \sim 1) \cong S^1$.



Restricting to an interval on the circle, we clearly see that it is homeomorphic to the product:



This bundle is often referred to as the *tautological line bundle*, and is written $\eta_1 \to S^1$. The reason for its name is that, by exercise 3.4.11 we know that $\mathbf{R}\mathrm{P}^1$ and S^1 are diffeomorphic, and over $\mathbf{R}\mathrm{P}^n$ we do have a "tautological line bundle" $\eta_n \to \mathbf{R}\mathrm{P}^n$ where η_n is the space of pairs (L, v) where L is a line (one dimensional linear subspace) in $\mathbf{R}^{n+1}$ and v a vector in L. The map is given by $(L, v) \mapsto L$. We will prove in exercise 6.4.4 that this actually defines a vector bundle.

Exercise 6.1.3 Consider the tautological line bundle (unbounded Möbius band)

$$\eta_1 \to S^1$$

from 6.1.2. Prove that there is no smooth map $f\colon \eta_1 \to \mathbf{R}$ such that the zero section is the inverse image of a regular value of f.

More generally, show that there is no map $f\colon \eta_1 \to N$ for any manifold N such that the zero section is the inverse image of a regular value of f.

Definition 6.1.4 Given an n-dimensional topological vector bundle $\pi\colon E \to X$, we call

$E_q = \pi^{-1}(q)$ the *fiber* over $q \in X$,

E the *total space* and

X the *base space* of the vector bundle.

The existence of the (h, U)s is referred to as the *local trivialization* of the bundle ("the bundle is *locally trivial*"), and the (h, U)s are called *bundle charts.* A *bundle atlas* is a collection $\mathcal{B}$ of bundle charts such that

$$X = \bigcup_{(h,U)\in\mathcal{B}} U$$

($\mathcal{B}$ "covers" X).

Note 6.1.5 Note the correspondence the definition spells out between h and h_q: for $r \in \pi^{-1}(U)$ we have

$$h(r) = (\pi(r), h_{\pi(r)}(r))$$

It is (bad taste, but) not uncommon to write just E when referring to the vector bundle $E \to X$.

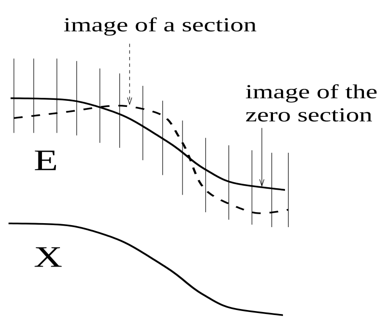
Example 6.1.6 Given a topological space X, the projection onto the first factor

$$\begin{array}{c} X \times \mathbf{R}^n \\ \downarrow{\scriptstyle \mathrm{pr}_X} \\ X \end{array}$$

is an n-dimensional topological vector bundle.

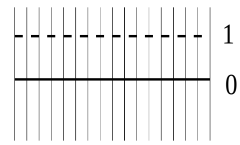
This example is so totally uninteresting that we call it the *trivial bundle over* X (or more descriptively, the *product bundle*). More generally, any vector bundle $\pi: E \to X$ with a bundle chart (h, X) is called trivial.

Definition 6.1.7 Let $\pi: E \to X$ be a vector bundle. A *section* to π is a continuous map $\sigma: X \to E$ such that $\pi\sigma(p) = p$ for all $p \in X$.



Example 6.1.8 Every vector bundle $\pi: E \to X$ has a section, namely the *zero section*, which is the map $\sigma_0: X \to E$ that sends $p \in X$ to zero in the vector space $\pi^{-1}(p)$. As for any section, the map onto its image $X \to \sigma_0(X)$ is a homeomorphism, and we will occasionally not distinguish between X and $\sigma_0(X)$ (we already did this when we talked informally about the unbounded Möbius band).

Example 6.1.9 The trivial bundle $X \times \mathbf{R}^n \to X$ has *nonvanishing sections* (i.e., a section whose image does not intersect the zero section), for instance $p \mapsto (p, 1)$ will do. The tautological line bundle $\eta_1 \to S^1$ (the unbounded Möbius band of example 6.1.2), however, does not. This follows by the intermediate value theorem: a function $f\colon [0,1] \to \mathbf{R}$ with $f(0) = -f(1)$ must have a zero.



The trivial bundle has nonvanishing sections.

We have to specify the maps connecting the vector bundles. They come in two types, according to whether we allow the base space to change. The more general is:

Definition 6.1.10 A *bundle morphism* from one bundle $\pi\colon E \to X$ to another $\pi'\colon E' \to X'$ is a pair of maps

$$f\colon X \to X' \text{ and } \tilde{f}\colon E \to E'$$

such that

$$\begin{array}{ccc} E & \xrightarrow{\tilde{f}} & E' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ X & \xrightarrow{f} & X' \end{array}$$

commutes, and such that

$$\tilde{f}|_{\pi^{-1}(p)}\colon \pi^{-1}(p) \to (\pi')^{-1}(f(p))$$

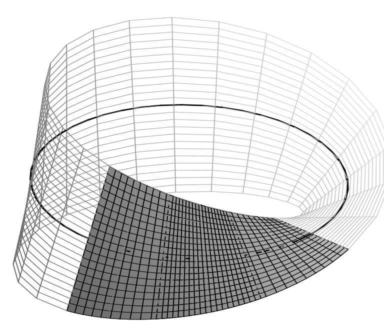
is a linear map.

6.2 Transition functions

We will need to endow our bundles with smooth structures, and in order to do this we will use the same trick as we used to define manifolds: transport everything down to issues in Euclidean spaces. Given two overlapping bundle charts (h, U) and (g, V), restricting to $\pi^{-1}(U \cap V)$ both define homeomorphisms

$$\pi^{-1}(U \cap V) \to (U \cap V) \times \mathbf{R}^n$$

which we may compose to give homeomorphisms of $(U \cap V) \times \mathbf{R}^n$ with itself. If the base space is a smooth manifold, we may ask whether this map is smooth.


Two bundle charts. Restricting to their intersection, how do the two homeomorphisms to $(U \cap V) \times \mathbf{R}^n$ compare?

We need some names to talk about this construction.

Definition 6.2.1 Let $\pi\colon E \to X$ be an n-dimensional topological vector bundle, and let $\mathcal{B}$ be a bundle atlas. If $(h,U),(g,V) \in \mathcal{B}$ then

$$gh^{-1}|_{(U\cap V)\times\mathbf{R}^n}\colon (U\cap V)\times\mathbf{R}^n \to (U\cap V)\times\mathbf{R}^n$$

are called the *bundle chart transformations*. The restrictions to each fiber

$$g_q h_q^{-1}\colon \mathbf{R}^n \to \mathbf{R}^n$$

are linear isomorphisms (i.e., elements in $\mathrm{GL}_n(\mathbf{R})$) and the associated functions

$$\begin{aligned} U\cap V &\to \mathrm{GL}_n(\mathbf{R}) \\ q &\mapsto g_q h_q^{-1} \end{aligned}$$

are called *transition functions*.

Again, visually bundle chart transformations are given by going up and down in

$$\begin{array}{ccc} & \pi^{-1}(U\cap V) & \\ {\scriptstyle h|_{\pi^{-1}(U\cap V)}} \swarrow & & \searrow {\scriptstyle g|_{\pi^{-1}(U\cap V)}} \\ (U\cap V)\times\mathbf{R}^n & & (U\cap V)\times\mathbf{R}^n \end{array}$$

The following lemma explains why giving the bundle chart transformations or the transition functions amounts to the same thing.

Lemma 6.2.2 *Let W be a topological space, and $f\colon W \to M_{m\times n}(\mathbf{R})$ a continuous function. Then the associated function*

$$\begin{aligned} f_*\colon W\times\mathbf{R}^n &\to \mathbf{R}^m \\ (w,v) &\mapsto f(w)\cdot v \end{aligned}$$

is continuous iff f is. If W is a smooth manifold, then f_* is smooth iff f is.

Proof: Note that f_* is the composite

$$W\times\mathbf{R}^n \xrightarrow{f\times id} M_{m\times n}(\mathbf{R})\times\mathbf{R}^n \xrightarrow{e} \mathbf{R}^m$$

where $e(A,v) = A\cdot v$. Since e is smooth, it follows that if f is continuous or smooth, then so is f_*.

Conversely, considered as a matrix, we have that

$$[f(w)] = [f_*(w,e_1),\ldots,f_*(w,e_n)]$$

If f_* is continuous (or smooth), then we see that each column of $[f(w)]$ depends continuously (or smoothly) on w, and so f is continuous (or smooth). ∎

So, requiring the bundle chart transformations to be smooth is the same as to require the transition functions to be smooth, and we will often take the opportunity to confuse this.

A nice formulation of the contents of Lemma 6.2.2 is that we have a bijection from the set of continuous functions $W \to M_{m\times n}\mathbf{R}$ to the set of bundle morphisms

$$\begin{array}{ccc} W\times\mathbf{R}^n & \longrightarrow & W\times\mathbf{R}^m \\ & {\scriptstyle \mathrm{pr}_W}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}_W} & \\ & W & \end{array},$$

by sending the function $g\colon W \to M_{m\times n}\mathbf{R}$ to the function

$$G(g)\colon W\times\mathbf{R}^n \xrightarrow{(p,v)\mapsto(p,g(p)\cdot v)} W\times\mathbf{R}^m.$$

Furthermore, if W is a smooth manifold, then g is smooth if and only if $G(g)$ is smooth.

Exercise 6.2.3 Show that any vector bundle $E \to [0,1]$ is trivial.

Exercise 6.2.4 Show that any 1-dimensional vector bundle (also called *line bundle*) $E \to S^1$ is either trivial, or $E \cong \eta_1$. Show the analogous statement for n-dimensional vector bundles.

6.3 Smooth vector bundles

Definition 6.3.1 Let M be a smooth manifold, and let $\pi\colon E \to M$ be a vector bundle. A bundle atlas is said to be *smooth* if all the transition functions are smooth.

Note 6.3.2 Spelling the differentiability out in full detail we get the following: Let $(M,\mathcal{A})$ be a smooth n-dimensional manifold, $\pi\colon E\to M$ a k-dimensional vector bundle, and $\mathcal{B}$ a bundle atlas. Then $\mathcal{B}$ is smooth if for all bundle charts $(h_1,U_1),(h_2,U_2)\in\mathcal{B}$ and all charts $(x_1,V_1),(x_2,V_2)\in\mathcal{A}$, the composites going up over and across

$$\begin{array}{ccccc} & & \pi^{-1}(U) & & \\ & {\scriptstyle h_1|_{\pi^{-1}(U)}}\swarrow & & \searrow{\scriptstyle h_2|_{\pi^{-1}(U)}} & \\ U\times\mathbf{R}^k & & & & U\times\mathbf{R}^k \\ \downarrow{\scriptstyle x_1|_U\times id} & & & & \downarrow{\scriptstyle x_2|_U\times id} \\ x_1(U)\times\mathbf{R}^k & & & & x_2(U)\times\mathbf{R}^k \end{array}$$

is a smooth function in $\mathbf{R}^{n+k}$, where $U = U_1\cap U_2\cap V_1\cap V_2$.

Example 6.3.3 If M is a smooth manifold, then the trivial bundle is a smooth vector bundle in an obvious manner.

Example 6.3.4 The tautological line bundle (unbounded Möbius strip of Example 6.1.2) $\eta_1 \to S^1$ is a smooth vector bundle. As a matter of fact, the trivial bundle and the tautological line bundle are the only one-dimensional smooth vector bundles over the circle (see example 6.2.4 for the topological case. The smooth case needs partitions of unity, which we will cover at a later stage, see exercise 6.3.15).

Note 6.3.5 Just as for atlases of manifolds, we have a notion of a *maximal (smooth) bundle atlas*, and to each smooth atlas we may associate a unique maximal one in exactly the same way as before.

Definition 6.3.6 A *smooth vector bundle* is a vector bundle equipped with a maximal smooth bundle atlas.

We will often suppress the bundle atlas from the notation, so a smooth vector bundle $(\pi\colon E \to M, \mathcal{B})$ will occasionally be written simply $\pi\colon E \to M$ (or even worse E), if the maximal atlas $\mathcal{B}$ is clear from the context.

Definition 6.3.7 A smooth vector bundle $(\pi\colon E \to M, \mathcal{B})$ is *trivial* if its (maximal smooth) atlas $\mathcal{B}$ contains a chart (h, M) with domain all of M.

Lemma 6.3.8 *The total space E of a smooth vector bundle $(\pi\colon E \to M, \mathcal{B})$ has a natural smooth structure, and π is a smooth map.*

Proof: Let M be n-dimensional with atlas $\mathcal{A}$, and let π be k-dimensional. Then the diagram in 6.3.2 shows that E is a smooth $(n+k)$-dimensional manifold. That π is smooth is the same as claiming that all the up over and across composites

$$\begin{array}{ccccc} & & \pi^{-1}(U) & & \\ & \swarrow^{h|_{\pi^{-1}(U)}} & & \searrow^{\pi|_{\pi^{-1}(U)}} & \\ U \times \mathbf{R}^k & & & & U \\ \downarrow{\scriptstyle x_1|_U \times id} & & & & \downarrow{\scriptstyle x_2|_U} \\ x_1(U) \times \mathbf{R}^k & & & & x_2(U) \end{array}$$

are smooth where $(x_1, V_1), (x_2, V_2) \in \mathcal{A}$, $(h, W) \in \mathcal{B}$ and $U = V_1 \cap V_2 \cap W$. But

$$\begin{array}{ccc} & \pi^{-1}(U) & \\ \swarrow^{h|_{\pi^{-1}(U)}} & & \searrow^{\pi|_{\pi^{-1}(U)}} \\ U \times \mathbf{R}^k & \xrightarrow[\mathrm{pr}_U]{} & U \end{array}$$

commutes, so the composite is simply

$$x_1(U) \times \mathbf{R}^k \xrightarrow{\mathrm{pr}_{x_1(U)}} x_1(U) \xleftarrow{x_1|_U} U \xrightarrow{x_2|_U} x_2(U)$$

which is smooth since $\mathcal{A}$ is smooth. ∎

Note 6.3.9 As expected, the proof shows that $\pi\colon E \to M$ locally looks like the projection

$$\mathbf{R}^n \times \mathbf{R}^k \to \mathbf{R}^n$$

(followed by a diffeomorphism).

Definition 6.3.10 A *smooth bundle morphism* is a bundle morphism

$$\begin{array}{ccc} E & \xrightarrow{\tilde f} & E' \\ {\scriptstyle\pi}\downarrow & & {\scriptstyle\pi'}\downarrow \\ M & \xrightarrow{f} & M' \end{array}$$

from a smooth vector bundle to another such that $\tilde f$ and f are smooth.

Definition 6.3.11 An *isomorphism* of two smooth vector bundles

$$\pi\colon E \to M \text{ and } \pi'\colon E' \to M$$

over the same base space M is an invertible smooth bundle morphism over the identity on M:

$$\begin{array}{ccc} E & \xrightarrow{\tilde f} & E' \\ {\scriptstyle\pi}\downarrow & & {\scriptstyle\pi'}\downarrow \\ M & = & M \end{array}$$

Checking whether a bundle morphism is an isomorphism reduces to checking that it is a bijection:

Lemma 6.3.12 *Let*

$$\begin{array}{ccc} E & \xrightarrow{\tilde f} & E' \\ {\scriptstyle\pi}\downarrow & & {\scriptstyle\pi'}\downarrow \\ M & = & M \end{array}$$

be a smooth (or continuous) bundle morphism. If $\tilde f$ is bijective, then it is a smooth (or continuous) isomorphism.

Proof: That $\tilde f$ is bijective means that it is a bijective linear map on every fiber, or in other words: a vector space isomorphism on every fiber. Choose charts (h, U) in E and (h', U) in E' around $p \in U \subseteq M$ (may choose the U's to be the same). Then

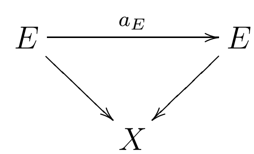
$$h'\tilde f h^{-1}\colon U \times \mathbf{R}^n \to U \times \mathbf{R}^n$$

is of the form $(u, v) \mapsto (u, \alpha_u v)$ where $\alpha_u \in \mathrm{GL}_n(\mathbf{R})$ depends smoothly (or continuously) on $u \in U$. But by Cramer's rule $(\alpha_u)^{-1}$ depends smoothly on α_u, and so the inverse

$$\left(h'\tilde f h^{-1}\right)^{-1}\colon U \times \mathbf{R}^n \to U \times \mathbf{R}^n, \qquad (u, v) \mapsto (u, (\alpha_u)^{-1} v)$$

is smooth (or continuous) proving that the inverse of $\tilde f$ is smooth (or continuous). ∎

Exercise 6.3.13 Let a be a real number and $E \to X$ a bundle. Show that multiplication by a in each fiber gives a bundle morphism



which is an isomorphism if and only if $a \neq 0$. If $E \to X$ is a smooth vector bundle, then a_E is smooth too.

Exercise 6.3.14 Show that any smooth vector bundle $E \to [0, 1]$ is trivial (smooth on the boundary means what you think it does: don't worry).

Exercise 6.3.15 Show that any 1-dimensional smooth vector bundle (also called *line bundle*) over S^1 is either trivial, or the unbounded Möbius band **??**. Show the analogous statement for n-dimensional vector bundles over S^1.

6.4 Pre-vector bundles

A smooth or topological vector bundle is a very structured object, and much of its structure is intertwined very closely. There is a sneaky way out of having to check topological properties all the time. As a matter of fact, the topology is determined by some of the other structure as soon as the claim that it is a vector bundle is made: specifying the topology on the total space is redundant!.

Definition 6.4.1 A *pre-vector bundle* of dimension n is

a set E (*total space*)

a topological space X (*base space*)

a surjective function $\pi\colon E \to X$

a vector space structure on the fiber $\pi^{-1}(q)$ for each $q \in X$

a *pre-bundle atlas* $\mathcal{B}$, i.e., a set $\mathcal{B}$ of pairs (h, U) with

U an open subset of X and

h a bijective function

$$\pi^{-1}(U) \xrightarrow{e \mapsto h(e)=(\pi(e), h_{\pi(e)}(e))} U \times \mathbf{R}^n$$

which is linear on each fiber,

such that

$\mathcal{B}$ covers X, and

the transition functions are continuous.

That $\mathcal{B}$ covers X means that $X = \bigcup_{(h,U)\in\mathcal{B}} U$, that h is linear on each fiber means that $h_q\colon \pi^{-1}(q) \to \mathbf{R}^n$ is linear for each $q \in X$, and that the transition functions of $\mathcal{B}$ are continuous means that if $(h, U), (h', U') \in \mathcal{B}$, then

$$U \cap U' \to \mathrm{GL}_n(\mathbf{R}), \quad q \mapsto h'_q h_q^{-1}$$

is continuous.

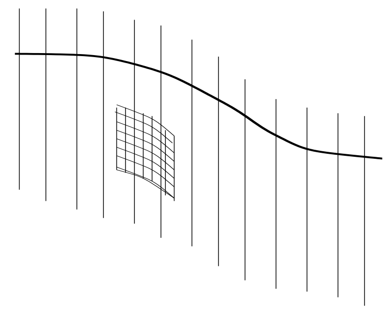
Definition 6.4.2 A *smooth pre-vector bundle* is a pre-vector bundle where the base space is a smooth manifold and the transition functions are smooth.

Lemma 6.4.3 *Given a pre-vector bundle, there is a unique vector bundle with underlying pre-vector bundle the given one. The same statement holds for the smooth case.*

Proof: Let $(\pi\colon E \to X, \mathcal{B})$ be a pre-vector bundle. We must equip E with a topology such that π is continuous and the bijections in the bundle atlas are homeomorphisms. The smooth case follows then immediately from the continuous case.

We must have that if $(h, U) \in \mathcal{B}$, then $\pi^{-1}(U)$ is an open set in E (for π to be continuous). The family of open sets $\{\pi^{-1}(U)\}_{U \subseteq X \text{ open}}$ covers E, so we only need to know what the open subsets of $\pi^{-1}(U)$ are, but this follows by the requirement that the bijection h should be a homeomorphism, That is $V \subseteq \pi^{-1}(U)$ is open if $V = h^{-1}(V')$ for some open $V' \subseteq U \times \mathbf{R}^k$. Ultimately, we get that

$$\left\{ h^{-1}(V_1 \times V_2) \,\middle|\, (h, U) \in \mathcal{B}, \begin{matrix} V_1 & \text{open in } U, \\ V_2 & \text{open in } \mathbf{R}^k \end{matrix} \right\}$$



A typical open set in $\pi^{-1}(U)$ gotten as h^{-1} of the product of an open set in U and an open set in $\mathbf{R}^k$

is a basis for the topology on E. ∎

Exercise 6.4.4 Let

$$\eta_n = \left\{ ([p], \lambda p) \in \mathbf{R}\mathrm{P}^n \times \mathbf{R}^{n+1} \,| p \in S^n, \lambda \in \mathbf{R} \right\}$$

Show that the projection

$$\begin{aligned} \eta_n &\to \mathbf{R}\mathrm{P}^n \\ ([p], \lambda p) &\mapsto [p] \end{aligned}$$

defines a non-trivial smooth vector bundle, called the *tautological line bundle*.

Exercise 6.4.5 Let $p \in \mathbf{R}\mathrm{P}^n$ and $X = \mathbf{R}\mathrm{P}^n \setminus \{p\}$. Show that X is diffeomorphic to the total space η_{n-1} of the tautological line bundle in exercise 6.4.4.

6.5 The tangent bundle

We define the tangent bundle as follows:

Definition 6.5.1 Let $(M, \mathcal{A})$ be a smooth n-dimensional manifold. The tangent bundle of M is defined by the following smooth pre-vector bundle

$TM = \coprod_{p \in M} T_p M$ (total space)

M (base space)

$\pi : TM \to M$ sends $T_p M$ to p

the pre-vector bundle atlas

$$\mathcal{B}_{\mathcal{A}} = \{(h_x, U) | (x, U) \in \mathcal{A}\}$$

where h_x is given by

$$\begin{aligned} h_x : \pi^{-1}(U) &\to U \times \mathbf{R}^n \\ [\gamma] &\mapsto (\gamma(0), (x\gamma)'(0)) \end{aligned}$$

Note 6.5.2 Since the tangent bundle is a smooth vector bundle, the total space TM is a smooth $2n$-dimensional manifold. To be explicit, its atlas is gotten from the smooth atlas on M as follows.

If (x, U) is a chart on M,

$$\pi^{-1}(U) \xrightarrow{h_x} U \times \mathbf{R}^n \xrightarrow{x \times id} x(U) \times \mathbf{R}^n$$

$$[\gamma] \mapsto (x\gamma(0), (x\gamma)'(0))$$

is a homeomorphism to an open subset of $\mathbf{R}^n \times \mathbf{R}^n$. It is convenient to have an explicit formula for the inverse. Let $(p, v) \in x(U) \times \mathbf{R}^n$. Define the germ

$$\gamma(p, v) \colon (\mathbf{R}, 0) \to (\mathbf{R}^n, p)$$

by sending t (in a sufficiently small open interval containing zero) to $p + tv$. Then the inverse is given by sending (p, v) to

$$[x^{-1}\gamma(p, v)] \in T_{x^{-1}(p)} M$$

Lemma 6.5.3 *Let $f \colon (M, \mathcal{A}_M) \to (N, \mathcal{A}_N)$ be a smooth map. Then*

$$[\gamma] \mapsto Tf[\gamma] = [f\gamma]$$

defines a smooth bundle morphism

$$\begin{array}{ccc} TM & \xrightarrow{Tf} & TN \\ \pi_M \downarrow & & \pi_N \downarrow \\ M & \xrightarrow{f} & N \end{array}$$

Proof: Since $Tf|_{\pi^{-1}(p)} = T_p f$ we have linearity on the fibers, and we are left with showing that Tf is a smooth map. Let $(x, U) \in \mathcal{A}_M$ and $(y, V) \in \mathcal{A}_N$. We have to show that up, across and down in

$$\begin{array}{ccc} \pi_M^{-1}(W) & \xrightarrow{Tf|} & \pi_N^{-1}(V) \\ {\scriptstyle h_x|_W}\downarrow & & \downarrow{\scriptstyle h_y} \\ W \times \mathbf{R}^m & & V \times \mathbf{R}^n \\ {\scriptstyle x|_W \times id}\downarrow & & \downarrow{\scriptstyle y \times id} \\ x(W) \times \mathbf{R}^m & & y(V) \times \mathbf{R}^n \end{array}$$

is smooth, where $W = U \cap f^{-1}(V)$ and $Tf|$ is Tf restricted to $\pi_M^{-1}(W)$. This composite sends $(p, v) \in x(W) \times \mathbf{R}^m$ to $[x^{-1}\gamma(p, v)] \in \pi_M^{-1}(W)$ to $[fx^{-1}\gamma(p, v)] \in \pi_N^{-1}(V)$ and finally to $(yfx^{-1}\gamma(p, v)(0), (yfx^{-1}\gamma(p, v))'(0) \in y(V) \times \mathbf{R}^n$ which is equal to

$$(yfx^{-1}(p), D(yfx^{-1})(p) \cdot v)$$

by the chain rule. Since yfx^{-1} is a smooth function, this is a smooth function too. ∎

Lemma 6.5.4 *If $f\colon M \to N$ and $g\colon N \to L$ are smooth, then*

$$TgTf = T(gf)$$

Proof: It is the chain rule 4.2.4 (made pleasant since the notation no longer has to specify over which point in your manifold you are). ∎

Note 6.5.5 The tangent space of $\mathbf{R}^n$ is trivial, since the identity chart induces a bundle chart

$$\begin{aligned} h_{id}\colon T\mathbf{R}^n &\to \mathbf{R}^n \times \mathbf{R}^n \\ [\gamma] &\mapsto (\gamma(0), \gamma'(0)) \end{aligned}$$

Definition 6.5.6 A manifold is often said to be *parallelizable* if its tangent bundle is trivial.

Example 6.5.7 The circle is parallelizable. This is so since the map

$$\begin{aligned} S^1 \times T_1 S^1 &\to TS^1 \\ (e^{i\theta}, [\gamma]) &\mapsto [e^{i\theta} \cdot \gamma] \end{aligned}$$

is a diffeomorphism (here $(e^{i\theta} \cdot \gamma)(t) = e^{i\theta} \cdot \gamma(t)$).

Exercise 6.5.8 The three-sphere S^3 is parallelizable

Exercise 6.5.9 All Lie groups are parallelizable. (A Lie group is a manifold with a smooth associative multiplication, with a unit and all inverses: skip this exercise if this sounds too alien to you).

Example 6.5.10 Let

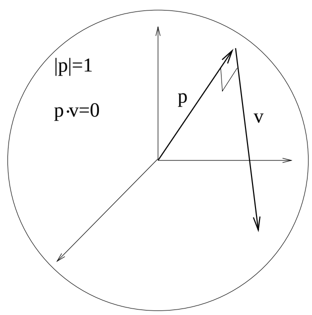
$$E = \{(p, v) \in \mathbf{R}^{n+1} \times \mathbf{R}^{n+1} \mid |p| = 1, p \cdot v = 0\}$$

Then

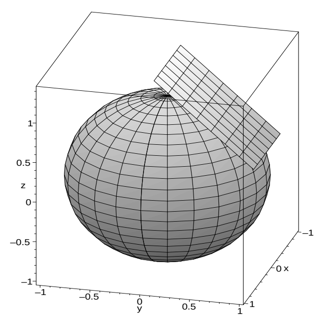
$$\begin{aligned} TS^n &\to E \\ [\gamma] &\mapsto (\gamma(0), \gamma'(0)) \end{aligned}$$

is a homeomorphism. The inverse sends $(p, v) \in E$ to the equivalence class of the germ associated to

$$t \mapsto \frac{p + tv}{|p + tv|}$$



A point in the tangent space of S^2 may be represented by a unit vector p together with an arbitrary vector v perpendicular to p.



We can't draw all the tangent planes simultaneously to illustrate the tangent space of S^2. The description we give is in $\mathbf{R}^6$.

More generally we have the following fact:

Lemma 6.5.11 *Let $f: M \to N$ be an imbedding. Then $Tf: TM \to TN$ is an imbedding.*

Proof: We may assume that f is the inclusion of a submanifold (the diffeomorphism part is taken care of by the chain rule which implies that Tf is a diffeomorphism if f is). Let $y: V \to V'$ be a chart on N such that $y(V \cap M) = V' \cap (\mathbf{R}^m \times \{0\})$. Since curves in $\mathbf{R}^m \times 0$ have derivatives in $\mathbf{R}^m \times 0$ we see that if $\bar{\gamma}: (\mathbf{R}^1, 0) \to (M, p)$ then

$$\begin{aligned} (y \times \mathrm{id}_{\mathbf{R}^m}) h_y(Tf[\gamma]) &= (y \times \mathrm{id}_{\mathbf{R}^m}) h_y[f\gamma] \\ &= (yf\gamma(0), (yf\gamma)'(0)) \in (V' \cap (\mathbf{R}^m \times \{0\})) \times (\mathbf{R}^m \times \{0\}) \end{aligned}$$

and so

$$\begin{aligned} (y \times \mathrm{id}_{\mathbf{R}^m}) h_y(Tf(\pi_M^{-1}(W \cap M)) &= (V' \cap (\mathbf{R}^m \times \{0\})) \times \mathbf{R}^m \times \{0\} \\ &\subseteq \mathbf{R}^m \times \mathbf{R}^k \times \mathbf{R}^m \times \mathbf{R}^k \end{aligned}$$

and by permuting the coordinates we have that Tf is the inclusion of a submanifold.

∎

Corollary 6.5.12 *If $M \subseteq \mathbf{R}^N$ is the inclusion of a smooth submanifold of an Euclidean space, then*

$$TM \cong \left\{ (p,v) \in \mathbf{M} \times \mathbf{R}^N \;\middle|\; \begin{matrix} v = \gamma'(0) \textit{ for some germ} \\ \bar{\gamma}\colon (\mathbf{R},0) \to (M,p) \end{matrix} \right\} \subseteq \mathbf{R}^N \times \mathbf{R}^N \cong T\mathbf{R}^N$$

(the derivation of γ happens in $\mathbf{R}^N$)

Exercise 6.5.13 There is an even groovier description of TS^n: prove that

$$E = \left\{ (z_0, \dots, z_n) \in \mathbf{C}^{n+1} | \sum_{i=0}^{n} z^2 = 1 \right\}$$

is the total space in a bundle isomorphic to TS^n.

Definition 6.5.14 Let M be a smooth manifold. A *vector field* on M is a section in the tangent bundle.

Exercise 6.5.15 Prove that the projection $S^n \to \mathbf{R}\mathrm{P}^n$ gives an isomorphism

$$T\mathbf{R}\mathrm{P}^n \cong \{(p,v) \in S^n \times \mathbf{R}^{n+1} \,|\, p \cdot v = 0\}/(p,v) \sim (-p,-v).$$

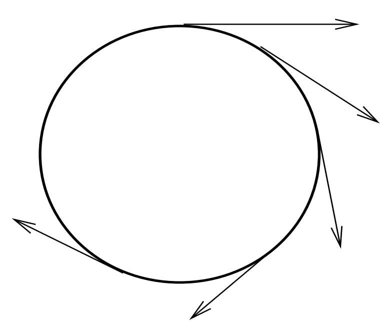
As we will see in Chapter 8 vector fields are closely related to differential equations. It is often of essence to know whether a manifold M supports *nonvanishing* vector fields, i.e., a vector field $s\colon M \to TM$ such that $s(p) \neq 0$ for all $p \in M$.

Example 6.5.16 The circle has nonvanishing vector fields. Let $[\gamma] \neq 0 \in T_1 S^1$, then

$$S^1 \to TS^1, \qquad e^{i\theta} \mapsto [e^{i\theta} \cdot \gamma]$$

is a vector field (since $e^{i\theta} \cdot \gamma(0) = e^{i\theta} \cdot 1$) and does not intersect the zero section since (viewed as a vector in $\mathbf{C}$)

$$|(e^{i\theta} \cdot \gamma)'(0)| = |e^{i\theta} \cdot \gamma'(0)| = |\gamma'(0)| \neq 0$$



The vector field spins around the circle with constant speed.

This is the same construction we used to show that S^1 was parallelizable. This is a general argument: an n dimensional manifold with n linearly independent vector fields has a trivial tangent bundle, and conversely.

Exercise 6.5.17 Construct three vector fields on S^3 that are linearly independent in all tangent spaces.

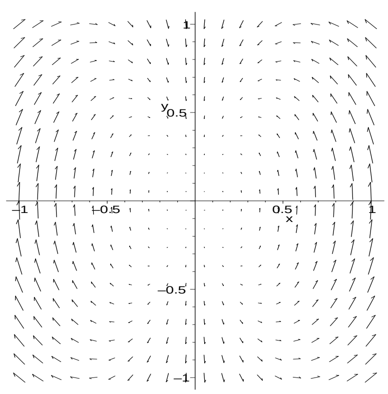
Exercise 6.5.18 Prove that $T(M \times N) \cong TM \times TN$.

Example 6.5.19 We have just seen that S^1 and S^3 (if you did the exercise) both have nonvanishing vector fields. It is a hard fact that S^2 does not: "you can't comb the hair on a sphere".

This has the practical consequence that when you want to confine the plasma in a fusion reactor by means of magnetic fields, you can't choose to let the plasma be in the interior of a sphere (or anything homeomorphic to it). At each point on the surface bounding the region occupied by the plasma, the component of the magnetic field parallel to the surface must be nonzero, or the plasma will leak out (if you remember your physics, there once was a formula saying something like $F = qv \times B$ where q is the charge of the particle, v its velocity and B the magnetic field: hence any particle moving nonparallel to the magnetic field will be deflected).

This problem is solved by letting the plasma stay inside a torus $S^1 \times S^1$ which does have nonvanishing vector fields (since S^1 has by 6.5.16, and since $T(S^1 \times S^1) \cong TS^1 \times TS^1$ by exercise 6.5.18).

Although there are no nonvanishing vector fields on S^2, there are certainly interesting ones that have only a few zeros. For instance "rotation around an axis" will give you a vector field with only two zeros. The "magnetic dipole" defines a vector field on S^2 with just one zero.



A magnetic dipole on S^2, seen by stereographic projection in a neighborhood of the only zero.

Exercise 6.5.20 Let M be an n-dimensional smooth manifold. For $p \in M$, let E_p be the set of germs

$$\bar{s}\colon (\mathbf{R}^2, 0) \to (M, p)$$

modulo the equivalence relation that $s \simeq s'$ if for any chart (x, U) with $p \in U$ we have that

$$D_1(xs)(0) = D_1(xs')(0), \quad D_2(xs)(0) = D_2(xs')(0) \text{ and } D_1D_2(xs)(0) = D_1D_2(xs')(0).$$

Let $E = \coprod_{p\in M} E_p$ and define a bijection $E \cong T(TM)$ making the projection $E \to TM$ sending $[s]$ to $[t \mapsto s(t, 0)]$ a smooth vector bundle isomorphic to the tangent bundle of TM.

6.6 The cotangent bundle[1]

Let M be a smooth m-dimensional manifold. Recall the definition of the cotangent spaces from 4.3, more precisely the definition 4.3.1. We will show that the cotangent spaces join to form a bundle, the *cotangent bundle* T^*M, by showing that they define a prevector bundle.

Let the total space T^*M be the set

$$T^*M = \{(p, d\phi) \,|\, p \in M, d\phi \in T_p^*M\},$$

and $\pi\colon T^*M \to M$ be the projection sending $(p, d\phi)$ to p. For a smooth chart (x, U) we have a bundle chart

$$h_x\colon \pi^{-1}(U) = T^*U \to U \times (\mathbf{R}^n)^*$$

gotten by sending $(p, d\phi)$ to $(p, D(\phi x^{-1})(x(p)\cdot)$. To get it in exactly the form of definition 6.1.1 we should choose an isomorphism $\mathrm{Hom}_{\mathbf{R}}(\mathbf{R}^m, \mathbf{R}) = (\mathbf{R}^m)^* \cong \mathbf{R}^m$ once and for all (e.g., transposing vectors), but it is convenient to postpone this translation as much as possible.

By the discussion in section 4.3, h_x induces a linear isomorphism $\pi^{-1}(p) = T_p^*M \cong \{p\} \times (\mathbf{R}^m)^*$ in each fiber. If (y, V) is another chart, the transition function is given by sending $p \in U \cap V$ to the linear isomorphism $(\mathbf{R}^m)^* \to (\mathbf{R}^m)^*$ induced by the linear isomorphism $\mathbf{R}^m \to \mathbf{R}^m$ given by multiplication by the Jacobi matrix $D(yx^{-1})(x(p))$. Since the Jacobi matrix $D(yx^{-1})(x(p))$ varies smoothly with p, we have shown that

$$T^*M \to M$$

is a smooth (pre)vector bundle, the *cotangent bundle*.

Exercise 6.6.1 Go through the details in the above discussion.

Definition 6.6.2 If M is a smooth manifold, a *one-form* is a smooth section of the cotangent bundle $T^*M \to M$.

[1]If you did not read about the cotangent space in section 4.3, you should skip this section

Example 6.6.3 Let $f\colon M \to \mathbf{R}$ be a smooth function. Recall the differential map $d\colon \mathcal{O}_{M,p} \to T_p^*M$ given by sending a function germ $\bar{\phi}$ to the cotangent vector represented by the germ of $q \mapsto \phi(q) - \phi(p)$. Correspondingly, we write $d_pf \in T_p^*M$ for the cotangent vector represented by $q \mapsto f(q) - f(p)$. Then the assignment $p \mapsto (p, d_pf) \in T^*M$ is a one-form, and we simply write

$$df\colon M \to T^*M.$$

To signify that this is just the beginning in a series of important vector spaces, let $\Omega^0M = \mathcal{C}^\infty(M, \mathbf{R})$ and let $\Omega^1(M)$ be the vector space of all one-forms on M. The differential is then a map

$$d\colon \Omega^0(M) \to \Omega^1(M).$$

Even though the differential as a map to each individual cotangent space $d\colon \mathcal{O}_{M,p} \to T_p^*M$ was surjective, this is not the case for $d\colon \Omega^0(M) \to \Omega^1(M)$. In fact, the one-forms in the image of d are the ones that are referred to as "exact" (this is classical notation coming from differential equations, the other relevant notion being "closed". It is the wee beginning of the study of the shapes of spaces through cohomological methods).

Example 6.6.4 If $x_1, x_2\colon S^1 \subseteq \mathbf{R}^2 \to \mathbf{R}$ are the projection to the first and second coordinate, respectively, then one can show that

$$x_1dx_2 - x_2dx_1$$

is a one-form that is not exact, a phenomenon related to the fact that the circle is not simply connected. As a matter of fact, the quotient $H^1(S^1) = \Omega^1(S^1)/d(\Omega^0(S^1))$ (which is known as the *first de Rham cohomology group* of the circle) is a one-dimensional real vector space, and so the image of the non-exact one-form displayed above generates $H^1(S^1)$.

Example 6.6.5 In physics the total space of the cotangent bundle is referred to as the *phase space*. If the manifold M is the collection of all possible positions of the physical system, the phase space T^*M is the collection of all positions and momenta. For instance, if we study a particle of mass m in euclidean 3-space, the position is given by three numbers x_1, x_2, x_3 (really, coordinates with respect to the standard basis) and the momentum by yet three numbers p_1, p_2, p_3 (coordinates with respect to the basis $\{dx_1, dx_2, dx_3\}$ in the cotangent space). See also example 4.3.19. We will come back to such matters when we have talked about Riemannian metrics.

6.6.6 The tautological one-form

If M is an m-dimensional smooth manifold, T^*M is a $2m$-dimensional smooth manifold. This manifold has an especially important one-form $\theta_M\colon T^*M \to T^*T^*M$, called the *tautological one-form* (or *canonical one-form* or *Liouville one-form* or *symplectic potential* – a dear child has many names). For each point $(p, d\phi) \in T^*M$ in the total space of the cotangent bundle we define an element in $T^*_{(p,d\phi)}T^*M$ as follows: consider the map $T\pi\colon T(T^*M) \to TM$ induced by the projection $\pi\colon T^*M \to M$. By the isomorphism

$\alpha_p(M)\colon T_p^*M \cong (T_pM)^*$, the cotangent vector $d\phi$ should be thought of as a linear map $T_pM \to \mathbf{R}$. By composing these maps

$$T_{(p,d\phi)}T^*M \to T_pM \to \mathbf{R}$$

we have an element in $\theta_M(p, d\phi) \in T^*_{(p,d\phi)}T^*M \cong (T_{(p,d\phi)}T^*M)^*$ (the isomorphism is the inverse of $\alpha_{T_p^*M,(p,d\phi)}$).

Exercise 6.6.7 Show that the procedure above gives a one-form θ_M on T^*M (that is a smooth section of the projection $T^*(T^*M) \to T^*M$).

Chapter 7

Constructions on vector bundles

A good way to think of vector bundles are as families of vector spaces indexed over a base space. All constructions we wish to perform on the individual vector spaces should conform with the indexation in that they should be allowed to vary continuously or smoothly from point to point.

This means that, in all essence, the "natural" (in a precise sense) constructions we know from linear algebra have their counterparts for bundles. The resulting theory gives deep information about the base space, as well as allowing us to construct some important mathematical objects. We start this study in this chapter.

7.1 Subbundles and restrictions

There are a variety of important constructions we need to address. The first of these have been lying underneath the surface for some time:

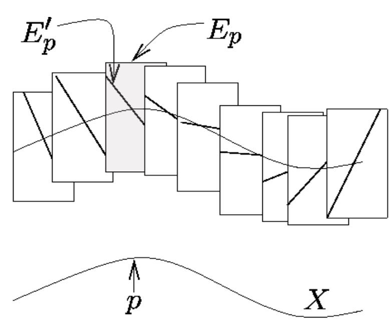
Definition 7.1.1 Let

$$\pi\colon E \to X$$

be an n-dimensional vector bundle. A k-dimensional *subbundle* of this vector bundle is a subset $E' \subseteq E$ such that around any point there is a bundle chart (h, U) such that

$$h(\pi^{-1}(U) \cap E') = U \times (\mathbf{R}^k \times \{0\}) \subseteq U \times \mathbf{R}^n$$

Note 7.1.2 It makes sense to call such a subset $E' \subseteq E$ a subbundle, since we see that the bundle charts, restricted to E', define a vector bundle structure on $\pi|_{E'}\colon E' \to X$ which is smooth if we start out with a smooth atlas.



A one-dimensional subbundle in a two-dimensional vector bundle: pick out a one-dimensional linear subspace of every fiber in a continuous (or smooth) manner.

Example 7.1.3 Consider the trivial bundle $S^1 \times \mathbf{C} \to S^1$. The tautological line bundle $\eta_1 \to \mathbf{R}\mathrm{P}^1 \cong S^1$ of Example 6.1.2 can be thought of as the subbundle given by

$$\{(e^{i\theta}, te^{i\theta/2}) \in S^1 \times \mathbf{C} | t \in \mathbf{R}\} \subseteq S^1 \times \mathbf{C}.$$

Exercise 7.1.4 Spell out the details of the previous example. Also show that

$$\eta_n = \left\{([p], \lambda p) \in \mathbf{R}\mathrm{P}^n \times \mathbf{R}^{n+1} \,|p \in S^n, \lambda \in \mathbf{R}\right\} \subseteq \mathbf{R}\mathrm{P}^n \times \mathbf{R}^{n+1}$$

is a subbundle of the trivial bundle $\mathbf{R}\mathrm{P}^n \times \mathbf{R}^{n+1} \to \mathbf{R}\mathrm{P}^n$.

Definition 7.1.5 Given a bundle $\pi\colon E \to X$ and a subspace $A \subseteq X$, the *restriction* to A is the bundle

$$\pi_A\colon E_A \to A$$

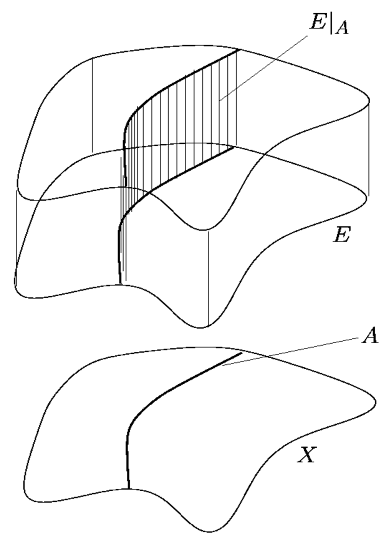
where $E_A = \pi^{-1}(A)$ and $\pi_A = \pi|_{\pi^{-1}(A)}$.

In the special case where A is a single point $p \in X$, we write $E_p = \pi^{-1}(p)$ (instead of $E_{\{p\}}$). Occasionally it is typographically convenient to write $E|_A$ instead of E_A (especially when the notation is already a bit cluttered).

Note 7.1.6 We see that the restriction is a new vector bundle, and the inclusion

$$\begin{array}{ccc} E_A & \xrightarrow{\subseteq} & E \\ {\scriptstyle \pi_A}\downarrow & & {\scriptstyle \pi}\downarrow \\ A & \xrightarrow{\subseteq} & X \end{array}$$

is a bundle morphism inducing an isomorphism on every fiber.

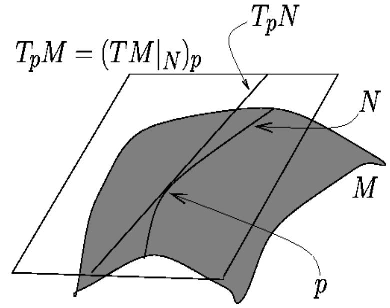


The restriction of a bundle $E \to X$ to a subset $A \subseteq X$.

Example 7.1.7 Let $N \subseteq M$ be a smooth submanifold. Then we can restrict the tangent bundle on M to N and get

$$(TM)|_N \to N$$

We see that $TN \subseteq TM|_N$ is a smooth subbundle.



In a submanifold $N \subseteq M$ the tangent bundle of N is naturally a subbundle of the tangent bundle of M restricted to N

Definition 7.1.8 A bundle morphism

$$\begin{array}{ccc} E_1 & \xrightarrow{f} & E_2 \\ \pi_1 \big\downarrow & & \pi_2 \big\downarrow \\ X_1 & \longrightarrow & X_2 \end{array}$$

is said to be of *constant rank r* if restricted to each fiber f is a linear map of rank r.

Note that this is a generalization of our concept of constant rank of smooth maps.

Theorem 7.1.9 *(Rank theorem for bundles) Consider a bundle morphism*

$$\begin{array}{ccc} E_1 & \xrightarrow{\quad f \quad} & E_2 \\ & {\scriptstyle \pi_1} \searrow \quad \swarrow {\scriptstyle \pi_2} & \\ & X & \end{array}$$

over a space X with constant rank r. Then around any point $p \in X$ there are bundle charts (h, U) and (g, U) such that

$$\begin{array}{ccc} E_1|_U & \xrightarrow{f|_U} & E_2|_U \\ h \big\downarrow & & g \big\downarrow \\ U \times \mathbf{R}^m & \xrightarrow{(u,(t_1,\dots,t_m)) \mapsto (u,(t_1,\dots,t_r,0,\dots,0))} & U \times \mathbf{R}^n \end{array}$$

commutes.

Furthermore if we are in a smooth situation, these bundle charts may be chosen to be smooth.

Proof: This is a local question, so translating via arbitrary bundle charts we may assume that we are in the trivial situation

$$\begin{array}{ccc} U' \times \mathbf{R}^m & \xrightarrow{\quad f \quad} & U' \times \mathbf{R}^n \\ & {\scriptstyle \mathrm{pr}_{U'}} \searrow \quad \swarrow {\scriptstyle \mathrm{pr}_{U'}} & \\ & U' & \end{array}$$

with $f(u,v) = (u, (f_u^1(v), \dots, f_u^n(v)))$, and $rkf_u = r$. By a choice of basis on $\mathbf{R}^m$ and $\mathbf{R}^n$ we may assume that f_u is represented by a matrix

$$\begin{bmatrix} A(u) & B(u) \\ C(u) & D(u) \end{bmatrix}$$

with $A(p) \in \mathrm{GL}_r(\mathbf{R})$ and $D(p) = C(p)A(p)^{-1}B(p)$ (the last equation follows as the rank rkf_p is r). We change the basis so that this is actually true in the standard basis.

Let $p \in U \subseteq U'$ be the open set $U = \{u \in U' | \det(A(u)) \neq 0\}$. Then again $D(u) = C(u)A(u)^{-1}B(u)$ on U.

Let

$$h\colon U \times \mathbf{R}^m \to U \times \mathbf{R}^m, \qquad h(u,v) = (u, h_u(v))$$

be the homeomorphism where h_u is given by the matrix

$$\begin{bmatrix} A(u) & B(u) \\ 0 & I \end{bmatrix}$$

Let

$$g\colon U \times \mathbf{R}^n \to U \times \mathbf{R}^n, \qquad g(u,w) = (u, g_u(w))$$

be the homeomorphism where g_u is given by the matrix

$$\begin{bmatrix} I & 0 \\ -C(u)A(u)^{-1} & I \end{bmatrix}$$

Then $gfh^{-1}(u,v) = (u, (gfh^{-1})_u(v))$ where $(gfh^{-1})_u$ is given by the matrix

$$\begin{aligned} &\begin{bmatrix} I & 0 \\ -C(u)A(u)^{-1} & I \end{bmatrix} \begin{bmatrix} A(u) & B(u) \\ C(u) & D(u) \end{bmatrix} \begin{bmatrix} A(u) & B(u) \\ 0 & I \end{bmatrix}^{-1} \\ =& \begin{bmatrix} I & 0 \\ -C(u)A(u)^{-1} & I \end{bmatrix} \begin{bmatrix} A(u) & B(u) \\ C(u) & D(u) \end{bmatrix} \begin{bmatrix} A(u)^{-1} & -A(u)^{-1}B(u) \\ 0 & I \end{bmatrix} \\ =& \begin{bmatrix} I & 0 \\ -C(u)A(u)^{-1} & I \end{bmatrix} \begin{bmatrix} I & 0 \\ C(u)A(u)^{-1} & 0 \end{bmatrix} \\ =& \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

as claimed (the right hand lower zero in the answer is really a $0 = -C(u)A(u)^{-1}B(u) + D(u)$). ∎

Recall that if $f\colon V \to W$ is a linear map of vector spaces, then the *kernel* (or *null space*) is the subspace

$$\ker\{f\} = \{v \in V \,|\, f(v) = 0\} \subseteq V$$

and the *image* (or *range*) is the subspace

$$\mathrm{Im}\{f\} = \{w \in W \,|\, \text{there is a } v \in V \text{ such that } w = f(v)\}$$

Corollary 7.1.10 *If*

$$\begin{array}{ccc} E_1 & \xrightarrow{f} & E_2 \\ & {\scriptstyle \pi_1}\searrow \quad \swarrow{\scriptstyle \pi_2} & \\ & X & \end{array}$$

is a bundle morphism of constant rank, then the kernel

$$\bigcup_{p\in X} \ker\{f_p\} \subseteq E_1$$

and image

$$\bigcup_{p\in X} \mathrm{Im}\{f_p\} \subseteq E_2$$

are subbundles.

Exercise 7.1.11 Let $\pi\colon E \to X$ be a vector bundle over a connected space X. Assume given a bundle morphism

$$\begin{array}{ccc} E & \xrightarrow{f} & E \\ & {\scriptstyle \pi}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & X & \end{array}$$

with $f \circ f = f$. Prove that f has constant rank.

Exercise 7.1.12 Let $\pi\colon E \to X$ be a vector bundle over a connected space X. Assume given a bundle morphism

$$\begin{array}{ccc} E & \xrightarrow{f} & E \\ & {\scriptstyle \pi}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & X & \end{array}$$

with $f \circ f = id_E$. Prove that the space of fixed points

$$E^{\{f\}} = \{e \in E | f(e) = e\}$$

is a subbundle of E.

Exercise 7.1.13 Consider the map $f\colon T\mathbf{R} \to T\mathbf{R}$ sending $[\gamma]$ to $[t \mapsto \gamma(0) \cdot \gamma(t)]$. Show that f is a bundle morphisms, but that f does not have constant rank and neither the kernel nor the image of f are subbundles.

Exercise 7.1.14 Let $f\colon E \to M$ be a smooth bundle of rank k. Show that the *vertical bundle*

$$V = \{v \in TE \,|\, Tf(v) = 0\} \subseteq TE$$

is a smooth subbundle of $TE \to E$.

7.2 The induced bundle

Definition 7.2.1 Assume given a bundle $\pi\colon E \to Y$ and a continuous map $f\colon X \to Y$. Let the *fiber product* of f and π be the space

$$f^*E = X \times_Y E = \{(x,e) \in X \times E | f(x) = \pi(e)\}$$

(topologized as a subspace of $X \times E$), and let *the induced bundle* be the projection

$$f^*\pi\colon f^*E \to X, \qquad (x,e) \mapsto x$$

Note 7.2.2 Note that the fiber over $x \in X$ may be identified with the fiber over $f(x) \in Y$.

The reader may recognize the fiber product $X \times_Y E$ from Exercise 5.7.10, where we showed that if the contributing spaces are smooth then the fiber product is often smooth too.

Lemma 7.2.3 *If $\pi\colon E \to Y$ is a vector bundle and $f\colon X \to Y$ a continuous map, then*

$$f^*\pi\colon f^*E \to X$$

is a vector bundle and the projection $f^*E \to E$ defines a bundle morphism

$$\begin{array}{ccc} f^*E & \longrightarrow & E \\ {\scriptstyle f^*\pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{f} & Y \end{array}$$

inducing an isomorphism on fibers. If the input is smooth the output is smooth too.

Proof: Let $p \in X$ and let (h,V) be a bundle chart

$$h\colon \pi^{-1}(V) \to V \times \mathbf{R}^k$$

such that $f(p) \in V$. Then $U = f^{-1}(V)$ is an open neighborhood of p. Note that

$$\begin{aligned} (f^*\pi)^{-1}(U) &= \{(u,e) \in X \times E | f(u) = \pi(e) \in V\} \\ &= \{(u,e) \in U \times \pi^{-1}(V) | f(u) = \pi(e)\} \\ &= U \times_V \pi^{-1}(V) \end{aligned}$$

and

$$U \times_V (V \times \mathbf{R}^k) \cong U \times \mathbf{R}^k$$

and we define

$$\begin{aligned} f^*h\colon (f^*\pi)^{-1}(U) = U \times_V \pi^{-1}(V) &\to U \times_V (V \times \mathbf{R}^k) \cong U \times \mathbf{R}^k \\ (u,e) &\mapsto (u, h(e)) \leftrightarrow (u, h_{\pi(e)}e) \end{aligned}$$

Since h is a homeomorphism f^*h is a homeomorphism (smooth if h is), and since $h_{\pi(e)}e$ is an isomorphism (f^*h) is an isomorphism on each fiber. The rest of the lemma now follows automatically. ∎

Theorem 7.2.4 *Let*

$$\begin{array}{ccc} E' & \xrightarrow{\tilde{f}} & E \\ {\scriptstyle \pi'}\big\downarrow & & {\scriptstyle \pi}\big\downarrow \\ X' & \xrightarrow{f} & X \end{array}$$

be a bundle morphism.

Then there is a factorization

$$\begin{array}{ccccc} E' & \longrightarrow & f^*E & \longrightarrow & E \\ {\scriptstyle \pi'}\big\downarrow & & {\scriptstyle f^*\pi}\big\downarrow & & {\scriptstyle \pi}\big\downarrow \\ X' & = & X' & \xrightarrow{f} & X \end{array}$$

Proof: Let

$$E' \to X' \times_X E = f^*E$$
$$e \mapsto (\pi'(e), \tilde{f}(e))$$

This is well defined since $f(\pi'(e)) = \pi(\tilde{f}(e))$. It is linear on the fibers since the composition

$$(\pi')^{-1}(p) \to (f^*\pi)^{-1}(p) \cong \pi^{-1}(f(p))$$

is nothing but $\tilde{f}_p$. ∎

Exercise 7.2.5 Let $i\colon A \subseteq X$ be an injective map and $\pi\colon E \to X$ a vector bundle. Prove that the induced and the restricted bundles are isomorphic.

Exercise 7.2.6 Show the following statement: if

$$\begin{array}{ccccc} E' & \xrightarrow{h} & \tilde{E} & \xrightarrow{g} & E \\ {\scriptstyle \pi'}\big\downarrow & & {\scriptstyle \tilde{\pi}}\big\downarrow & & {\scriptstyle \pi}\big\downarrow \\ X' & = & X' & \xrightarrow{f} & X \end{array}$$

is a factorization of $(f, \tilde{f})$, then there is a unique bundle map

$$\begin{array}{ccc} \tilde{E} & \longrightarrow & f^*E \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

such that

$$\begin{array}{ccccc} E' & \longrightarrow & \tilde{E} & & \\ & \searrow & \big\downarrow & \searrow & \\ & & f^*E & \longrightarrow & E \end{array}$$

commutes.

As a matter of fact, you could characterize the induced bundle by this property.

Exercise 7.2.7 Show that if $E \to X$ is a trivial vector bundle and $f\colon Y \to X$ a map, then $f^*E \to Y$ is trivial.

Exercise 7.2.8 Let $E \to Z$ be a vector bundle and let

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

be maps. Show that $((gf)^*E \to X) \cong (f^*(g^*E) \to X)$.

Exercise 7.2.9 Let $\pi\colon E \to X$ be a vector bundle, $\sigma_0\colon X \to E$ the zero section, and $\pi_0\colon E \setminus \sigma_0(X) \to X$ be the restriction of π. Construct a nonvanishing section on $\pi_0^*E \to E \setminus \sigma_0(X)$.

7.3 Whitney sum of bundles

Natural constructions you can perform on vector spaces, pass to to constructions on vector bundles by applying the constructions on each fiber. As an example, we consider the sum $\oplus$. You should check that you believe the constructions, since we plan to be sketchier in future examples.

Definition 7.3.1 If V_1 and V_2 are vector spaces, then $V_1 \oplus V_2 = V_1 \times V_2$ is the vector space of pairs (v_1, v_2) with $v_j \in V_j$. If $f_j\colon V_j \to W_j$ is a linear map $j = 1, 2$, then

$$f_1 \oplus f_2\colon V_1 \oplus V_2 \to W_1 \oplus W_2$$

is the linear map which sends (v_1, v_2) to $(f_1(v_1), f_2(v_2))$.

Note that not all linear maps $V_1 \oplus V_2 \to W_1 \oplus W_2$ are of the form $f_1 \oplus f_2$. For instance, if $V_1 = V_2 = W_1 = W_2 = \mathbf{R}$, then the set of linear maps $\mathbf{R} \oplus \mathbf{R} \to \mathbf{R} \oplus \mathbf{R}$ may be identified (by choosing the standard basis) with the set of 2×2-matrices, whereas the maps of the form $f_1 \oplus f_2$ correspond to the diagonal matrices.

Definition 7.3.2 Let $(\pi_1\colon E_1 \to X, \mathcal{A}_1)$ and $(\pi_2\colon E_2 \to X, \mathcal{A}_2)$ be vector bundles over a common space X. Let

$$E_1 \oplus E_2 = \coprod_{x \in X} \pi_1^{-1}(x) \oplus \pi_2^{-1}(x)$$

and let $\pi_1 \oplus \pi_2\colon E_1 \oplus E_2 \to X$ send all points in the x'th summand to $x \in X$. If $(h_1, U_1) \in \mathcal{A}_1$ and $(h_2, U_2) \in \mathcal{A}_2$ then

$$h_1 \oplus h_2\colon (\pi_1 \oplus \pi_2)^{-1}(U_1 \cap U_2) \to (U_1 \cap U_2) \times (\mathbf{R}^{n_1} \oplus \mathbf{R}^{n_2})$$

is $h_1 \oplus h_2$ on each fiber (i.e., over the point $p \in X$ it is $(h_1)_p \oplus (h_2)_p\colon \pi_1^{-1}(p) \oplus \pi_2^{-1}(p) \to \mathbf{R}^{n_1} \oplus \mathbf{R}^{n_2}$).

This defines a pre-vector bundle, and the associated vector bundle is called the *Whitney sum* of the two vector bundles.

If

$$\begin{array}{ccc} E_j & \xrightarrow{f_j} & E'_j \\ {\scriptstyle \pi_j}\searrow & & \swarrow{\scriptstyle \pi'_j} \\ & X & \end{array}$$

are bundle morphisms over X, then

$$\begin{array}{ccc} E_1 \oplus E_2 & \xrightarrow{f_1 \oplus f_2} & E'_1 \oplus E'_2 \\ {\scriptstyle \pi_1 \oplus \pi_2}\searrow & & \swarrow{\scriptstyle \pi'_1 \oplus \pi'_2} \\ & X & \end{array}$$

is a bundle morphism defined as $f_1 \oplus f_2$ on each fiber.

Exercise 7.3.3 Check that if all bundles and morphisms are smooth, then the Whitney sum is a smooth bundle too, and that $f_1 \oplus f_2$ is a smooth bundle morphism over X.

Note 7.3.4 Although $\oplus = \times$ for vector spaces, we must not mix them for vector bundles, since $\times$ is reserved for another construction: the product of two bundles $E_1 \times E_2 \to X_1 \times X_2$.

As a matter of fact, as a *space* $E_1 \oplus E_2$ is the fiber product $E_1 \times_X E_2$.

Exercise 7.3.5 Let

$$\epsilon = \{(p, \lambda p) \in \mathbf{R}^{n+1} \times \mathbf{R}^{n+1} |\, |p| = 1, \lambda \in \mathbf{R}\}$$

Show that the projection down to S^n defines a trivial bundle.

Definition 7.3.6 A bundle $E \to X$ is called *stably trivial* if there is a trivial bundle $\epsilon \to X$ such that $E \oplus \epsilon \to X$ is trivial.

Exercise 7.3.7 Show that the tangent bundle of the sphere $TS^n \to S^n$ is stably trivial.

Exercise 7.3.8 Show that the sum of two trivial bundles is trivial. Also that the sum of two stably trivial bundles is stably trivial.

Exercise 7.3.9 Given three bundles $\pi_i \colon E_i \to X$, $i = 1, 2, 3$. Show that the set of pairs (f_1, f_2) of bundle morphisms

$$\begin{array}{ccc} E_i & \xrightarrow{f_i} & E_3 \\ {\scriptstyle \pi_i}\searrow & & \swarrow{\scriptstyle \pi_3} \\ & X & \end{array}$$

$(i = 1, 2)$ is in one-to-one correspondence with the set of bundle morphisms

$$\begin{array}{ccc} E_1 \oplus E_2 & \longrightarrow & E_3 \\ {\scriptstyle \pi_1 \oplus \pi_2}\searrow & & \swarrow{\scriptstyle \pi_3} \\ & X & \end{array}$$

7.4 More general linear algebra on bundles

There are many constructions on vector spaces that pass on to bundles. We list a few. The examples 1-4 and 8-9 will be used in the text, and the others are listed for reference, and for use in exercises.

7.4.1 Constructions on vector spaces

1. *The (Whitney) sum.* If V_1 and V_2 are vector spaces, then $V_1 \oplus V_2$ is the vector space of pairs (v_1, v_2) with $v_j \in V_j$. If $f_j \colon V_j \to W_j$ is a linear map $j = 1, 2$, then

$$f_1 \oplus f_2 \colon V_1 \oplus V_2 \to W_1 \oplus W_2$$

is the linear map which sends (v_1, v_2) to $(f_1(v_1), f_2(v_2))$.

2. *The quotient.* If $W \subseteq V$ is a linear subspace we may define the quotient V/W as the set of equivalence classes $V/\sim$ under the equivalence relation that $v \sim v'$ if there is a $w \in W$ such that $v = v' + w$. The equivalence class containing $v \in V$ is written $\bar{v}$. We note that V/W is a vector space with

$$a\bar{v} + b\bar{v'} = \overline{av + bv'}$$

If $f \colon V \to V'$ is a linear map with $f(W) \subseteq W'$ then f defines a linear map

$$\bar{f} \colon V/W \to V'/W'$$

via the formula $\bar{f}(\bar{v}) = \overline{f(v)}$ (check that this makes sense).

3. *The hom-space.* Let V and W be vector spaces, and let

$$\mathrm{Hom}(V, W)$$

be the set of linear maps $f \colon V \to W$. This is a vector space via the formula $(af + bg)(v) = af(v) + bg(v)$. Note that

$$\mathrm{Hom}(\mathbf{R}^m, \mathbf{R}^n) \cong M_{n\times m}(\mathbf{R})$$

Also, if $R \colon V \to V'$ and $S \colon W \to W'$ are linear maps, then we get a linear map

$$\mathrm{Hom}(V', W) \xrightarrow{\mathrm{Hom}(R,S)} \mathrm{Hom}(V, W')$$

by sending $f \colon V' \to W$ to

$$V \xrightarrow{R} V' \xrightarrow{f} W \xrightarrow{S} W'$$

(note that the direction of R is turned around!).

4. *The dual space.* This is a special case of the example above (and was discussed thoroughly in section following Definition 4.3.10): if V is a vector space, then the dual space is the vector space

$$V^* = \mathrm{Hom}(V, \mathbf{R}).$$

5. *The tensor product.* Let V and W be vector spaces. Consider the set of bilinear maps from $V \times W$ to some other vector space V'. The *tensor product*

$$V \otimes W$$

is the vector space codifying this situation in the sense that giving a bilinear map $V \times W \to V'$ is the same as giving a linear map $V \otimes W \to V'$. With this motivation it is possible to write down explicitly what $V \otimes W$ is: as a set it is the set of all finite linear combinations of symbols $v \otimes w$ where $v \in V$ and $w \in W$ subject to the relations

$$\begin{aligned} a(v \otimes w) =&(av) \otimes w = v \otimes (aw) \\ (v_1 + v_2) \otimes w =&v_1 \otimes w + v_2 \otimes w \\ v \otimes (w_1 + w_2) =&v \otimes w_1 + v \otimes w_2 \end{aligned}$$

where $a \in \mathbf{R}$, $v, v_1, v_2 \in V$ and $w, w_1, w_2 \in W$. This is a vector space in the obvious manner, and given linear maps $f\colon V \to V'$ and $g\colon W \to W'$ we get a linear map

$$f \otimes g\colon V \otimes W \to V' \otimes W'$$

by sending $\sum_{i=1}^k v_i \otimes w_i$ to $\sum_{i=1}^k f(v_i) \otimes g(w_i)$ (check that this makes sense!).

Note that

$$\mathbf{R}^m \otimes \mathbf{R}^n \cong \mathbf{R}^{mn}$$

and that there are isomorphisms

$$\mathrm{Hom}(V \otimes W, V') \cong \{\text{bilinear maps } V \times W \to V'\}$$

The bilinear map associated to a linear map $f\colon V \otimes W \to V'$ sends $(v, w) \in V \times W$ to $f(v \otimes w)$. The linear map associated to a bilinear map $g\colon V \times W \to V'$ sends $\sum v_i \otimes w_i \in V \otimes W$ to $\sum g(v_i, w_i)$.

6. *The exterior power.* Let V be a vector space. The kth exterior power $\Lambda^k V$ is defined as the quotient of the k-fold tensor product $V \otimes \cdots \otimes V$ by the subspace generated by the elements $v_1 \otimes v_2 \otimes \cdots \otimes v_k$ where $v_i = v_j$ for some $i \neq j$. The image of $v_1 \otimes v_2 \otimes \cdots \otimes v_k$ in $\Lambda^k V$ is written $v_1 \wedge v_2 \wedge \cdots \wedge v_k$. Note that it follows that $v_1 \wedge v_2 = -v_2 \wedge v_1$ since

$$0 = (v_1 + v_2) \wedge (v_1 + v_2) = v_1 \wedge v_1 + v_1 \wedge v_2 + v_2 \wedge v_1 + v_2 \wedge v_2 = v_1 \wedge v_2 + v_2 \wedge v_1$$

and similarly for more $\wedge$-factors: swapping two entries changes sign.

Note that the dimension of $\Lambda^k \mathbf{R}^n$ is $\binom{n}{k}$. There is a particularly nice isomorphism $\Lambda^n \mathbf{R}^n \to R$ given by the determinant function.

7. *The symmetric power.* Let V be a vector space. The kth symmetric power S^kV is defined as the quotient of the k-fold tensor product $V \otimes \cdots \otimes V$ by the subspace generated by the elements $v_1 \otimes v_2 \otimes \cdots \otimes v_i \otimes \cdots \otimes v_j \otimes \cdots \otimes v_k - v_1 \otimes v_2 \otimes \cdots \otimes v_j \otimes \cdots \otimes v_i \otimes \cdots \otimes v_k$.

8. *Alternating forms.* The space of alternating forms $\mathrm{Alt}^k(V)$ on a vector space V is defined to be $\left(\Lambda^k V\right)^*$, the dual of the exterior power $\Lambda^k V$ 6. That is $\mathrm{Alt}^k(V)$ consists of the multilinear maps
$$V \times \cdots \times V \to \mathbf{R}$$
in k V-variables which are zero on inputs with repeated coordinates.

   The alternating forms on the tangent space is the natural home of the symbols like $dxdydz$ you'll find in elementary multivariable analysis.

9. *Symmetric bilinear forms.* Let V be a vector space. The space of $SB(V)$ symmetric bilinear forms is the space of bilinear maps $f\colon V \times V \to R$ such that $f(v,w) = f(w,v)$. In other words, the space of symmetric bilinear forms is $SB(V) = (S^2V)^*$.

7.4.2 Constructions on vector bundles

When translating these constructions to vector bundles, it is important not only to bear in mind what they do on each individual vector space but also what they do on linear maps. Note that some of the examples "turn the arrows around". The Hom-space in Section 7.4.1(3) is a particular example of this: it "turns the arrows around" in the first variable, but not in the second.

Instead of giving the general procedure for translating such constructions to bundles in general, we do it on the Hom-space which exhibit all the potential difficult points.

Example 7.4.3 Let $(\pi\colon E \to X, \mathcal{B})$ and $(\pi'\colon E' \to X, \mathcal{B}')$ be vector bundles of dimension m and n. We define a pre-vector bundle
$$\mathrm{Hom}(E, E') = \coprod_{p \in X} \mathrm{Hom}(E_p, E'_p) \to X$$
of dimension mn as follows. The projection sends the pth summand to p, and given bundle charts $(h, U) \in \mathcal{B}$ and $(h', U') \in \mathcal{B}'$ we define a bundle chart $(\mathrm{Hom}(h^{-1}, h'), U \cap U')$. On the fiber above $p \in X$,
$$\mathrm{Hom}(h^{-1}, h')_p\colon\ \mathrm{Hom}(E_p, E'_p) \to \mathrm{Hom}(\mathbf{R}^m, \mathbf{R}^n) \cong \mathbf{R}^{mn}$$
is given by sending $f\colon E_p \to E'_p$ to
$$\begin{array}{ccc} \mathbf{R}^m & & \mathbf{R}^n \\ {\scriptstyle h_p^{-1}}\downarrow & & \uparrow{\scriptstyle h'_p} \\ E_p & \xrightarrow{f} & E'_p \end{array}$$

If $(g,V)\in\mathcal{B}$ and $(g',V')\in\mathcal{B}'$ are two other bundle charts, the transition function becomes

$$p \mapsto \operatorname{Hom}(g_p^{-1},g'_p)\left(\operatorname{Hom}(h_p^{-1},h'_p)\right)^{-1} = \operatorname{Hom}(h_pg_p^{-1},g'_p(h'_p)^{-1}),$$

sending $f\colon \mathbf{R}^m\to\mathbf{R}^n$ to

$$\begin{array}{ccc}
\mathbf{R}^m & & \mathbf{R}^n \\
{\scriptstyle g_p^{-1}}\downarrow & & \uparrow{\scriptstyle g'_p} \\
E_p & & E'_p \\
{\scriptstyle h_p}\downarrow & & \uparrow{\scriptstyle (h'_p)^{-1}} \\
\mathbf{R}^m & \xrightarrow{f} & \mathbf{R}^n.
\end{array}$$

That is, if $W=U\cap U'\cap V\cap V'$, then the transition function

$$W \longrightarrow \operatorname{GL}(\operatorname{Hom}(\mathbf{R}^m,\mathbf{R}^n))\cong \operatorname{GL}_{mn}(\mathbf{R})$$

is the composite of

1. the diagonal $W\to W\times W$ sending p to (p,p),
2. the product of the transition functions
$$W\times W\to\operatorname{GL}(\mathbf{R}^m)\times\operatorname{GL}(\mathbf{R}^n),$$
sending (p,q) to $(g_ph_p^{-1},g'_p(h'_p)^{-1})$
3. the map
$$\operatorname{GL}(\mathbf{R}^m)\times\operatorname{GL}(\mathbf{R}^n)\to\operatorname{GL}(\operatorname{Hom}(\mathbf{R}^m,\mathbf{R}^n)),$$
sending (A,B) to $\operatorname{Hom}(A^{-1},B)$.

The first two are continuous or smooth depending on whether the bundles are topological or smooth. The last map $\operatorname{GL}(\mathbf{R}^m)\times\operatorname{GL}(\mathbf{R}^n)\to\operatorname{GL}(\operatorname{Hom}(\mathbf{R}^m,\mathbf{R}^n))$ is smooth ($C\mapsto BCA^{-1}$ is a linear transformation on $\operatorname{Hom}(\mathbf{R}^m,\mathbf{R}^n)$ which depends smoothly on A and B by Cramer's rule (to invert A) and the fact that the algebraic operations are continuous).

In effect, the transition functions of $\operatorname{Hom}(E,E')\to X$ are smooth (resp. continuous) if the transition functions of $E\to X$ and $E'\to X$ are smooth (resp. continuous).

Exercise 7.4.4 Let $E\to X$ and $E'\to X$ be vector bundles. Show that there is a one-to-one correspondence between bundle morphisms

$$\begin{array}{ccc}
E & \xrightarrow{f} & E' \\
& \searrow \quad \swarrow & \\
& X &
\end{array}$$

and sections of $\operatorname{Hom}(E,E')\to X$.

Exercise 7.4.5 Convince yourself that the construction of $\mathrm{Hom}(E, E') \to X$ outlined above really gives a vector bundle, and that if

$$\begin{array}{ccc} E & \xrightarrow{f} & E_1 \\ & \searrow \quad \swarrow & \\ & X & \end{array} \quad , \text{ and } \quad \begin{array}{ccc} E' & \xrightarrow{f'} & E'_1 \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

are bundle morphisms, we get another

$$\begin{array}{ccc} \mathrm{Hom}(E_1, E') & \xrightarrow{\mathrm{Hom}(f,f')} & \mathrm{Hom}(E, E'_1) \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

Exercise 7.4.6 Write out the definition of the *quotient bundle*, and show that if

$$\begin{array}{ccc} E & \xrightarrow{f} & E' \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

is a bundle map, $F \subseteq E$ and $F' \subseteq E'$ are subbundles such that $\mathrm{Im}\{f|_F\} \subseteq F'$, then we get a bundle morphism

$$\begin{array}{ccc} E/F & \xrightarrow{\bar{f}} & E'/F' \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

Example 7.4.7 Given a bundle $E \to X$, the *dual bundle* $E^* \to X$ is important in many situations. If (h, U) is a bundle chart, then we get a bundle chart for the dual bundle

$$(E^*)_U = \coprod_{p\in U} E_p^* \xrightarrow{\coprod (h_p^{-1})^*} \coprod_{p\in U} (\mathbf{R}^k)^* = U \times (\mathbf{R}^k)^*$$

(choose a fixed isomorphism $(\mathbf{R}^k)^* \cong \mathbf{R}^k$).

Exercise 7.4.8 Check that the bundle charts proposed for the dual bundle actually give a bundle atlas, and that this atlas is smooth if the original bundle was smooth.

Exercise 7.4.9 For those who read the section on the cotangent bundle $T^*M \to M$ associated with a smooth n-manifold M: prove that the maps of Proposition 4.3.12

$$\alpha_p \colon T_p^*M \to (TM)^*, \quad d\phi \mapsto \{[\gamma] \mapsto ()'(0)\}$$

induces an isomorphism from the cotangent bundle to the dual of the tangent bundle.

Given Exercise 7.4.9, the ones who have *not* studied the cotangent bundle is free to define it in the future as the dual of the tangent bundle. Recall that the elements of the cotangent bundle are called 1-*forms*.

Exercise 7.4.10 Given a bundle $E \to X$, write out the definition of the associated *symmetric bilinear forms bundle* $SB(E) \to X$

Example 7.4.11 An alternating *k-form* (or just k-form) is an element in $\mathrm{Alt}^k(TM)$ (see 7.4.1(**??**)). These are the main object of study when doing analysis of manifolds (integrations etc.).

Exercise 7.4.12 Write out the definition of the bundle of alternating k-forms, and if you are still not bored stiff, do some more examples. If you are really industrious, find out on what level of generality these ideas really work, and prove it there.

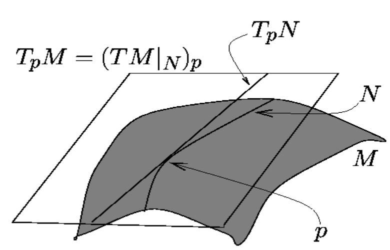
Exercise 7.4.13 Let $L \to M$ be a line bundle (one-dimensional vector bundle). Show that the tensor product $L \otimes L \to M$ is also a line bundle and that all the transition functions in the associated bundle atlas on $L \otimes L \to M$ have values in the positive real numbers $\{t \in \mathbf{R} \mid t > 0\} \subset \mathrm{GL}_1(\mathbf{R})$.

7.5 Normal bundles

We will later discuss Riemannian structures and more generally fiber metrics over smooth manifolds. This will give us the opportunity to discuss inner products, and in particular questions pertaining to orthogonality, in the fibers. That such structures exists over smooth manifolds is an artifact of the smooth category, in which local smooth data occasionally can be patched together to global smooth structures.

However, there is a formulation of these phenomena which does not depend on inner products, but rather uses quotient bundles.

Definition 7.5.1 Let $N \subseteq M$ be a smooth submanifold. The *normal bundle* $\perp N \to N$ is defined as the quotient bundle $(TM|_N)/TN \to N$ (see exercise 7.4.6).



In a submanifold $N \subseteq M$ the tangent bundle of N is naturally a subbundle of the tangent bundle of M restricted to N, and the normal bundle is the quotient on each fiber, or isomorphically in each fiber: the normal space

More generally, if $f\colon N \to M$ is an imbedding, we define the normal bundle $\perp^f N \to N$ to be the bundle $(f^*TM)/TN \to N$.

It turns out that there is an important consequence of transversality pertaining to normal bundles:

Theorem 7.5.2 *Assume $f\colon N \to M$ is transverse to a k-codimensional submanifold $L \subseteq M$ and that $f(N) \cap L \neq \emptyset$. Then $f^{-1}(L) \subseteq N$ is a k-codimensional submanifold and there is an isomorphism*

$$\begin{array}{ccc} \perp f^{-1}(L) & \xrightarrow{\cong} & f^*(\perp L) \\ & \searrow \quad \swarrow & \\ & f^{-1}(L) & \end{array}$$

Proof: The first part is simply Theorem 5.5.3. For the statement about normal bundles, consider the diagram

$$\begin{array}{ccc} T(f^{-1}(L)) & \longrightarrow & TL \\ \downarrow & & \downarrow \\ TN|_{f^{-1}(L)} & \longrightarrow & TM|_L \\ \downarrow & & \downarrow \\ (TN|_{f^{-1}(L)})/T(f^{-1}(L)) = \perp f^{-1}(L) & \longrightarrow & (TM|_L)/TL = \perp L \end{array}$$

Transversality gives that the map from $TN|_{f^{-1}(L)}$ to $(TM|_L)/TL$ is surjective on every fiber, and so – for dimensional reasons – $\perp f^{-1}(L) \to \perp L$ is an isomorphism on every fiber. This then implies that $\perp f^{-1}(L) \to f^*(\perp L)$ must be an isomorphism by lemma 6.3.12.

∎

Corollary 7.5.3 *Consider a smooth map $f\colon N \to M$ and a regular value $q \in M$. Then the normal bundle $\perp f^{-1}(q) \to f^{-1}(q)$ is trivial.*

Note 7.5.4 In particular, this shows that the normal bundle of $S^n \subseteq \mathbf{R}^{n+1}$ is trivial. Also it shows that the normal bundle of $O(n) \subseteq M_n(\mathbf{R})$ is trivial, and all the other manifolds we constructed in Chapter 5 as the inverse image of regular values.

In Exercise 7.3.7 we showed that the tangent bundle of S^n is stably trivial, and an analysis of that proof gives an isomorphism between $T\mathbf{R}^{n+1}|_{S^n}$ and $TS^n \oplus \perp S^n$. This "splitting" is a general phenomenon and is a result of the flexibility of the smooth category alluded to at the beginning of this section. We will return to such issues in Section 9.3. When we discuss Riemannian structures.

7.6 Orientations[1]

The space of alternating forms $\mathrm{Alt}^k(V)$ on a vector space V is defined to be $\left(\Lambda^k V\right)^* = \mathrm{Hom}(\Lambda^k V, \mathbf{R})$ (see 7.4.1(**??**)), or alternatively, $\mathrm{Alt}^k(V)$ consists of the multilinear maps

$$V \times \cdots \times V \to \mathbf{R}$$

in k V-variables which are zero on inputs with repeated coordinates.

In particular, if $V = \mathbf{R}^k$ we have the *determinant function*

$$\det \in \mathrm{Alt}^k(\mathbf{R}^k)$$

given by sending $v_1 \wedge \cdots \wedge v_k$ to the determinant of the $k \times k$-matrix $[v_1 \dots v_k]$ you get by considering v_i as the ith column.

In fact, $\det\colon \Lambda^k \mathbf{R}^k \to \mathbf{R}$ is an isomorphism.

Exercise 7.6.1 Check that the determinant actually is an alternating form and an isomorphism.

Definition 7.6.2 An *orientation* on a k-dimensional vector space V is an equivalence class of bases on V, where $(v_1, \dots v_k)$ and $(w_1, \dots w_k)$ are equivalent if $v_1 \wedge \cdots \wedge v_k$ and $w_1 \wedge \cdots \wedge w_k$ differ by a positive scalar. The equivalence class, or *orientation class*, represented by a basis $(v_1, \dots v_k)$ is written $[v_1, \dots v_k]$.

Note 7.6.3 That two bases $v_1 \wedge \cdots \wedge v_k$ and $w_1 \wedge \cdots \wedge w_k$ in $\mathbf{R}^k$ define the same orientation class can be formulated by means of the determinant:

$$\det(v_1 \dots v_k)/\det(w_1 \dots w_k) > 0.$$

As a matter of fact, this formula is valid for any k-dimensional vector space if you choose an isomorphism $V \to \mathbf{R}^k$ (the choice turns out not to matter).

Note 7.6.4 On a vector space V there are exactly two orientations. For instance, on $\mathbf{R}^k$ the two orientations are $[e_1, \dots, e_k]$ and $[-e_1, e_2, \dots, e_k] = [e_2, e_1, e_3 \dots, e_k]$.

Note 7.6.5 An *isomorphism* of vector spaces $f\colon V \to W$ sends an orientation $\mathcal{O} = [v_1, \dots, v_k]$ to the orientation $f\mathcal{O} = [f(v_1), \dots, f(v_k)]$.

Definition 7.6.6 An *oriented vector space* is a vector space together with a chosen orientation. An isomorphism of oriented vector spaces either *preserve* or *reverse* the orientation.

[1] This section is not used anywhere else and may safely be skipped.

Definition 7.6.7 Let $E \to X$ be a vector bundle. An *orientation* on $E \to X$ is a family $\mathcal{O} = \{\mathcal{O}_p\}_{p \in X}$ such that $\mathcal{O}_p$ is an orientation on the fiber E_p, and such that around any point $p \in X$ there is a bundle chart (h, U) such that for all $q \in U$ the isomorphism

$$h_q \colon E_q \to \mathbf{R}^k$$

sends $\mathcal{O}_q$ to $h_p \mathcal{O}_p$.

Definition 7.6.8 A vector bundle is *orientable* if it can be equipped with an orientation.

Example 7.6.9 A trivial bundle is orientable.

Example 7.6.10 Not all bundles are orientable, for instance, the tautological line bundle $\eta_1 \to S^1$ of example 6.1.2 is not orientable: start choosing orientations, run around the circle, and have a problem.

Definition 7.6.11 A manifold M is *orientable* if the tangent bundle is orientable.

7.7 The generalized Gauss map[2]

The importance of the Grassmann manifolds to bundle theory stems from the fact that in a certain precise sense the bundles over a given manifold M is classified by a set of equivalence classes (called homotopy classes) from M into Grassmann manifolds. This is really cool, but unfortunately beyond the scope of our current investigations. We offer a string of exercises as a vague orientation into interesting stuff we can't pursue to the depths it deserves.

Exercise 7.7.1 Recall the Grassmann manifold G_n^k of all k-dimensional linear subspaces of $\mathbf{R}^n$ defined in Example 3.3.13. Define the *canonical k-plane bundle* over the Grassmann manifold

$$\gamma_n^k \to G_n^k$$

by setting

$$\gamma_n^k = \{(E, v) | E \in G_n^k, v \in E\}$$

(note that $\gamma_n^1 = \eta_n \to \mathbf{R}\mathrm{P}^n = G_n^1$). (hint: use the charts in Example 3.3.13, and let

$$h_g \colon \pi^{-1}(U_g) \to U_g \times \mathbf{E}_g$$

send (E, v) to $(E, \mathrm{pr}_{E_g} v)$).

Note 7.7.2 The Grassmann manifolds are important because there is a neat way to describe vector bundles as maps from manifolds into Grassmann manifolds, which makes their global study much more transparent. We won't have the occasion to study this phenomenon, but we include the following example.

[2]This section is not used anywhere else and may safely be skipped.

Exercise 7.7.3 Let $M \subseteq \mathbf{R}^n$ be a smooth k-dimensional manifold. Then we define the *generalized Gauss map*

$$\begin{array}{ccc} TM & \longrightarrow & \gamma_n^k \\ \downarrow & & \downarrow \\ M & \longrightarrow & G_n^k \end{array}$$

by sending $p \in M$ to $T_pM \in G_n^k$ (we consider T_pM as a subspace of $\mathbf{R}^n$ under the standard identification $T_p\mathbf{R}^n = \mathbf{R}^n$), and $[\gamma] \in TM$ to $(T_{\gamma(0)}M, [\gamma])$. Check that it is a bundle morphism and displays the tangent bundle of M as the induced bundle of the tautological k-plane bundle under $M \to G_n^k$.

Chapter 8

Integrability

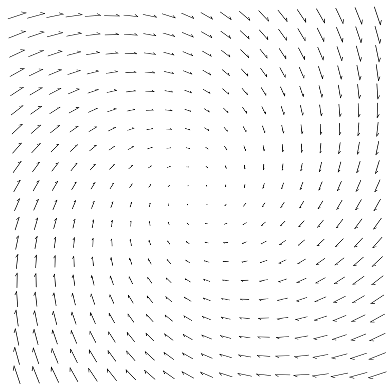
Many applications lead to situations where you end up with a differential equation on some manifold. Solving these are no easier than it is in the flat case. However, the language of tangent bundles can occasionally make it clearer what is going on, and where the messy formulas actually live.

Furthermore, the existence of solutions to differential equations are essential to show that the deformations we intuitively are inclined to perform on manifolds, actually make sense smoothly. This is reflected in that the flows we construct are smooth.

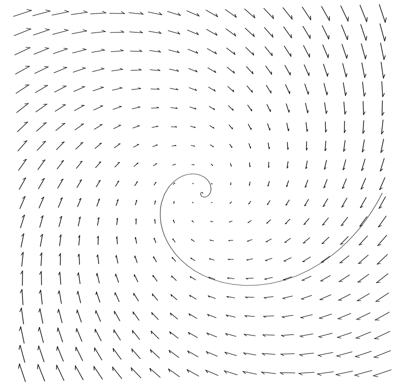
Example 8.0.4 In the flat case, we are used to draw "flow charts". E.g., given a first order differential equation

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = f(x(t), y(t))$$

we associate to each point (x, y) the vector $f(x, y)$. In this fashion a first order ordinary differential equation may be identified with a vector field. Each vector would be the velocity vector of a solution to the equation passing through the point (x, y). If f is smooth, the vectors will depend smoothly on the point (it is a smooth vector field), and the picture would resemble a flow of a liquid, where each vector would represent the velocity of the particle at the given point. The paths of each particle would be solutions of the differential equation, and assembling all these solutions, we could talk about the flow of the liquid.



The vector field resulting from a system of ordinary differential equations (here: a predator-prey system with a stable equilibrium).



A solution to the differential equation is a curve whose derivative equals the corresponding vector field.

8.1 Flows and velocity fields

If we are to talk about differential equations on manifolds, the confusion of where the velocity fields live (as opposed to the solutions) has to be sorted out. The place of velocity vectors is the tangent bundle, and a differential equation can be represented by a vector field, that is a section in the tangent bundle $TM \to M$, and its solutions by a "flow":

Definition 8.1.1 Let M be a smooth manifold. A *(global) flow* is a smooth map

$$\Phi\colon \mathbf{R} \times M \to M$$

such that for all $p \in M$ and $s, t \in \mathbf{R}$

- $\Phi(0, p) = p$
- $\Phi(s, \Phi(t, p)) = \Phi(s + t, p)$

We are going to show that on a compact manifold there is a one-to-one correspondence between vector fields and global flows. In other words, first order ordinary differential equations have unique solutions on compact manifolds. This statement is true also for non-compact manifolds, but then we can't expect the flows to be defined on all of $\mathbf{R} \times M$ anymore, and we have to talk about *local flows*. We will return to this later, but first we will familiarize ourselves with global flows.

Definition 8.1.2 Let $M = \mathbf{R}$, let

$$L\colon \mathbf{R} \times \mathbf{R} \to \mathbf{R}$$

be the flow given by $L(s, t) = s + t$.

Example 8.1.3 Consider the map

$$\Phi\colon \mathbf{R}\times\mathbf{R}^2 \to \mathbf{R}^2$$

given by

$$\left(t, \begin{bmatrix} p \\ q \end{bmatrix}\right) \mapsto e^{-t/2} \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix}$$

Exercise 8.1.4 Check that this actually **is** a global flow!

For fixed p and q this is the solution to the initial value problem

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1/2 & 1 \\ -1 & -1/2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \qquad \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix}$$

whose corresponding vector field was used in the figures in example 8.0.4.

A flow is a very structured representation of a vector field:

Definition 8.1.5 Let Φ be a flow on the smooth manifold M. The *velocity field* of Φ is defined to be the vector field

$$\vec{\Phi}\colon M \to TM$$

where $\vec{\Phi}(p) = [t \mapsto \Phi(t,p)]$.

The surprise is that *every* vector field is the velocity field of a flow (see the integrability theorems 8.2.2 and 8.4.1)

Example 8.1.6 Consider the global flow of 8.1.2. Its velocity field

$$\vec{L}\colon \mathbf{R} \to T\mathbf{R}$$

is given by $s \mapsto [L_s]$ where L_s is the curve $t \mapsto L_s(t) = L(s,t) = s+t$. Under the diffeomorphism

$$T\mathbf{R} \to \mathbf{R}\times\mathbf{R}, \qquad [\omega] \mapsto (\omega(0), \omega'(0))$$

we see that $\vec{L}$ is the non-vanishing vector field corresponding to picking out 1 in every fiber.

Example 8.1.7 Consider the flow Φ in 8.1.3. Under the diffeomorphism

$$T\mathbf{R}^2 \to \mathbf{R}^2\times\mathbf{R}^2, \qquad [\omega] \mapsto (\omega(0), \omega'(0))$$

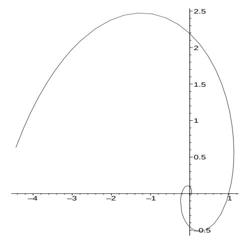
the velocity field $\vec{\Phi}\colon \mathbf{R}^2 \to T\mathbf{R}^2$ corresponds to

$$\mathbf{R}^2 \to \mathbf{R}^2\times\mathbf{R}^2, \qquad \begin{bmatrix} p \\ q \end{bmatrix} \mapsto \left(\begin{bmatrix} p \\ q \end{bmatrix}, \begin{bmatrix} -1/2 & 1 \\ -1 & -1/2 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} \right)$$

Definition 8.1.8 Let Φ be a global flow on a smooth manifold M, and $p \in M$. The curve

$$\mathbf{R} \to M, \qquad t \mapsto \Phi(t,p)$$

is called the *flow line* of Φ through p. The image $\Phi(\mathbf{R}, p)$ of this curve is called the *orbit* of p.



The orbit of the point $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ of the flow of example 8.1.3.

The orbits split the manifold into disjoint sets:

Exercise 8.1.9 Let $\Phi\colon \mathbf{R} \times M \to M$ be a flow on a smooth manifold M. Then

$$p \sim q \Leftrightarrow \text{ there is a } t \text{ such that } \Phi(t,p) = q$$

defines an equivalence relation on M. hence, every point in M lies in a unique orbit: no orbits intersect.

Example 8.1.10 The flow line through 0 of the flow L of definition 8.1.2 is the identity on $\mathbf{R}$. The only orbit is $\mathbf{R}$.

More interesting: the flow lines of the flow of example 8.1.3 are of two types: the constant flow line at the origin, and the spiraling flow lines filling out the rest of the space.

Exercise 8.1.11 Given $w = re^{i\theta}$, let $\phi\colon \mathbf{R} \times \mathbf{C} \to \mathbf{C}$ be the flow $\Phi(t,z) = r^t e^{it\theta}$. Describe the flow lines when (r,θ) is i) $(1,0)$, ii) $(1,\pi/2)$ and iii) $(1/2,0)$.

Note 8.1.12 (Contains important notation, and a reinterpretation of the term "global flow"). Writing $\Phi_t(p) = \Phi(t,p)$ we get another way of expressing a flow. To begin with we have

- $\Phi_0 =$ identity
- $\Phi_{s+t} = \Phi_s \circ \Phi_t$

We see that for each t the map Φ_t is a diffeomorphism (with inverse Φ_{-t}) from M to M. The assignment $t \mapsto \Phi_t$ sends sum to composition of diffeomorphisms and so defines a "group homomorphism"

$$\mathbf{R} \to \mathrm{Diff}(M)$$

from the additive group of real numbers to the group of diffeomorphism (under composition) on M.

We have already used this notation in connection with the flow L of Definition 8.1.2: $L_s(t) = L(s,t) = s + t$.

Lemma 8.1.13 *Let Φ be a global flow on M and $s \in \mathbf{R}$. Then the diagram*

$$\begin{array}{ccc} TM & \xrightarrow[\cong]{T\Phi_s} & TM \\ {\scriptstyle\vec{\Phi}}\uparrow & & \uparrow{\scriptstyle\vec{\Phi}} \\ M & \xrightarrow[\cong]{\Phi_s} & M \end{array}$$

commutes.

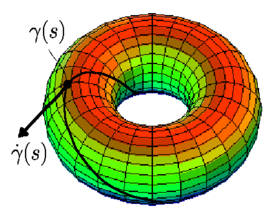
Proof: One of the composites sends $q \in M$ to $[t \mapsto \Phi(s, \Phi(t,q))]$ and the other sends $q \in M$ to $[t \mapsto \Phi(t, \Phi(s,q))]$. ∎

Definition 8.1.14 Let

$$\gamma\colon \mathbf{R} \to M$$

be a smooth curve on the manifold M. The *velocity vector* $\dot\gamma(s) \in T_{\gamma(s)}M$ of γ at $s \in \mathbf{R}$ is defined as the tangent vector

$$\dot\gamma(s) = T\gamma\,\vec{L}(s) = [\gamma L_s] = [t \mapsto \gamma(s+t)]$$



The velocity vector $\dot\gamma(s)$ of the curve γ at s lives in $T_{\gamma(s)}M$.

Note 8.1.15 The curve γL_s is given by $t \mapsto \gamma(s+t)$ and $(L_s)'(0) = 1$. So, if (x, U) is a chart with $\gamma(s) \in U$ we get that $\dot\gamma(s) \in T_{\gamma(s)}M$ corresponds to $(x\gamma L_s)'(0) = (x\gamma)'(s)$ under the isomorphism $T_{\gamma(s)} \cong \mathcal{R}^m$ induced by x, explaining the term "velocity vector".

The following diagram can serve as a reminder for the construction and will be used later:

$$\begin{array}{ccc} T\mathbf{R} & \xrightarrow{T\gamma} & TM \\ {\scriptstyle\vec{L}}\uparrow & \overset{\dot\gamma}{\nearrow} & \downarrow \\ \mathbf{R} & \xrightarrow{\gamma} & M \end{array}$$

The velocity field and the flow are intimately connected, and the relation can be expressed in many ways. Here are some:

Lemma 8.1.16 *Let Φ be a flow on the smooth manifold M, $p \in M$. Let ϕ_p be the flow line through p given by $\phi_p(s) = \Phi(s,p)$. Then the diagrams*

$$\begin{array}{ccc} \mathbf{R} & \xrightarrow{\phi_p} & M \\ {\scriptstyle L_s}\big\downarrow{\scriptstyle\cong} & & {\scriptstyle \Phi_s}\big\downarrow{\scriptstyle\cong} \\ \mathbf{R} & \xrightarrow{\phi_p} & M \end{array}$$

and

$$\begin{array}{ccc} T\mathbf{R} & \xrightarrow{T\phi_p} & TM \\ {\scriptstyle \vec{L}}\big\uparrow & \overset{\dot{\phi}_p}{\nearrow} & \big\uparrow{\scriptstyle \vec{\Phi}} \\ \mathbf{R} & \xrightarrow[\phi_p]{} & M \end{array}$$

commutes. For future reference, we have for all $s \in \mathbf{R}$ that

$$\begin{aligned} \dot{\phi}_p(s) &= \vec{\Phi}(\phi_p(s)) \\ &= T\Phi_s[\phi_p] \end{aligned}$$

Proof: All these claims are variations of the fact that

$$\Phi(s+t,q) = \Phi(s,\Phi(t,q)) = \Phi(t,\Phi(s,q))$$

∎

Proposition 8.1.17 *Let Φ be a flow on a smooth manifold M, and $p \in M$. If*

$$\gamma\colon \mathbf{R} \to M$$

is the flow line of Φ through p (i.e., $\gamma(t) = \Phi(t,p)$) then either

- *γ is an injective immersion*
- *γ is a periodic immersion (i.e., there is a $T > 0$ such that $\gamma(s) = \gamma(t)$ if and only if there is an integer k such that $s = t + kT$), or*
- *γ is constant.*

Proof: Note that since $T\Phi_s\dot{\gamma}(0) = T\Phi_s[\gamma] = \dot{\gamma}(s)$ and Φ_s is a diffeomorphism $\dot{\gamma}(s)$ is either zero for all s or never zero at all.

If $\dot{\gamma}(s) = 0$ for all s, this means that γ is constant since if (x,U) is a chart with $\gamma(s_0) \in U$ we get that $(x\gamma)'(s) = 0$ for all s close to s_0, hence $x\gamma(s)$ is constant for all s close to s_0 giving that γ is constant.

If $\dot{\gamma}(s) = T\gamma[L_s]$ is never zero we get that $T\gamma$ is injective (since $[L_s] \neq 0 \in T_s\mathbf{R} \cong \mathbf{R}$), and so γ is an immersion. Either it is injective, or there are two numbers $s > s'$ such that $\gamma(s) = \gamma(s')$. This means that

$$\begin{aligned} p = \gamma(0) = \Phi(0, p) &= \Phi(s - s, p) = \Phi(s, \Phi(-s, p)) \\ &= \Phi(s, \gamma(-s)) = \Phi(s, \gamma(-s')) = \Phi(s - s', p) \\ &= \gamma(s - s') \end{aligned}$$

Since γ is continuous $\gamma^{-1}(p) \subseteq \mathbf{R}$ is closed and not empty (it contains 0 and $s - s' > 0$ among others). As γ is an immersion it is a local imbedding, so there is an $\epsilon > 0$ such that

$$(-\epsilon, \epsilon) \cap \gamma^{-1}(0) = \{0\}$$

Hence

$$S = \{t > 0 | p = \gamma(t)\} = \{t \geq \epsilon | p = \gamma(t)\}$$

is closed and bounded below. This means that there is a smallest positive number T such that $\gamma(0) = \gamma(T)$. Clearly $\gamma(t) = \gamma(t + kT)$ for all $t \in \mathbf{R}$ and any integer k.

On the other hand we get that $\gamma(t) = \gamma(t')$ **only if** $t - t' = kT$ for some integer k. For if $(k-1)T < t - t' < kT$, then $\gamma(0) = \gamma(kT - (t - t'))$ with $0 < kT - (t - t') < T$ contradicting the minimality of T. ∎

Note 8.1.18 In the case the flow line is a periodic immersion we note that γ must factor through an imbedding $f\colon S^1 \to M$ with $f(e^{it}) = \gamma(tT/2\pi)$. That it is an imbedding follows since it is an injective immersion from a compact space.

In the case of an injective immersion there is no reason to believe that it is an imbedding.

Example 8.1.19 The flow lines in example 8.1.3 are either constant (the one at the origin) or injective immersions (all the others). The flow

$$\Phi\colon \mathbf{R} \times \mathbf{R}^2 \to \mathbf{R}^2, \qquad \left(t, \begin{bmatrix} x \\ y \end{bmatrix}\right) \mapsto \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

has periodic flow lines (except at the origin).

Exercise 8.1.20 Display an injective immersion $f\colon \mathbf{R} \to \mathbf{R}^2$ which is not the flow line of a flow.

8.2 Integrability: compact case

A difference between vector fields and flows is that vector fields can obviously be added, which makes it easy to custom-build vector fields for a particular purpose. That this is true also for flows is far from obvious, but is one of the nice consequences of the integrability theorem 8.2.2 below. The importance of the theorem is that we may custom-build flows for particular purposes simply by specifying their velocity fields.

Going from flows to vector fields is simple: just take the velocity field. The other way is harder, and relies on the fact that first order ordinary differential equations have unique solutions.

Definition 8.2.1 Let $X\colon M \to TM$ be a vector field. A *solution curve* is a curve $\gamma\colon J \to M$ (where J is an open interval) such that $\dot\gamma(t) = X(\gamma(t))$ for all $t \in J$.

We note that the equation

$$\dot{\phi}_p(s) = \vec{\Phi}(\phi_p(s))$$

of lemma 8.1.16 says that "the flow lines are solution curves to the velocity field". This is the key to proof of the integrability theorem:

Theorem 8.2.2 *Let M be a smooth compact manifold. Then the velocity field gives a natural bijection between the sets*

$$\{\textit{global flows on } M\} \leftrightarrows \{\textit{vector fields on } M\}$$

Before we prove the Integrability theorem, recall a the basic existence and uniqueness theorem for ordinary differential equations. For a nice proof giving just continuity see Spivak's book [12] chapter 5. For a complete proof, see e.g., one of the analysis books of Lang.

Theorem 8.2.3 *Let $f\colon U \to \mathbf{R}^n$ be a smooth map where $U \subseteq \mathbf{R}^n$ is an open subset and $p \in U$.*

- *(Existence of solution) There is a neighborhood $p \in V \subseteq U$ of p, a neighborhood J of $0 \in \mathbf{R}$ and a smooth map*

  $$\Phi\colon J \times V \to U$$

  such that

  - $\Phi(0, q) = q$ *for all* $q \in V$ *and*
  - $\frac{\partial}{\partial t}\Phi(t, q) = f(\Phi(t, q))$ *for all* $(t, q) \in J \times V$.

- *(Uniqueness of solution) If γ_i are smooth curves in U satisfying $\gamma_1(0) = \gamma_2(0) = p$ and*

  $$\gamma_i'(t) = f(\gamma(t)), \qquad i = 1, 2$$

  then $\gamma_1 = \gamma_2$ where they both are defined.

Notice that uniqueness gives that the Φ satisfies the condition $\Phi(s+t, q) = \Phi(s, \Phi(t, q))$ for small s and t. In more detail, for sufficiently small, but fixed t let $\gamma_1(s) = \Phi(s + t, q)$ and $_2(s) = \Phi(s, \Phi(t, q))$. Then $\gamma_1(0) = \gamma_2(0)$ and $\gamma_k'(s) = f(\gamma_k(s))$ for $k = 1, 2$, so $\gamma_1 = \gamma_2$
Proof: To prove the Integrability theorem 8.2.2 we construct an inverse to the function given by the velocity field. That is, given a vector field X on M we will produce a unique flow Φ whose vector field is $\vec{\Phi} = X$.

Our problem hinges on a local question which we refer away to analysis (although the proof contains nice topological stuff). Given a point $p \in M$ choose a chart $x = x_p \colon U \to U'$ with $p \in U$. Let $X^p \colon U' \to TU'$ be section given by the composite $X^p = Tx_p \circ X|_U \circ x_p^{-1}$ (i.e., so that the diagram

$$\begin{array}{ccc} TU' & \xleftarrow[\cong]{Tx_p} & TU \\ {\scriptstyle X^p}\uparrow & & \uparrow{\scriptstyle X|_U} \\ U' & \xleftarrow[\cong]{x_p} & U \end{array}$$

commutes) and define $f = f_p \colon U' \to \mathbf{R}^n$ as the composite

$$U' \xrightarrow{X^p} TU' \xrightarrow[\cong]{[\nu]\mapsto(\nu(0),\nu'(0))} U' \times \mathbf{R}^n \xrightarrow{\mathrm{pr}_{\mathbf{R}^n}} \mathbf{R}^n.$$

Then we get that claiming that a curve $\gamma \colon J \to U$ is a solution curve to X, i.e., it satisfies the equation

$$\dot{\gamma}(t) = X(\gamma(t)),$$

is equivalent to claiming that

$$(x\gamma)'(t) = f(x\gamma(t)).$$

By the existence and uniqueness theorem for first order differential equations 8.2.3 there is a neighborhood $J_p \times V'_p$ around $(0, x(p)) \in \mathbf{R} \times U'$ for which there exists a smooth map

$$\Psi = \Psi_p \colon J_p \times V'_p \to U'_p$$

such that

- $\Psi(0, q) = q$ for all $q \in V'_p$ and
- $\frac{\partial}{\partial t}\Psi(t, q) = f(\Psi(t, q))$ for all $(t, q) \in J_p \times V'_p$.

and furthermore for each $q \in V'_p$ the curve $\Psi(-, q) \colon J_p \to U'_p$ is unique with respect to this property.

The set of open sets of the form $x_p^{-1}V'_p$ is an open cover of M, and hence we may choose a finite subcover. Let J be the intersection of the J_p's corresponding to this finite cover. Since it is a finite intersection J contains an open interval $(-\epsilon, \epsilon)$ around 0.

What happens to f_p when we vary p? Let $q \in U = U_p \cap U_{p'}$ and consider the commutative diagram

$$\begin{array}{ccccc} x_pU & \xleftarrow{x_p} & U & \xrightarrow{x_{p'}} & x_{p'}U \\ \downarrow{\scriptstyle X^p} & & \downarrow{\scriptstyle X} & & \downarrow{\scriptstyle X^{p'}} \\ T(x_pU) & \xleftarrow{Tx_p} & T(U) & \xrightarrow{Tx_{p'}} & T(x_{p'}U) \\ \downarrow{\scriptstyle\cong} & & & & \downarrow{\scriptstyle\cong} \\ x_pU \times \mathbf{R}^n & & \xrightarrow{(r,v)\mapsto(x_{p'}x_p^{-1}(r),D(x_{p'}x_p^{-1})(r)\cdot v)} & & x_{p'}U \times \mathbf{R}^n \end{array}$$

(restrictions suppressed). Hence, from the definition of f_p, we get that $f_{p'}x_{p'}x_p^{-1}(r) = D(x_{p'}x_p^{-1})(r)\cdot f_p(r)$ for $r\in x_pU$. So, if we set $P(t,q) = x_{p'}x_p^{-1}\Psi_p(t, x_px_{p'}^{-1}(q))$, the flat chain rule gives that $\frac{\partial}{\partial t}P(t,q) = f_{p'}(P(t,q))$. Since in addition $P(0,q)=0$, we get that both P and $\Psi_{p'}$ are solutions to the initial value problem (with $f_{p'}$), and so by uniqueness of solution $P=\Psi_{p'}$ on the domain of definition, or in other words

$$x_p^{-1}\Psi_p(t, x_p(q)) = x_{p'}^{-1}\Psi_{p'}(t, x_{p'}(q)), \qquad q\in U, \quad t\in J.$$

Hence we may define a smooth map

$$\tilde\Phi\colon J\times M\to M$$

by $\tilde\Phi(t,q) = x_p^{-1}\Psi_p(t,x_pq)$ if $q\in x_p^{-1}V_p'$.

Note that the uniqueness of solution also gives that

$$\tilde\Phi(t,\tilde\Phi(s,q)) = \tilde\Phi(s+t,q)$$

for $|s|$, $|t|$ and $|s+t|$ less than ϵ.

But this also means that we may extend the domain of definition to get a map

$$\Phi\colon \mathbf{R}\times M\to M$$

since for any $t\in\mathbf{R}$ there is a natural number k such that $|t/k|<\epsilon$, and we simply define $\Phi(t,q)$ as $\tilde\Phi_{t/k}$ applied k times to q. ∎

The condition that M was compact was crucial to this proof. A similar statement is true for noncompact manifolds, and we will return to that statement later.

Exercise 8.2.4 Given two flows Φ_N and Φ_S on the sphere S^2. Why does there exist a flow Φ with $\Phi(t,q)=\Phi_N(t,q)$ for small t and q close to the North pole, and $\Phi(t,q)=\Phi_S(t,q)$ for small t and q close to the South pole?

Exercise 8.2.5 Construct vector fields on the torus such that the solution curves are all either

- imbedded circles, or
- dense immersions.

Exercise 8.2.6 Let $O(n)$ be the orthogonal group, and recall from exercise 5.4.10 the isomorphism between the tangent bundle of $O(n)$ and the projection on the first factor

$$E = \{(g,A)\in O(n)\times M_n(\mathbf{R})|A^t = -g^tAg^t\}\to O(n).$$

Choose a skew matrix $A\in M_n(\mathbf{R})$ (i.e., such that $A^t=-A$), and consider the vector field $X\colon O(n)\to TO(n)$ induced by

$$\begin{aligned} O(n) &\to E\\ g &\mapsto (g, gA)\end{aligned}$$

Show that the flow associated to X is given by $\Phi(s,g) = ge^{sA}$ where the exponential is defined as usual by $e^B = \sum_{n=0}^\infty \frac{B^n}{n!}$.

8.3 Local flows

We now make the necessary modifications for the non-compact case.

On manifolds that are not compact, the concept of a (global) flow is not the correct one. This can be seen by considering a global flow Φ on some manifold M and restricting it to some open submanifold U. Then some of the flow lines may leave U after finite time. To get a "flow" Φ_U on U we must then accept that Φ_U is only defined on some open subset of $\mathbf{R} \times U$ containing $\{0\} \times U$.

Also, if we jump ahead a bit, and believe that flows should correspond to general solutions to first order ordinary differential equations (that is, vector fields), you may consider the differential equation

$$y' = y^2, \qquad y(0) = y_0$$

on $M = \mathbf{R}$ (the corresponding vector field is $\mathbf{R} \to T\mathbf{R}$ given by $s \mapsto [t \mapsto s + s^2 t]$). Here the solution is of the type

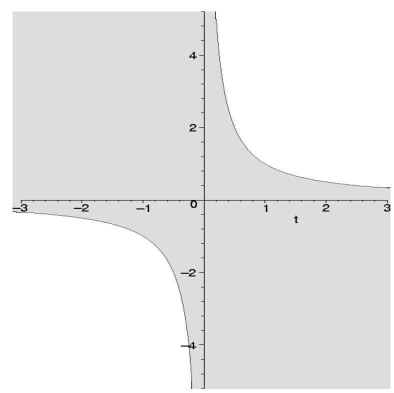
$$y(t) = \begin{cases} \frac{1}{1/y_0 - t} & \text{if } y_0 \neq 0 \\ 0 & \text{if } y_0 = 0 \end{cases}$$

and the domain of the "flow"

$$\Phi(t,p) = \begin{cases} \frac{1}{1/p - t} & \text{if } p \neq 0 \\ 0 & \text{if } p = 0 \end{cases}$$

is

$$A = \{(t,p) \in \mathbf{R} \times \mathbf{R} \,|\, pt < 1\}$$



The domain A of the "flow". It contains an open neighborhood around $\{0\} \times M$

Definition 8.3.1 Let M be a smooth manifold. A *local flow* is a smooth map

$$\Phi\colon A \to M$$

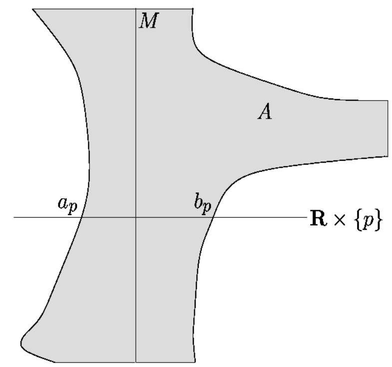
where $A \subseteq \mathbf{R} \times M$ is open and contains $\{0\} \times M$, such that for each $p \in M$

$$J_p \times \{p\} = A \cap (\mathbf{R} \times \{p\})$$

is connected (so that J_p is an open interval containing 0) and such that

- $\Phi(0, p) = p$
- $\Phi(s, \Phi(t, p)) = \Phi(s + t, p)$

for all $p \in M$ such that (t, p), $(s + t, p)$ and $(s, \Phi(t, p))$ are in A.
For each $p \in M$ we define $-\infty \le a_p < 0 < b_p \le \infty$ by $J_p = (a_p, b_p)$.



The "horizontal slice" J_p of the domain of a local flow is an open interval containing zero.

Definition 8.3.2 A local flow $\Phi\colon A \to M$ is *maximal* if there is no local flow $\Psi\colon B \to M$ such that $A \subsetneq B$ and $\Psi|_A = \Phi$.

Note that maximal flows that are not global must leave any given compact subset within finite time:

Lemma 8.3.3 *Let $K \subset M$ be a compact subset of a smooth manifold M, and let Φ be a maximal local flow on M such that $b_p < \infty$. Then there is an $\epsilon > 0$ such that $\Phi(t, p) \notin K$ for $t > b_p - \epsilon$.*

Proof: Since K is compact there is an $\epsilon > 0$ such that

$$[-\epsilon, \epsilon] \times K \subseteq A \cap (\mathbf{R} \times K)$$

If $\Phi(t, p) \in K$ for $t < T$ where $T > b_p - \epsilon$ then we would have that Φ could be extended to $T + \epsilon > b_p$ by setting

$$\Phi(t, p) = \Phi(\epsilon, \Phi(t - \epsilon, p))$$

for all $T \le t < T + \epsilon$. ∎

Note 8.3.4 The definitions of the *velocity field*

$$\vec{\Phi}\colon M \to TM$$

(the tangent vector $\vec{\Phi}(p) = [t \mapsto \Phi(t, p)]$ only depends on the values of the curve in a small neighborhood of 0), the *flow lines*

$$\Phi(-, p)\colon J_p \to M, \qquad t \mapsto \Phi(t, p)$$

and the *orbits*

$$\Phi(J_p, p) \subseteq M$$

make just as much sense for a local flow Φ.

However, we can't talk about "the diffeomorphism Φ_t" since there may be $p \in M$ such that $(t, p) \notin A$, and so Φ_t is not defined on all of M.

Example 8.3.5 Check that the proposed flow

$$\Phi(t,p) = \begin{cases} \frac{1}{1/p-t} & \text{if } p \neq 0 \\ 0 & \text{if } p = 0 \end{cases}$$

is a local flow with velocity field $\vec{\Phi}\colon \mathbf{R} \to T\mathbf{R}$ given by $s \mapsto [t \mapsto \Phi(t,s)]$ (which under the standard trivialization

$$T\mathbf{R} \xrightarrow{[\omega]\mapsto(\omega(0),\omega'(0))} \mathbf{R}\times\mathbf{R}$$

correspond to $s \mapsto (s, s^2)$ – and so $\vec{\Phi}(s) = [t \mapsto \Phi(t,s)] = [t \mapsto s + s^2 t]$) with domain

$$A = \{(t,p) \in \mathbf{R}\times\mathbf{R} | pt < 1\}$$

and so $a_p = 1/p$ for $p < 0$ and $a_p = -\infty$ for $p \geq 0$. Note that Φ_t only makes sense for $t = 0$.

8.4 Integrability

Theorem 8.4.1 *Let M be a smooth manifold. Then the velocity field gives a natural bijection between the sets*

$$\{\text{maximal local flows on } M\} \leftrightarrows \{\text{vector fields on } M\}$$

Proof: The essential idea is the same as in the compact case, but we have to worry a bit more about the domain of definition of our flow. The local solution to the ordinary differential equation means that we have unique maximal solution curves

$$\phi_p\colon J_p \to M$$

for all p. This also means that the curves $t \mapsto \phi_p(s+t)$ and $t \mapsto \phi_{\phi_p(s)}(t)$ agree (both are solution curves through $\phi_p(s)$), and we define

$$\Phi\colon A \to M$$

by setting

$$A = \bigcup_{p\in M} J_p \times \{p\}, \text{ and } \Phi(t,p) = \phi_p(t)$$

The only questions are whether A is open and Φ is smooth. But this again follows from the local existence theorems: around any point in A there is a neighborhood on which Φ corresponds to the unique local solution (see [3] page 82 and 83 for more details).

∎

Note 8.4.2 Some readers may worry about the fact that we do not consider "time dependent" differential equations, but by a simple trick as in [12] page 226, these are covered by our present considerations.

Exercise 8.4.3 Find a nonvanishing vector field on $\mathbf{R}$ whose solution curves are only defined on finite intervals.

8.5 Second order differential equations[1]

We give a brief and inadequate sketch of second order differential equations. This is important for a wide variety of applications, in particular for the theory of geodesics which will be briefly discussed in section 9.2.7 after partitions of unity has been covered.

For a smooth manifold M let $\pi_M \colon TM \to M$ be the tangent bundle (just need a decoration on π to show its dependence on M).

Definition 8.5.1 A *second order differential equation* on a smooth manifold M is a smooth map

$$\xi \colon TM \to TTM$$

such that

$$\begin{array}{ccccc} & & TTM & & \\ & \swarrow^{T\pi_M} & \uparrow{\scriptstyle \xi} & \searrow^{\pi_{TM}} & \\ TM & \xleftarrow{=} & TM & \xrightarrow{=} & TM \end{array}$$

commutes.

Note 8.5.2 The $\pi_{TM}\xi = id_{TM}$ just says that ξ is a vector field on TM, it is the other relation $(T\pi_M)\xi = id_{TM}$ which is crucial.

Exercise 8.5.3 The flat case: reference sheet. Make sense of the following remarks, write down your interpretation and keep it for reference.

A curve in TM is an equivalence class of "surfaces" in M, for if $\beta \colon J \to TM$ then to each $t \in J$ we have that $\beta(t)$ must be an equivalence class of curves, $\beta(t) = [\omega(t)]$ and we

[1]This section is not referred to later in the book except in the example on the exponential map 9.2.7

may think of $t \mapsto \{s \mapsto \omega(t)(s)\}$ as a surface if we let s and t move simultaneously. If $U \subseteq \mathbf{R}^n$ is open, then we have the trivializations

$$TU \xrightarrow[\cong]{[\omega]\mapsto(\omega(0),\omega'(0))} U \times \mathbf{R}^n$$

with inverse $(p, v) \mapsto [t \mapsto p + tv]$ (the directional derivative at p in the vth direction) and

$$\begin{aligned} T(TU) & \xrightarrow[\cong]{[\beta]\mapsto(\beta(0),\beta'(0))} T(U) \times (\mathbf{R}^n \times \mathbf{R}^n) \\ & \xrightarrow[\cong]{(\beta(0),\beta'(0))\mapsto((\omega(0,0),D_2\omega(0,0)),(D_1\omega(0,0),D_2D_1\omega(0,0)))} (U \times \mathbf{R}^n) \times (\mathbf{R}^n \times \mathbf{R}^n) \end{aligned}$$

with inverse $(p, v_1, v_2, v_3) \mapsto [t \mapsto [s \mapsto \omega(t)(s)]]$ with

$$\omega(t)(s) = p + sv_1 + tv_2 + stv_3$$

Hence if $\gamma\colon J \to U$ is a curve, then $\dot{\gamma}$ correspond to the curve

$$J \xrightarrow{t\mapsto(\gamma(t),\gamma'(t))} U \times \mathbf{R}^n$$

and if $\beta\colon J \to TU$ corresponds to $t \mapsto (x(t), v(t))$ then $\dot{\beta}$ corresponds to

$$J \xrightarrow{t\mapsto(x(t),v(t),x'(t),v'(t))} U \times \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^n$$

This means that $\ddot{\gamma} = \dot{\dot{\gamma}}$ corresponds to

$$J \xrightarrow{t\mapsto(\gamma(t),\gamma'(t),\gamma'(t),\gamma''(t))} U \times \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^n$$

Exercise 8.5.4 Show that our definition of a second order differential equation corresponds to the usual notion of a second order differential equation in the case $M = \mathbf{R}^n$.

Definition 8.5.5 Given a second order differential equation

$$\xi\colon TM \to TTM$$

A curve $\gamma\colon J \to M$ is called a *solution curve* for ξ on M if $\dot{\gamma}$ is a solution curve to ξ "on TM".

Note 8.5.6 Spelling this out we have that

$$\ddot{\gamma}(t) = \xi(\dot{\gamma}(t))$$

for all $t \in J$. Note the bijection

$$\{\text{solution curves } \beta\colon J \to TM\} \leftrightarrows \{\text{solution curves } \gamma\colon J \to M\}, \qquad \begin{matrix} \dot{\gamma} \leftarrow \gamma \\ \beta \mapsto \pi_M\beta \end{matrix}$$

Chapter 9

Local phenomena that go global

In this chapter we define partitions of unity. They are smooth devices making it possible to patch together some types of local information into global information. They come in the form of “bump functions” such that around any given point there are only finitely many of them that are nonzero, and such that the sum of their values is 1.

This can be applied for instance to patch together the nice local structure of a manifold to an imbedding into an Euclidean space (we do it in the compact case, see Theorem 9.2.6), construct sensible metrics on the tangent spaces (so-called Riemannian metrics 9.3), and in general to construct smooth functions with desirable properties. We will also use it to prove Ehresmann’s fibration theorem 9.5.6.

9.1 Refinements of covers

In order to patch local phenomena together, we will be using that manifolds can be covered by chart domains in a very orderly fashion, by means of what we will call “good” atlases. This section gives the technical details needed.

If $0 < r$ let $E^n(r) = \{x \in \mathbf{R}^n | |x| < r\}$ be the open n-dimensional ball of radius r centered at the origin.

Lemma 9.1.1 *Let M be an n-dimensional manifold. Then there is a countable atlas $\mathcal{A}$ such that $x(U) = E^n(3)$ for all $(x, U) \in \mathcal{A}$ and such that*

$$\bigcup_{(x,U)\in\mathcal{A}} x^{-1}(E^n(1)) = M$$

If M is smooth all charts may be chosen to be smooth.

Proof: Let $\mathcal{B}$ be a countable basis for the topology on M. For every $p \in M$ there is a chart (x, U) with $x(p) = 0$ and $x(U) = E^n(3)$. The fact that $\mathcal{B}$ is a basis for the topology gives that there is a $V \in \mathcal{B}$ with

$$p \in V \subseteq x^{-1}(E^n(1))$$

For each such $V \in \mathcal{B}$ choose just **one** such chart (x, U) with $x(U) = E^n(3)$ and

$$x^{-1}(0) \in V \subseteq x^{-1}(E^n(1))$$

The set of these charts is the desired countable $\mathcal{A}$.

If M were smooth we just append "smooth" in front of every "chart" in the proof above. ∎

Lemma 9.1.2 *Let M be a manifold. Then there is a sequence $A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots$ of compact subsets of M such that for every $i \geq 1$ the compact subset A_i is contained in the interior of A_{i+1} and such that $\bigcup_i A_i = M$*

Proof: Let $\{(x_i, U_i)\}_{i=1,\dots}$ be the countable atlas of the lemma above, and let

$$A_k = \bigcup_{i=1}^{k} x_i^{-1}(\overline{E^n(2 - 1/k)})$$

∎

Definition 9.1.3 Let $\mathcal{U}$ be an open cover of a space X. We say that another cover $\mathcal{V}$ is a *refinement* of $\mathcal{U}$ if every member of $\mathcal{V}$ is contained in a member of $\mathcal{U}$.

Definition 9.1.4 Let $\mathcal{U}$ be an open cover of a space X. We say that $\mathcal{U}$ is *locally finite* if each $p \in X$ has a neighborhood which intersects only finitely many sets in $\mathcal{U}$.

Definition 9.1.5 Let M be a manifold and let $\mathcal{U}$ be an open cover of M. A *good atlas* subordinate to $\mathcal{U}$ is a countable atlas $\mathcal{A}$ on M such that

1) the cover $\{V\}_{(x,V)\in\mathcal{A}}$ is a locally finite refinement of $\mathcal{U}$,

2) $x(V) = E^n(3)$ for each $(x, V) \in \mathcal{A}$ and

3) $\bigcup_{(x,V)\in\mathcal{A}} x^{-1}(E^n(1)) = M$.

Theorem 9.1.6 *Let M be a manifold and let $\mathcal{U}$ be an open cover of M. Then there exists a good atlas $\mathcal{A}$ subordinate to $\mathcal{U}$. If M is smooth, then $\mathcal{A}$ may be chosen smooth too.*

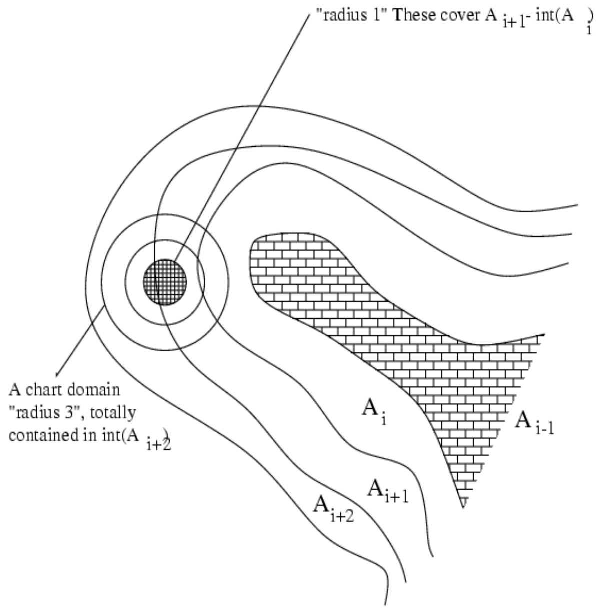
Proof: The remark about the smooth situation will follow by the same proof. Choose a sequence

$$A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots$$

of compact subsets of M such that for every $i \geq 1$ the compact subset A_i is contained in the interior of A_{i+1} and such that $\bigcup_i A_i = M$.
For every point

$$p \in A_{i+1} - int(A_i)$$

choose a $U_p \in \mathcal{U}$ with $p \in U_p$ and choose a chart (y_p, W_p) such that $p \in W_p$ and $y_p(p) = 0$.



The positioning of the charts

Since $int(A_{i+2}) - A_{i-1}$, $y_p(W_p)$ and U_p are open there is an $\epsilon_p > 0$ such that

$$E^n(\epsilon_p) \subseteq y_p(W_p), \qquad y_p^{-1}(E^n(\epsilon_p)) \subseteq (int(A_{i+2}) - A_{i-1}) \cap U_p$$

Let $V_p = y_p^{-1}(E^n(\epsilon_p))$ and

$$x_p = \frac{3}{\epsilon_p} y_p|_{V_p} : V_p \to E^n(3)$$

Then $\{x_p^{-1}(E^n(1))\}_p$ covers the compact set $A_{i+1} - int(A_i)$, and we may choose a finite set of points $p_1, \dots, p_k$ such that

$$\{x_{p_j}^{-1}(E^n(1))\}_{j=1,\dots,k}$$

still cover $A_{i+1} - int(A_i)$.

Letting $\mathcal{A}$ consist of the (x_{p_j}, V_{p_j}) as i and j vary we have proven the theorem. ■

9.2 Partition of unity

Recall that if $f\colon X \to \mathbf{R}$ is a continuous function, then the support of f is $\text{supp}(f) = \overline{f^{-1}(0)}$.

Definition 9.2.1 A family of continuous function

$$\phi_\alpha : X \to [0,1]$$

is called a *partition of unity* if

the collection of subsets $\{ \{p \in X | \phi_\alpha(p) \neq 0\} \}$ is a locally finite (9.1.4) open cover of X,

for all $p \in X$ the (finite) sum $\sum_\alpha \phi_\alpha(p) = 1$.

The partition of unity is said to be *subordinate* to a cover $\mathcal{U}$ of X if in addition

for every ϕ_α there is a $U \in \mathcal{U}$ with $\mathrm{supp}(\phi_\alpha) \subseteq U$.

Given a space that is not too big and complicated (for instance if it is a compact manifold) it may not be surprising that we can build a partition of unity on it. What is more surprising is that on smooth manifolds we can build **smooth** partitions of unity (that is, all the ϕ_α's are smooth).

In order to this we need smooth bump functions, in particular, we will use the smooth bump function

$$\gamma_{(1,1)} : \mathbf{R}^n \to \mathbf{R}$$

defined in Lemma 4.1.16 which has the property that $\gamma_{(1,1)}(p) = 1$ for all p with $|p| \leq 1$ and $\gamma_{(1,1)}(p) = 0$ for all $p \in \mathbf{R}^n$ with $|p| \geq 2$.

Theorem 9.2.2 *Let M be a differentiable manifold, and let $\mathcal{U}$ be a cover of M. Then there is a smooth partition of unity of M subordinate to $\mathcal{U}$.*

Proof: To the good atlas $\mathcal{A} = \{(x_i, V_i)\}$ subordinate to $\mathcal{U}$ constructed in theorem 9.1.6 we may assign functions $\{\psi_i\}$ as follows

$$\psi_i(q) = \begin{cases} \gamma_{(1,1)}(x_i(q)) & \text{for } q \in V_i = x_i^{-1}(E^n(3)) \\ 0 & \text{otherwise.} \end{cases}$$

The function ψ_i has support $x_i^{-1}(E^n(2))$ and is obviously smooth. Since $\{V_i\}$ is locally finite, around any point $p \in M$ there is an open set such there are only finitely many ψ_i's with nonzero values, and hence the expression

$$\sigma(p) = \sum_i \psi_i(p)$$

defines a smooth function $M \to \mathbf{R}$ with everywhere positive values. The partition of unity is then defined by

$$\phi_i(p) = \psi_i(p)/\sigma(p)$$

■

Exercise 9.2.3 Let M be a smooth manifold, $f\colon M \to \mathbf{R}$ a continuous function and ϵ a positive real number. Then there is a smooth $g\colon M \to \mathbf{R}$ such that for all $p \in M$

$$|f(p) - g(p)| < \epsilon.$$

You may use without proof Weierstrass' theorem which says the following: Suppose $f\colon K \to \mathbf{R}$ is a continuous function with $K \subseteq \mathbf{R}^m$ compact. For every $\epsilon > 0$, there exists a polynomial g such that for all $x \in K$ we have $|f(x) - p(x)| < \epsilon$.

Exercise 9.2.4 Let $L \to M$ be a line bundle (one-dimensional smooth vector bundle over the smooth manifold M). Show that $L \otimes L \to M$ is trivial.

9.2.5 Imbeddings in Euclidean space

As an application of partitions of unity, we will prove the easy version of Whitney's imbedding theorem. The hard version states that any manifold may be imbedded in the Euclidean space of the double dimension. As a matter of fact, with the appropriate topology, the space of imbeddings $M \to \mathbf{R}^{2n+1}$ is dense in the space of all smooth maps $M \to \mathbf{R}^{2n+1}$ (see e.g. [5, 2.1.0], or the more refined version [5, 2.2.13]). We will only prove:

Theorem 9.2.6 *Let M be a compact smooth manifold. Then there is an imbedding $M \to \mathbf{R}^N$ for some N.*

Proof: Assume M has dimension m. Choose a finite good atlas

$$\mathcal{A} = \{x_i, V_i\}_{i=1,\dots,r}$$

Define $\psi_i\colon M \to \mathbf{R}$ and $k_i\colon M \to \mathbf{R}^m$ by

$$\psi_i(p) = \begin{cases} \gamma_{(1,1)}(x_i(p)) & \text{for } p \in V_i \\ 0 & \text{otherwise} \end{cases}$$

$$k_i(p) = \begin{cases} \psi_i(p) \cdot x_i(p) & \text{for } p \in V_i \\ 0 & \text{otherwise} \end{cases}$$

Consider the map

$$\begin{aligned} f\colon M &\to \prod_{i=1}^{r} \mathbf{R}^m \times \prod_{i=1}^{r} \mathbf{R} \\ p &\mapsto ((k_1(p), \dots, k_r(p)), (\psi_1(p), \dots, \psi_r(p))) \end{aligned}$$

We shall prove that this is an imbedding by showing that it is an immersion inducing a homeomorphism onto its image.

Firstly, f is an immersion, because for every $p \in M$ there is a j such that $T_p k_j$ has rank m.

Secondly, assume $f(p) = f(q)$ for two points $p, q \in M$. Assume $p \in x_j^{-1}(E^m(1))$. Then we must have that q is also in $x_j^{-1}(E^m(1))$ (since $\psi_j(p) = \psi_j(q)$). But then we have that $k_j(p) = x_j(p)$ is equal to $k_j(q) = x_j(q)$, and hence $p = q$ since x_j is a bijection.

Since M is compact, f is injective (and so $M \to f(M)$ is bijective) and $\mathbf{R}^N$ Hausdorff, $M \to f(M)$ is a homeomorphism by theorem 10.7.8. ∎

Techniques like this are used to construct imbeddings. However, occasionally it is important to know when imbeddings are not possible, and then these techniques are of no use. For instance, why can't we imbed $\mathbf{R}\mathrm{P}^2$ in $\mathbf{R}^3$? Proving this directly is probably very hard. For such problems algebraic topology is needed.

9.2.7 The exponential map[1]

This section gives a quick definition of the exponential map from the tangent space to the manifold.

Exercise 9.2.8 (The existence of "geodesics") The differential equation $T\mathbf{R}^n \to TT\mathbf{R}^n$ corresponding to the map

$$\mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^n, \qquad (x, v) \mapsto (x, v, v, 0)$$

has solution curves given by the straight line $t \mapsto x + tv$ (a straight curve has zero second derivative). Prove that you may glue together these straight lines by means of charts and partitions of unity to get a second order differential equation (see Definition 8.5.1)

$$\xi\colon TM \to TTM$$

with the property that

$$\begin{array}{ccc} TTM & \xrightarrow{Ts} & TTM \\ {\scriptstyle s\xi}\uparrow & & \uparrow{\scriptstyle \xi} \\ TM & \xrightarrow{s} & TM \end{array}$$

for all $s \in \mathbf{R}$ where $s\colon TM \to TM$ is multiplication by s in each fiber.

The significance of the diagram in the previous exercise on geodesics is that "you may speed up (by a factor s) along a geodesic, but the orbit won't change".

Exercise 9.2.9 (Definition of the exponential map). Given a second order differential equation $\xi\colon TM \to TTM$ as in exercise 9.2.8, consider the corresponding local flow $\Phi\colon A \to TM$, define the open neighborhood of the zero section

$$\mathcal{T} = \{[\omega] \in TM \,|1 \in A \cap (\mathbf{R} \times \{[\omega]\})\}$$

[1]Geodesics and the exponential map is important for many applications, but is not used later on in these notes, so this section may be skipped without disrupting the flow.

and you may define the *exponential map*

$$\exp\colon \mathcal{T} \to M$$

by sending $[\omega] \in TM$ to $\pi_M \Phi(1, [\omega])$.

Essentially exp says: for a tangent vector $[\omega] \in TM$ start out in $\omega(0) \in M$ in the direction on $\omega'(0)$ and travel a unit in time along the corresponding geodesic.

The exponential map depends on on ξ. Alternatively we could have given a definition of exp using a choice of a Riemannian metric, which would be more in line with the usual treatment in differential geometry.

9.3 Riemannian structures

In differential geometry one works with more highly structured manifolds than in differential topology. In particular, all manifolds should come equipped with metrics on the tangent spaces which vary smoothly from point to point. This is what is called a Riemannian manifold, and is crucial to many applications.

In this section we will show that all smooth manifolds have a structure of a Riemannian manifold. However, the reader should notice that there is a huge difference between merely saying that a given manifold has *some* Riemannian structure, and actually working with manifolds with a *chosen* Riemannian structure.

Recall from 7.4.1(9) that if V is a vector space, then $SB(V)$ is the vector space of all symmetric bilinear forms $g\colon V \times V \to \mathbf{R}$, i.e., functions g such that $g(v, w) = g(w, v)$ and which are linear in each variable.

Recall that this lifts to the level of bundles: if $\pi\colon E \to X$ is a bundle, we get an associated symmetric bilinear forms bundle $SB(\pi)\colon SB(E) \to X$ (see Exercise 7.4.10). A more involved way of saying this is $SB(E) = (S^2E)^* \to X$ in the language of 7.4.1(4) and 7.4.1(7).

Definition 9.3.1 Let V be a vector space. An *inner product* is a symmetric bilinear form $g \in SB(V)$ which is *positive definite*, i.e., we have that $g(v, v) \geq 0$ for all $v \in V$ and $g(v, v) = 0$ only if $v = 0$.

Example 9.3.2 So, if A is a symmetric $n \times n$-matrix, then $\langle v, w\rangle_A = v^t A w$ defines an inner product $\langle, \rangle_A \in SB(\mathbf{R}^n)$. In particular, if A is the identity matrix we get the standard inner product on $\mathbf{R}^n$.

Definition 9.3.3 A *fiber metric* on a vector bundle $\pi\colon E \to X$ is a section $g\colon X \to SB(E)$ on the associated symmetric bilinear forms bundle, such that for every $p \in X$ the associated symmetric bilinear form $g_p\colon E_p \times E_p \to \mathbf{R}$ is positive definite. The fiber metric is smooth if $E \to X$ and the section g are smooth.

A fiber metric is often called a *Riemannian metric*, although many authors reserve this notion for a fiber metric on the tangent bundle of a smooth manifold.

Definition 9.3.4 A *Riemannian manifold* is a smooth manifold with a smooth fiber metric on the tangent bundle.

Theorem 9.3.5 *Let M be a differentiable manifold and let $E \to M$ be an n-dimensional smooth bundle with bundle atlas $\mathcal{B}$. Then there is a fiber metric on $E \to M$*

Proof: Choose a good atlas $\mathcal{A} = \{(x_i, V_i)\}_{i \in \mathbf{N}}$ subordinate to $\{U | (h, U) \in \mathcal{B}\}$ and a smooth partition of unity $\{\phi_i : M \to \mathbf{R}\}$ with $\operatorname{supp}(\phi_i) \subset V_i$ as given by the proof of theorem 9.2.2.

Since for any of the V_i's, there is a bundle chart (h, U) in $\mathcal{B}$ such that $V_i \subseteq U$, the bundle restricted to any V_i is trivial. Hence we may choose a fiber metric, i.e., a section

$$\sigma_i : V_i \to SB(E)|_{V_i}$$

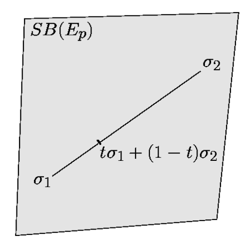
such that σ_i is (bilinear, symmetric and) positive definite on every fiber. For instance we may let $\sigma_i(p) \in SB(E_p)$ be the positive definite symmetric bilinear map

$$E_p \times E_p \xrightarrow{h_p \times h_p} \mathbf{R}^n \times \mathbf{R}^n \xrightarrow{(v,w) \mapsto v \cdot w = v^T w} \mathbf{R}$$

Let $g_i : M \to SB(E)$ be defined by

$$g_i(p) = \begin{cases} \phi_i(p)\sigma_i(p) & \text{if } p \in V_i \\ 0 & \text{otherwise} \end{cases}$$

and let $g : M \to SB(E)$ be given as the sum $g(p) = \sum_i g_i(p)$. The property "positive definite" is *convex*, i.e., if σ_1 and σ_2 are two positive definite forms on a vector space and $t \in [0, 1]$, then $t\sigma_1 + (1 - t)\sigma_2$ is also positive definite (since $t\sigma_1(v, v) + (1 - t)\sigma_2(v, v)$ must obviously be nonnegative, and can be zero only if $\sigma_1(v, v) = \sigma_2(v, v) = 0$). By induction we get that $g(p)$ is positive definite since all the $\sigma_i(p)$'s were positive definite.



In the space of symmetric bilinear forms, all the points on the straight line between two positive definite forms are positive definite.

∎

Corollary 9.3.6 *All smooth manifolds possess Riemannian metrics.*

Note 9.3.7 A fiber metric on a bundle $E \to M$ gives rise to an isomorphism between $E \to M$ and its dual bundle $E^* \to M$ as follows. If V is a finite dimensional vector space and $\langle -, - \rangle$ is an inner product, we define an isomorphism $V \to V^*$ by sending $v \in V$ to the linear map $\langle v, - \rangle : V \to \mathbf{R}$ sending $w \in V$ to $\langle v, w \rangle$. The bilinearity of the inner product ensures that the map $V \to V^*$ is linear and well defined. The nondegenerate property of the inner product is equivalent to the injectivity of $V \to V^*$, and since any injective linear map of vector spaces of equal finite dimension is an isomorphism, $V \to V^*$ is an isomorphism.

Now, given a vector bundle $E \to M$ with a chosen fiber metric g we define

$$\begin{array}{ccc} E & \xrightarrow[\cong]{g_*} & E^* \\ & \searrow \quad \swarrow & \\ & M & \end{array}$$

by using the inner product $g_p\colon E_p \otimes E_p \to \mathbf{R}$ to define $g_*\colon E_p \cong (E_p)^*$ with $g_*(v) = g_p(v, -)$. Since g_p varies smoothly in p this assembles to the desired isomorphism of bundles (exercise: check that this actually works).

Ultimately, we see that a Riemannian manifold comes with an isomorphism

$$TM \xrightarrow[\cong]{g_*} > (TM)^*$$

between the tangent and the cotangent bundles.

Example 9.3.8 In applications the fiber metric is often given by physical considerations. Consider a particle moving on a manifold M, defining, a smooth curve $\gamma\colon \mathbf{R} \to M$. At each point the velocity of the curve defines a tangent vector, and so the curve lifts to a curve on the tangent space $\gamma\colon \mathbf{R} \to TM$ (see 8.1.14 for careful definitions). The dynamics is determined by the energy, and the connection between the metric and the energy is that the norm associated with the metric g at a given point is twice the kinetic energy T. The "generalized" or "conjugate momentum" in mechanics is then nothing but g_* of the velocity, living in the cotangent bundle T^*M which is often referred to as the "phase space".

For instance, if $M = \mathbf{R}^n$ (with the identity chart) and the mass of the particle is m, the kinetic energy of a particle moving with velocity $v \in T_pM$ at $p \in M$ is $\frac{1}{2}m|v|^2$, and so the appropriate metric is m times the usual Euclidean metric $g_p(v, w) = m \cdot \langle v, w\rangle$ (and in particular independent of p) and the generalized momentum is $m\langle v, -\rangle \in T_p^*M$.

9.4 Normal bundles

With the knowledge that the existence of fiber metrics is not such an uncommon state of affairs, we offer a new take on normal bundles. Normal bundles in general were introduced in Section 7.5.

Definition 9.4.1 Given a bundle $\pi\colon E \to X$ with a chosen fiber metric g and a subbundle $F \subseteq E$, then we define the *normal bundle with respect to g* of $F \subseteq E$ to be the subset

$$F^\perp = \coprod_{p \in X} F_p^\perp$$

given by taking the orthogonal complement of $F_p \in E_p$ (relative to the metric $g(p)$).

Lemma 9.4.2 *Given a bundle $\pi\colon E \to X$ with a fiber metric g and a subbundle $F \subset E$, then*

1. *the normal bundle $F^\perp \subseteq E$ is a subbundle*
2. *the composite*
$$F^\perp \subseteq E \to E/F$$
is an isomorphism of bundles over X.
3. *the bundle morphism $F \oplus F^\perp \to E$ induced by the inclusions is an isomorphism over X.*

Proof: Choose a bundle chart (h, U) such that

$$h(F|_U) = U \times (\mathbf{R}^k \times \{0\}) \subseteq U \times \mathbf{R}^n$$

Let $v_j(p) = h^{-1}(p, e_j) \in E_p$ for $p \in U$. Then $(v_1(p), \ldots, v_n(p))$ is a basis for E_p whereas $(v_1(p), \ldots, v_k(p))$ is a basis for F_p. We can then perform the Gram-Schmidt process with respect to the metric $g(p)$ to transform these bases to orthogonal bases $(v_1'(p), \ldots, v_n'(p))$ for E_p, $(v_1'(p), \ldots, v_k'(p))$ for F_p and $(v_{k+1}'(p), \ldots, v_n'(p))$ for $F_p^\perp$.

We can hence define a new bundle chart (h', U) given by

$$\begin{aligned} h' \colon E|_U &\to U \times \mathbf{R}^n \\ \sum_{i=1}^n a_i v_i'(p) &\mapsto (p, (a_1, \ldots, a_n)) \end{aligned}$$

(it is a bundle chart since the metric varies continuously with p, and so the basis change from $\{v_i\}$ to $\{v_i'\}$ is not only an isomorphism on each fiber, but a homeomorphism) which gives $F^\perp|_U$ as $U \times (\{0\} \times \mathbf{R}^{n-k})$.

For the second claim, observe that the dimension of $F^\perp$ is equal to the dimension of E/F, and so the claim follows if the map $F^\perp \subseteq E \to E/F$ is injective on every fiber, but this is true since $F_p \cap F_p^\perp = \{0\}$.

For the last claim, note that the map in question induces a linear map on every fiber which is an isomorphism, and hence by lemma 6.3.12 the map is an isomorphism of bundles.

∎

Note 9.4.3 Note that the bundle chart h' produced in the lemma above is orthogonal on every fiber (i.e., $g(x)(e, e') = (h'(e)) \cdot (h'(e'))$). This means that all the transition functions between maps produced in this fashion would map to the orthogonal group $O(n) \subseteq \mathrm{GL}_n(\mathbf{R})$.

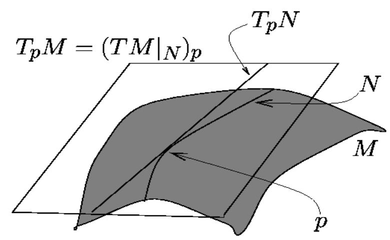
In conclusion:

Corollary 9.4.4 *Every bundle over a (smooth) manifold possesses an atlas whose transition functions maps to the orthogonal group.*

Note 9.4.5 This is an example of the notion of *reduction of the structure group*, in this case from $\mathrm{GL}_n(\mathbf{R})$ to $O(n)$. Another example is gotten when it is possible to choose an atlas whose transition functions land in the special linear group $\mathrm{SL}_n(\mathbf{R})$: then the bundle is *orientable*, see Section 7.6. If all transition functions are the identity matrix, then the bundle is trivial. If $n = 2m$, then $\mathrm{GL}_m(\mathbf{C}) \subseteq \mathrm{GL}_n(\mathbf{R})$, and a reduction to $\mathrm{GL}_m(\mathbf{C})$ is called a *complex structure* on the bundle.

Generally, a reduction of the structure group provides important information about the bundle. In particular, a reduction of the structure group for the tangent bundle provides important information about the manifold.

Definition 9.4.6 Let $N \subseteq M$ be a smooth submanifold. The *normal bundle* $\perp N \to N$ is defined as the quotient bundle $(TM|_N)/TN \to N$ (see Exercise 7.4.6).



In a submanifold $N \subseteq M$ the tangent bundle of N is naturally a subbundle of the tangent bundle of M restricted to N, and the normal bundle is the quotient on each fiber, or isomorphically in each fiber: the normal space

More generally, if $f\colon N \to M$ is an imbedding, we define the normal bundle $\perp^f N \to N$ to be the bundle $(f^*TM)/TN \to N$.

Note 9.4.7 With respect to some Riemannian structure on M, we note that the normal bundle $\perp N \to N$ of $N \subseteq M$ is isomorphic to $(TN)^\perp \to N$.

Exercise 9.4.8 Let $M \subseteq \mathbf{R}^n$ be a smooth submanifold. Prove that $\perp M \oplus TM \to M$ is trivial.

Exercise 9.4.9 Consider S^n as a smooth submanifold of $\mathbf{R}^{n+1}$ in the usual way. Prove that the normal bundle is trivial.

Exercise 9.4.10 Let M be a smooth manifold, and consider M as a submanifold by imbedding it as the diagonal in $M \times M$ (i.e., as the set $\{(p,p) \in M \times M\}$: show that it is a smooth submanifold). Prove that the normal bundle $\perp M \to M$ is isomorphic to $TM \to M$.

Exercise 9.4.11 The tautological line bundle $\eta_n \to \mathbf{R}\mathrm{P}^n$ is a subbundle of the trivial bundle $\mathrm{pr}\colon \mathbf{R}\mathrm{P}^n \times \mathbf{R}^{n+1} \to \mathbf{R}\mathrm{P}^n$:

$$\eta_n = \{(L, v) \in \mathbf{R}\mathrm{P}^n \times \mathbf{R}^{n+1} \,|\, v \in L\} \subseteq \mathbf{R}\mathrm{P}^n \times \mathbf{R}^{n+1} = \epsilon.$$

Let

$$\eta_1^\perp = \{\{(L,v) \in \mathbf{R}\mathrm{P}^n \times \mathbf{R}^{n+1} \,|\, v \in L^\perp\} \subseteq \mathbf{R}\mathrm{P}^n \times \mathbf{R}^{n+1} = \epsilon$$

be the orthogonal complement.

Prove that the Hom bundle $\mathrm{Hom}(\eta_n, \eta_n^\perp) \to$ is isomorphic to the tangent bundle $T^n \to \mathbf{R}\mathrm{P}^n$.

9.5 Ehresmann's fibration theorem

We have studied quite intensely what consequences it has that a map $f\colon M \to N$ is an immersion. In fact, adding the point set topological property that $M \to f(M)$ is a homeomorphism we got in Theorem 5.7.4 that f was an imbedding.

We are now ready to discuss submersions (which by definition said that all points were regular). It turns out that adding a point set property we get that submersions are also rather special: they look like vector bundles, except that the fibers are not vector spaces, but manifolds!

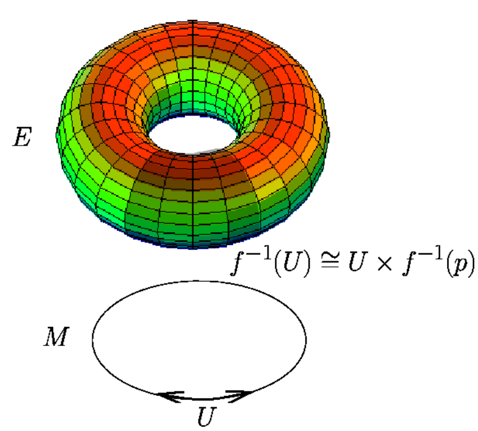
Definition 9.5.1 Let $f\colon E \to M$ be a smooth map. We say that f is a *locally trivial fibration* if for each $p \in M$ there is an open neighborhood U and a diffeomorphism

$$h\colon f^{-1}(U) \to U \times f^{-1}(p)$$

such that

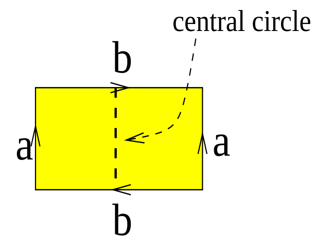
$$\begin{array}{ccc} f^{-1}(U) & \xrightarrow{\quad h \quad} & U \times f^{-1}(p) \\ & {\scriptstyle f|_{f^{-1}(U)}} \searrow \qquad \swarrow {\scriptstyle \mathrm{pr}_U} & \\ & U & \end{array}$$

commutes.



Over a small $U \in M$ a locally trivial fibration looks like the projection $U \times f^{-1}(p) \to U$ (the picture is kind of misleading, since the projection $S^1 \times S^1 \to S^1$ is **globally** of this kind).

Example 9.5.2 The projection of the torus down to a circle which is illustrated above is kind of misleading since the torus is globally a product. However, due to the scarcity of compact two-dimensional manifolds, the torus is the only example of a total space of a locally trivial fibration with non-discrete fibers that can be imbedded in $\mathbf{R}^3$.
However, there are nontrivial examples we can envision: for instance, the projection of the Klein bottle onto its "central circle" (see illustration to the right) is a nontrivial example.



The projection from the Klein bottle onto its "central circle" is a locally trivial fibration

If we allow discrete fibers there are many examples small enough to to be pictured. For instance, the squaring operation $z \mapsto z^2$ in complex numbers gives a locally trivial fibration $S^1 \to S^1$: the fiber of any point $z \in S^1$ is the set consisting of the two complex square roots of z (it is what is called a *double cover*). However, the fibration is not trivial (since S^1 is not homeomorphic to $S^1 \coprod S^1$)!

The last example is of a kind one encounters frequently: if $E \to M$ is a vector bundle endowed with some fiber metric, one can form the so-called *sphere bundle* $S(E) \to M$ by letting $S(E) = \{v \in E \mid |v| = 1\}$. The double cover above is exactly the sphere bundle associated to the infinite Möbius band.

Exercise 9.5.3 Let $E \to M$ be a vector bundle. Show that $E \to M$ has a non-vanishing vector field if and only if the associated sphere bundle (with respect to some fiber metric) $S(E) \to M$ has a section.

Exercise 9.5.4 In a locally trivial smooth fibration over a connected smooth manifold all fibers are diffeomorphic.

Definition 9.5.5 A continuous map $f\colon X \to Y$ is *proper* if the inverse image of compact subsets are compact.

Theorem 9.5.6 *(Ehresmann's fibration theorem) Let $f\colon E \to M$ be a proper submersion. Then f is a locally trivial fibration.*

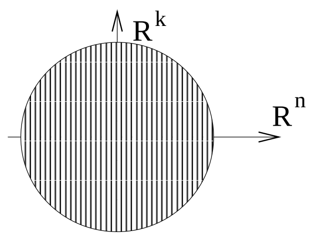
Proof: Since the question is local in M, we may start out by assuming that $M = \mathbf{R}^n$. The theorem then follows from lemma 9.5.8 below. ∎

Note 9.5.7 Before we start with the proof, it is interesting to see what the ideas are.

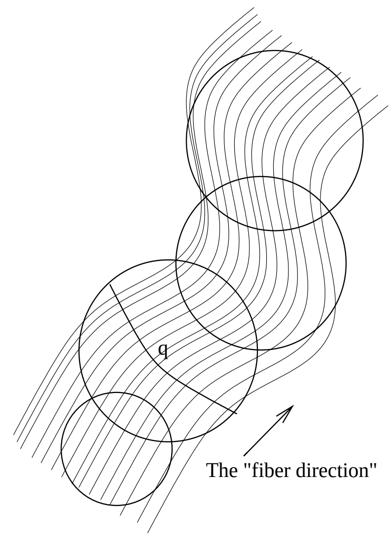
By the rank theorem a submersion looks locally (in E and M) as a projection

$$\mathbf{R}^{n+k} \to \mathbf{R}^n \times \{0\} \cong \mathbf{R}^n$$

and so locally all submersions are trivial fibrations. We will use flows to glue all these pieces together using partitions of unity.



Locally a submersion looks like the projection from $\mathbf{R}^{n+k}$ down onto $\mathbf{R}^n$.



The idea of the proof: make "flow" that flows transverse to the fibers: locally OK, but can we glue these pictures together?

The clue is then that a point $(t, q) \in \mathbf{R}^n \times f^{-1}(p)$ should correspond to what you get if you flow away from q, first a time t_1 in the first coordinate direction, then a time t_2 in the second and so on.

Lemma 9.5.8 *Let $f\colon E \to \mathbf{R}^n$ be a proper submersion. Then there is a diffeomorphism $h\colon E \to \mathbf{R}^n \times f^{-1}(0)$ such that*

$$\begin{array}{ccc} E & \xrightarrow{h} & \mathbf{R}^n \times f^{-1}(0) \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}_{\mathbf{R}^n}} & \\ & \mathbf{R}^n & \end{array}$$

commutes.

Proof: If E is empty, the lemma holds vacuously since then $f^{-1}(0) = \emptyset$, and $\emptyset = \mathbf{R}^n \times \emptyset$. Disregarding this rather uninteresting case, let $p_0 \in E$ and $r_0 = f(p_0) \in \mathbf{R}^n$. The third part of the rank theorem 5.3.1 guarantees that for all $p \in f^{-1}(r_0)$ there are charts $x_p\colon U_p \to U'_p$ such that

$$\begin{array}{ccc} U_p & \xrightarrow{f|_{U_p}} & \mathbf{R}^n \\ {\scriptstyle x_p}\downarrow & & \| \\ U'_p \subseteq \mathbf{R}^{n+k} & \xrightarrow{\mathrm{pr}} & \mathbf{R}^n \end{array}$$

commutes (the map $\mathrm{pr}\colon \mathbf{R}^{n+k} \to \mathbf{R}^n$ is the projection onto the first n coordinates.

Choose a partition of unity (see theorem 9.2.2) $\{\phi_j\}$ subordinate to $\{U_p\}$. For every j choose p such that $\mathrm{supp}(\phi_j) \subseteq U_p$, and let $x_j = x_p$ (so that we now are left with only countably many charts).

Define the vector fields (the ith partial derivative in the jth chart)

$$X_{i,j}\colon U_j \to TU_j, \qquad i = 1, \dots, n$$

by $X_{i,j}(q) = [\omega_{i,j}(q)]$ where

$$\omega_{i,j}(q)(t) = x_j^{-1}(x_j(q) + e_i t)$$

and where $e_i \in \mathbf{R}^n$ is the ith unit vector. Let

$$X_i = \sum_j \phi_j X_{i,j}\colon E \to TE, \qquad i = 1, \dots, n$$

(a "global version" of the i-th partial derivative).

Notice that since $f(u) = \mathrm{pr}\, x_j(u)$ for $u \in U_j$ we get that

$$\begin{aligned} f\omega_{i,j}(q)(t) &= f x_j^{-1}(x_j(q) + e_i t) = \mathrm{pr}\, x_j x_j^{-1}(x_j(q) + e_i t) = \mathrm{pr}\,(x_j(q) + e_i t) \\ &= f(q) + e_i t \end{aligned}$$

(the last equality uses that $i \le n$), which is independent of j. Since $\sum_j \phi_j(q) = 1$ for all q this gives that

$$\begin{aligned} Tf X_i(q) &= \sum_j \phi_j(q)[f\omega_{i,j}(q)] = \sum_j \phi_j(q)[t \mapsto f(q) + e_i t] \\ &= [t \mapsto f(q) + e_i t], \end{aligned}$$

or in other words

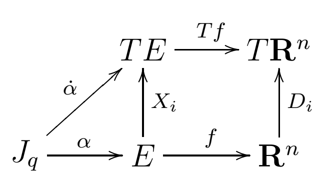
$$\begin{array}{ccc} TE & \xrightarrow{Tf} & T\mathbf{R}^n \\ {\scriptstyle X_i}\Big\uparrow & & \Big\uparrow{\scriptstyle D_i} \\ E & \xrightarrow{f} & \mathbf{R}^n \end{array}$$

commutes for all $i = 1, \dots, n$.

Fix the index i for a while. Notice that the curve $\beta\colon \mathbf{R} \to \mathbf{R}^n$ given by $\beta(t) = u + te_i$ is the unique solution to the initial value problem $\beta'(t) = e_i$, $\beta(0) = u$ (see Theorem 8.2.3), or in terms of the velocity vector $\dot\beta\colon \mathbf{R} \to T\mathbf{R}^n$ given by $\dot\beta(t) = [s \mapsto \beta(s+t)]$ of 8.1.14: β is unique with respect to the fact that $\dot\beta = D_i\beta$ and $\beta(0) = u$.

Let $\Phi_i\colon A_i \to E$ be the local flow corresponding to X_i, and let J_q be the slice of A_i at $q \in E$ (i.e., $A_i \cap (\mathbf{R} \times \{q\}) = J_q \times \{q\}$).

Fix q (and i), and consider the flow line $\alpha(t) = \Phi(t, q)$. Since flow lines are solution curves, the triangle in



commutes, and since $Tf(\dot{\alpha}) = (\dot{f\alpha})$ and $(f\alpha)(0) = f(q)$ we get by uniqueness that

$$f\Phi_i(t,q) = f\alpha(t) = f(q) + te_i.$$

We want to show that $A_i = \mathbf{R} \times E$. Since $f\Phi_i(t,q) = f(q) + e_i t$ we see that the image of a finite open interval under $f\Phi_i(-,q)$ must be contained in a compact, say K. Hence the image of the finite open interval under $\Phi_i(-,q)$ must be contained in $f^{-1}(K)$ which is compact since f is proper. But if $J_q \neq \mathbf{R}$, then Corollary 8.3.3 tells us that $\Phi_i(-,q)$ will leave any given compact in finite time leading to a contradiction.

Hence all the Φ_i defined above are global and we define the diffeomorphism

$$\phi\colon \mathbf{R}^n \times f^{-1}(r_0) \to E$$

by

$$\phi(t,q) = \Phi_1(t_1, \Phi_2(t_2, \dots, \Phi_n(t_n, q) \dots)), \qquad t = (t_1, \dots, t_n) \in \mathbf{R}^n, \quad q \in f^{-1}(r_0).$$

The inverse is given by

$$\begin{aligned} &E \to \mathbf{R}^n \times f^{-1}(r_0) \\ &q \mapsto (f(q) - r_0, \Phi_n((r_0)_n - f_n(q), \dots, \Phi_1((r_0)_1 - f_1(q), q) \dots)). \end{aligned}$$

Finally, we note that we have also proven that f is surjective, and so we are free in our choice of $r_0 \in \mathbf{R}^n$. Choosing $r_0 = 0$ gives the formulation stated in the lemma. ∎

Corollary 9.5.9 *(Ehresmann's fibration theorem, compact case) Let $f\colon E \to M$ be a submersion of compact smooth manifolds. Then f is a locally trivial fibration.*

Proof: We only need to notice that E being compact forces f to be proper: if $K \subset M$ is compact, it is closed (since M is Hausdorff), and $f^{-1}(K) \subseteq E$ is closed (since f is continuous). But a closed subset of a compact space is compact. ∎

Exercise 9.5.10 Check in all detail that the proposed formula for the inverse of ϕ given at the end of the proof of Ehresmann's fibration theorem 9.5.6 is correct.

Exercise 9.5.11 Consider the projection

$$f\colon S^3 \to \mathbf{C}\mathrm{P}^1$$

Show that f is a locally trivial fibration. Consider the map

$$\ell\colon S^1 \to \mathbf{C}\mathrm{P}^1$$

given by sending $z \in S^1 \subseteq \mathbf{C}$ to $[1, z]$. Show that ℓ is an imbedding. Use Ehresmann's fibration theorem to show that the inverse image

$$f^{-1}(\ell S^1)$$

is diffeomorphic to the torus $S^1 \times S^1$. (note: there is a diffeomorphism $S^2 \to \mathbf{C}\mathrm{P}^1$ given by $(a,z) \mapsto [1+a, z]$, and the composite $S^3 \to S^2$ induced by f is called the *Hopf fibration* and has many important properties. Among other things it has the truly counter-intuitive property of detecting a "three-dimensional hole" in S^2!)

Exercise 9.5.12 Let $\gamma\colon \mathbf{R} \to M$ be a smooth curve and $f\colon E \to M$ a proper submersion. Let $p \in f^{-1}(\gamma(0))$. Show that there is a smooth curve $\sigma\colon \mathbf{R} \to E$ such that

$$\begin{array}{ccc} & & E \\ & \overset{\sigma}{\nearrow} & \downarrow \\ \mathbf{R} & \xrightarrow{\gamma} & M \end{array}$$

commutes and $\sigma(0) = p$. Show that if the dimensions of E and M agree, then σ is unique. In particular, study the cases and $S^n \to \mathbf{R}\mathrm{P}^n$ and $S^{2n+1} \to \mathbf{C}\mathrm{P}^n$.

Chapter 10

Appendix: Point set topology

I have collected a few facts from point set topology. The main focus of this note is to be short and present exactly what we need in the manifold course. Point set topology may be your first encounter of real mathematical abstraction, and can cause severe distress to the novice, but it is kind of macho when you get to know it a bit better. However: keep in mind that the course is about manifold theory, and point set topology is only a means of expressing some (obvious?) properties these manifolds should possess. Point set topology is a powerful tool when used correctly, but it is not our object of study.

The concept that typically causes most concern is the quotient space. This construction is used widely whenever working with manifolds and must be taken seriously. However, the abstraction involved should be eased by the many concrete examples (like the flat torus in the robot's arm example 2.1). For convenience I have repeated the definition of equivalence relations at the beginning of section 10.6.

If you need more details, consult any of the excellent books listed in the references. The real classics are [2] and [6], but the most widely used these days is [10]. There are also many on-line textbooks, some of which you may find at the Topology Atlas' "Education" web site

`http://at.yorku.ca/topology/educ.htm`

Most of the exercises are not deep and are just rewritings of definitions (which may be hard enough if you are new to the subject) and the solutions short.

If I list a fact without proof, the result **may** be deep and its proof (much too) hard.

At the end, or more precisely in section 10.10, I have included a few standard definitions and statements about sets that are used frequently in both the text and in the exercises. The purpose of collecting them in a section at the end, is that whereas they certainly should not occupy central ground in the note (even in the appendix), the reader will still find the terms in the index and be referred directly to a definition, if she becomes uncertain about them at some point or other.

10.1 Topologies: open and closed sets

Definition 10.1.1 A *topology* is a family of sets $\mathcal{U}$ closed under finite intersection and arbitrary unions, that is if

if $U, U' \in \mathcal{U}$, then $U \cap U' \in \mathcal{U}$

if $\mathcal{I} \subseteq \mathcal{U}$, then $\bigcup_{U \in \mathcal{I}} U \in \mathcal{U}$.

Note 10.1.2 Note that the set $X = \bigcup_{U \in \mathcal{U}} U$ and $\emptyset = \bigcup_{U \in \emptyset} U$ automatically are members of $\mathcal{U}$.

Definition 10.1.3 We say that $\mathcal{U}$ is a *topology on* X, or that $(X, \mathcal{U})$ is a *topological space*. Frequently we will even refer to X as a topological space when $\mathcal{U}$ is evident from the context.

Definition 10.1.4 The members of $\mathcal{U}$ are called the *open sets* of X with respect to the topology $\mathcal{U}$.

A subset C of X is *closed* if the complement $X \setminus C = \{x \in X | x \notin C\}$ is open.

Example 10.1.5 An open set on the real line $\mathbf{R}$ is a (possibly empty) union of open intervals. Check that this defines a topology on $\mathbf{R}$. Check that the closed sets do **not** form a topology on $\mathbf{R}$.

Definition 10.1.6 A subset of X is called a *neighborhood* of $x \in X$ if it contains an open set containing x.

Lemma 10.1.7 *Let $(X, \mathcal{T})$ be a topological space. Prove that a subset $U \subseteq X$ is open if and only if for all $p \in U$ there is an open set V such that $p \in V \subseteq U$.*

Proof: Exercise! ∎

Definition 10.1.8 Let $(X, \mathcal{U})$ be a space and $A \subseteq X$ a subset. Then the *interior* $\operatorname{int} A$ of A in X is the union of all open subsets of X contained in A. The *closure* $\bar{A}$ of A in X is the intersection of all closed subsets of X containing A.

Exercise 10.1.9 Prove that $\operatorname{int} A$ is the biggest open set $U \in \mathcal{U}$ such that $U \subseteq A$, and that $\bar{A}$ is the smallest closed set C in X such that $A \subseteq C$.

Example 10.1.10 If (X, d) is a metric space (i.e., a set X and a symmetric positive definite function

$$d : X \times X \to \mathbf{R}$$

satisfying the triangle inequality), then X may be endowed with the *metric topology* by letting the open sets be arbitrary unions of open balls (note: given an $x \in X$ and a positive real number $\epsilon > 0$, the *open ϵ-ball centered in x* is the set $B(x, \epsilon) = \{y \in X | d(x, y) < \epsilon\}$). Exercise: show that this actually defines a topology.

In particular, *Euclidean n-space* is defined to be $\mathbf{R}^n$ with the metric topology.

Exercise 10.1.11 The metric topology coincides with the topology we have already defined on $\mathbf{R}$.

10.2 Continuous maps

Definition 10.2.1 A *continuous map* (or simply a map)

$$f : (X, \mathcal{U}) \to (Y, \mathcal{V})$$

is a function $f : X \to Y$ such that for every $V \in \mathcal{V}$ the inverse image

$$f^{-1}(V) = \{x \in X | f(x) \in V\}$$

is in $\mathcal{U}$

In other words: f is continuous if *the inverse images of open sets are open.*

Exercise 10.2.2 Prove that a continuous map on the real line is just what you expect.

More generally, if X and Y are metric spaces, considered as topological spaces by giving them the metric topology as in 10.1.10: show that a map $f : X \to Y$ is continuous iff the corresponding $\epsilon - \delta$-horror is satisfied.

Exercise 10.2.3 Let $f : X \to Y$ and $g : Y \to Z$ be continuous maps. Prove that the composite $gf : X \to Z$ is continuous.

Example 10.2.4 Let $f : \mathbf{R}^1 \to S^1$ be the map which sends $p \in \mathbf{R}^1$ to $e^{ip} = (\cos p, \sin p) \in S^1$. Since $S^1 \subseteq \mathbf{R}^2$, it is a metric space, and hence may be endowed with the metric topology. Show that f is continuous, and also that the image of open sets are open.

Definition 10.2.5 A *homeomorphism* is a continuous map $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ with a continuous inverse, that is a continuous map $g : (Y, \mathcal{V}) \to (X, \mathcal{U})$ with $f(g(y)) = y$ and $g(f(x)) = x$ for all $x \in X$ and $y \in Y$.

Exercise 10.2.6 Prove that $\tan : (-\pi/2, \pi/2) \to \mathbf{R}$ is a homeomorphism.

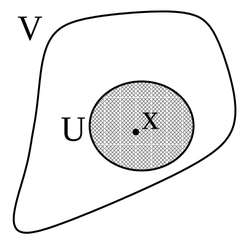
Note 10.2.7 Note that being a homeomorphism is **more than** being bijective and continuous. As an example let X be the set of real numbers endowed with the metric topology, and let Y be the set of real numbers, but with the “indiscrete topology”: only $\emptyset$ and Y are open. Then the identity map $X \to Y$ (sending the real number x to x) is continuous and bijective, but it is **not** a homeomorphism: the identity map $Y \to X$ is not continuous.

Definition 10.2.8 We say that two spaces are *homeomorphic* if there exists a homeomorphism from one to the other.

10.3 Bases for topologies

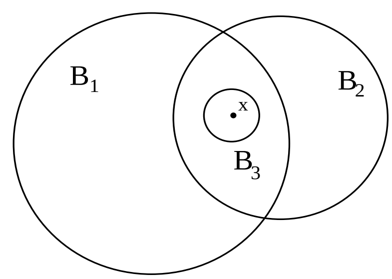
Definition 10.3.1 If $(X, \mathcal{U})$ is a topological space, a subfamily $\mathcal{B} \subseteq \mathcal{U}$ is a *basis for the topology* $\mathcal{U}$ if for each $x \in X$ and each $V \in \mathcal{U}$ with $x \in V$ there is a $U \in \mathcal{B}$ such that

$$x \in U \subseteq V$$



Note 10.3.2 This is equivalent to the condition that each member of $\mathcal{U}$ is a union of members of $\mathcal{B}$.

Conversely, given a family of sets $\mathcal{B}$ with the property that if $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$ then there is a $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$, then $\mathcal{B}$ is a basis for the topology on $X = \bigcup_{U \in \mathcal{B}} U$ given by declaring the open sets to be arbitrary unions from $\mathcal{B}$. We say that the basis $\mathcal{B}$ *generates* the topology on X.



Exercise 10.3.3 The real line has a countable basis for its topology.

Exercise 10.3.4 The balls with rational radius and whose center have coordinates that all are rational form a countable basis for $\mathbf{R}^n$.

Just to be absolutely clear: a topological space $(X, \mathcal{U})$ has a *countable basis* for its topology iff there exist a countable subset $\mathcal{B} \subseteq \mathcal{U}$ which is a basis.

Exercise 10.3.5 Let (X, d) be a metric space. Then the open balls form a basis for the metric topology.

Exercise 10.3.6 Let X and Y be topological spaces, and $\mathcal{B}$ a basis for the topology on Y. Show that a function $f: X \to Y$ is continuous if $f^{-1}(V) \subseteq X$ is open for all $V \in \mathcal{B}$.

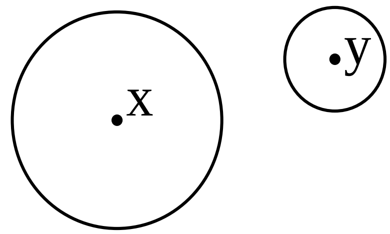
10.4 Separation

There are zillions of separation conditions, but we will only be concerned with the most intuitive of all: Hausdorff spaces.

Definition 10.4.1 A topological space $(X,\mathcal{U})$ is *Hausdorff* if for any two distinct $x, y \in X$ there exist disjoint neighborhoods of x and y.

Example 10.4.2 The real line is Hausdorff.

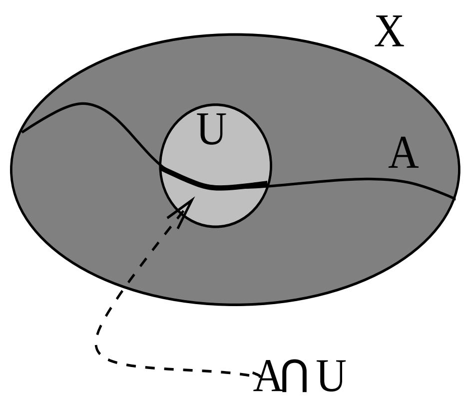
Example 10.4.3 More generally, the metric topology is always Hausdorff.



The two points x and y are contained in disjoint open sets.

10.5 Subspaces

Definition 10.5.1 Let $(X,\mathcal{U})$ be a topological space. A *subspace* of $(X,\mathcal{U})$ is a subset $A \subset X$ with the topology given letting the open sets be $\{A \cap U | U \in \mathcal{U}\}$.



Exercise 10.5.2 Show that the subspace topology **is** a topology.

Exercise 10.5.3 Prove that a map to a subspace $Z \to A$ is continuous iff the composite

$$Z \to A \subseteq X$$

is continuous.

Exercise 10.5.4 Prove that if X has a countable basis for its topology, then so has A.

Exercise 10.5.5 Prove that if X is Hausdorff, then so is A.

Corollary 10.5.6 *All subspaces of* $\mathbf{R}^n$ *are Hausdorff, and have countable bases for their topologies.*

Definition 10.5.7 If $A \subseteq X$ is a subspace, and $f\colon X \to Y$ is a map, then the composite

$$A \subseteq X \to Y$$

is called the *restriction of f to A*, and is written $f|_A$.

10.6 Quotient spaces

Before defining the quotient topology we recall the concept of equivalence relations.

Definition 10.6.1 Let X be a set. An *equivalence relation* on X is a subset E of of the set $X \times X = \{(x_1, x_2) | x_1, x_2 \in X\}$ satisfying the following three conditions

$$\begin{array}{rl} \text{(reflexivity)} & (x,x) \in E \text{ for all } x \in X \\ \text{(symmetry)} & \text{If } (x_1,x_2) \in E \text{ then } (x_2,x_1) \in E \\ \text{(transitivity)} & \text{If } (x_1,x_2) \in E\ (x_2,x_3) \in E \text{ then } (x_1,x_3) \in E \end{array}$$

We often write $x_1 \sim x_2$ instead of $(x_1, x_2) \in E$.

Definition 10.6.2 Given an equivalence relation E on a set X we may for each $x \in X$ define the *equivalence class* of x to be the subset $[x] = \{y \in X | x \sim y\}$.

This divides X into a collection of nonempty, mutually disjoint subsets.
The set of equivalence classes is written $X/\sim$, and we have a surjective function

$$X \to X/\sim$$

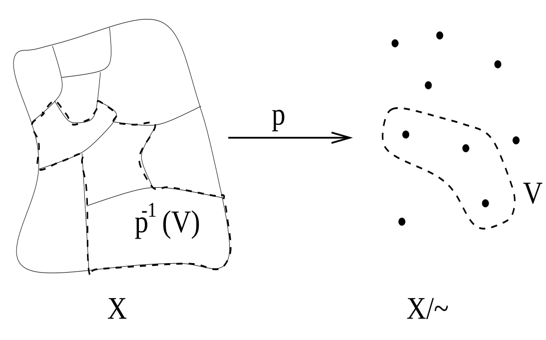
sending $x \in X$ to its equivalence class $[x]$.

Definition 10.6.3 Let $(X, \mathcal{U})$ be a topological space, and consider an equivalence relation $\sim$ on X. The *quotient space* space with respect to the equivalence relation is the set $X/\sim$ with the *quotient topology*. The quotient topology is defined as follows: Let

$$p \colon X \to X/\sim$$

be the projection sending an element to its equivalence class. A subset $V \subseteq X/\sim$ is open iff $p^{-1}(V) \subseteq X$ is open.

Exercise 10.6.4 Show that the quotient topology **is** a topology on $X/\sim$.



Exercise 10.6.5 Prove that a map from a quotient space $(X/\sim) \to Y$ is continuous iff the composite

$$X \to (X/\sim) \to Y$$

is continuous.

Exercise 10.6.6 The projection $\mathbf{R}^1 \to S^1$ given by $p \mapsto e^{ip}$ shows that we may view S^1 as the set of equivalence classes of real number under the equivalence $p \sim q$ if there is an integer n such that $p = q + 2\pi n$. Show that the quotient topology on S^1 is the same as the subspace topology you get by viewing S^1 as a subspace of $\mathbf{R}^2$.

10.7 Compact spaces

Definition 10.7.1 A *compact space* is a space $(X,\mathcal{U})$ with the following property: in any set $\mathcal{V}$ of open sets covering X (i.e., $\mathcal{V} \subseteq \mathcal{U}$ and $\bigcup_{V\in\mathcal{V}} V = X$) there is a finite subset that also covers X.

Exercise 10.7.2 If $f: X \to Y$ is a continuous map and X is compact, then $f(X)$ is compact.

We list without proof the results

Theorem 10.7.3 *(Heine–Borel) A subset of* $\mathbf{R}^n$ *is compact iff it is closed and of finite size.*

Example 10.7.4 Hence the unit sphere $S^n = \{p \in \mathbf{R}^{n+1} | \, |p| = 1\}$ (with the subspace topology) is a compact space.

Exercise 10.7.5 The *real projective space* $\mathbf{R}\mathrm{P}^n$ is the quotient space $S^n/\sim$ under the equivalence relation $p \sim -p$ on the unit sphere S^n. Prove that $\mathbf{R}\mathrm{P}^n$ is a compact Hausdorff space with a countable basis for its topology.

Theorem 10.7.6 *If* X *is a compact space, then all closed subsets of* X *are compact spaces.*

Theorem 10.7.7 *If* X *is a Hausdorff space and* $C \subseteq X$ *is a compact subspace, then* $C \subseteq X$ *is closed.*

A very important corollary of the above results is the following:

Theorem 10.7.8 *If* $f: C \to X$ *is a continuous map where* C *is compact and* X *is Hausdorff, then* f *is a homeomorphism if and only if it is bijective.*

Exercise 10.7.9 Prove 10.7.8 using the results preceding it

Exercise 10.7.10 Prove in three or fewer lines the standard fact that a continuous function $f\colon [a,b] \to \mathbf{R}$ has a maximum value.

A last theorem sums up some properties that are preserved under formation of quotient spaces (under favorable circumstances). It is not optimal, but will serve our needs. You can extract a proof from the more general statement given in [6, p. 148].

Theorem 10.7.11 *Let X be a compact space, and let $\sim$ be an equivalence relation on X. Let $p\colon X \to X/\sim$ be the projection and assume that if $K \subseteq X$ is closed, then $p^{-1}p(K) \subseteq X$ is closed too.*

If X is Hausdorff, then so is $X/\sim$.

If X has a countable basis for its topology, then so has $X/\sim$.

10.8 Product spaces

Definition 10.8.1 If $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ are two topological spaces, then their *product* $(X \times Y, \mathcal{U} \times \mathcal{V})$ is the set $X \times Y = \{(x,y) | x \in X, y \in Y\}$ with a basis for the topology given by products of open sets $U \times V$ with $U \in \mathcal{U}$ and $V \in \mathcal{V}$.

There are two projections $\mathrm{pr}_X\colon X \times Y \to X$ and $\mathrm{pr}_Y\colon X \times Y \to Y$. They are clearly continuous.

Exercise 10.8.2 A map $Z \to X \times Y$ is continuous iff both the composites with the projections

$$\begin{aligned} &Z \to X \times Y \to X, \text{ and} \\ &Z \to X \times Y \to Y \end{aligned}$$

are continuous.

Exercise 10.8.3 Show that the metric topology on $\mathbf{R}^2$ is the same as the product topology on $\mathbf{R}^1 \times \mathbf{R}^1$, and more generally, that the metric topology on $\mathbf{R}^n$ is the same as the product topology on $\mathbf{R}^1 \times \cdots \times \mathbf{R}^1$.

Exercise 10.8.4 If X and Y have countable bases for their topologies, then so has $X \times Y$.

Exercise 10.8.5 If X and Y are Hausdorff, then so is $X \times Y$.

10.9 Connected spaces

Definition 10.9.1 A space X is *connected* if the only subsets that are both open and closed are the empty set and the set X itself.

Exercise 10.9.2 The natural generalization of the intermediate value theorem is "If $f : X \to Y$ is continuous and X connected, then $f(X)$ is connected". Prove this.

Definition 10.9.3 Let $(X_1,\mathcal{U}_1)$ and $(X_2,\mathcal{U}_2)$ be topological spaces. The *disjoint union* $X_1 \coprod X_2$ is the union of disjoint copies of X_1 and X_2 (i.e., the set of pairs (k, x) where $k \in \{1, 2\}$ and $x \in X_k$), where an open set is a union of open sets in X and Y.

Exercise 10.9.4 Show that the disjoint union of two nonempty spaces X_1 and X_2 is not connected.

Exercise 10.9.5 A map $X_1 \coprod X_2 \to Z$ is continuous iff both the composites with the injections

$$X_1 \subseteq X_1 \coprod X_2 \to Z$$
$$X_2 \subseteq X_1 \coprod X_2 \to Z$$

are continuous.

10.10 Set theoretical stuff

The only purpose of this section is to provide a handy reference for some standard results in elementary set theory.

Definition 10.10.1 Let $A \subseteq X$ be a subset. The *complement* of A in X is the subset

$$X \setminus A = \{x \in X | x \notin A\}$$

Definition 10.10.2 Let $f : X \to Y$ be a function. We say that f is *injective* (or one-to-one) if $f(x_1) = f(x_2)$ implies that $x_1 = x_2$. We say that f is *surjective* (or onto) if for every $y \in Y$ there is an $x \in X$ such that $y = f(x)$. We say that f is *bijective* if it is both surjective and injective.

Definition 10.10.3 Let $A \subseteq X$ be a subset and $f : X \to Y$ a function. The *image* of A under f is the set

$$f(A) = \{y \in Y | \exists a \in A \text{ s.t. } y = f(a)\}$$

The subset $f(X) \subseteq Y$ is simply called the image of f.

If $B \subseteq Y$ is a subset, then the *inverse image* (or *preimage*) of B under f is the set

$$f^{-1}(B) = \{x \in X | f(x) \in B\}$$

The subset $f^{-1}(Y) \subseteq X$ is simply called the preimage of f.

Exercise 10.10.4 Prove that $f(f^{-1}(B)) \subseteq B$ and $A \subseteq f^{-1}(f(A))$.

Exercise 10.10.5 Prove that $f\colon X \to Y$ is surjective iff $f(X) = Y$ and injective iff for all $y \in Y$ $f^{-1}(\{y\})$ consists of a single element.

Lemma 10.10.6 *(***De Morgan's formulae***) Let X be a set and $\{A_i\}_{i\in I}$ be a family of subsets. Then*

$$X \setminus \bigcup_{i\in I} A_i = \bigcap_{i\in I}(X \setminus A_i)$$
$$X \setminus \bigcap_{i\in I} A_i = \bigcup_{i\in I}(X \setminus A_i)$$

Apology: the use of the term *family* is just phony: to us a family is nothing but a set (so a "family of sets" is nothing but a set of sets).

Exercise 10.10.7 Let $B_1, B_2 \subseteq Y$ and $f\colon X \to Y$ be a function. Prove that

$$f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2) \tag{10.1}$$
$$f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2) \tag{10.2}$$
$$f^{-1}(Y \setminus B_1) = X \setminus f^{-1}(B_1) \tag{10.3}$$

If in addition $A_1, A_2 \subseteq X$ then

$$f(A_1 \cup A_2) = f(A_1) \cup f(A_2) \tag{10.4}$$
$$f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2) \tag{10.5}$$
$$Y \setminus f(A_1) \subseteq f(X \setminus A_1) \tag{10.6}$$
$$B_1 \cap f(A_1) = f(f^{-1}(B_1) \cap A_1) \tag{10.7}$$

Chapter 11

Hints or solutions to the exercises

Below you will find hints for all the exercises. Some are very short, and some are almost complete solutions. Ignore them if you can, but if you are stuck, take a peek and see if you can get some inspiration.

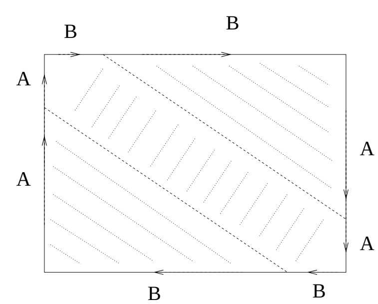
Chapter 2

Exercise 2.4.5

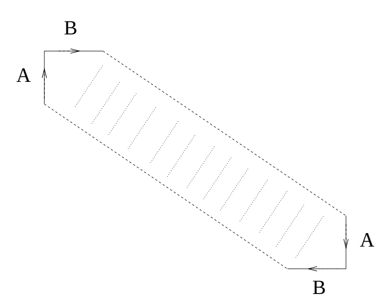
Draw a hexagon with identifications so that it represents a handle attached to a Möbius band. Try your luck at cutting and pasting this figure into a (funny looking) hexagon with identifications so that it represents three Möbius bands glued together (remember that the cuts may cross your identified edges).

Exercise 2.4.6

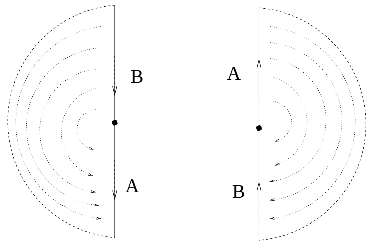
First, notice that any line through the origin intersects the unit sphere S^2 in two antipodal points, so that $\mathbf{R}\mathrm{P}^2$ can be identified with $S^2/p \sim -p$. Since any point on the Southern hemisphere is in the same class as one on the northern hemisphere we may disregard (in standard imperialistic fashion) all the points on the Southern hemisphere, so that $\mathbf{R}\mathrm{P}^2$ can be identified with the Northern hemisphere with antipodal points on the equator identified. Smashing down the Northern hemisphere onto a closed disk, we get that $\mathbf{R}\mathrm{P}^2$ can be identified with a disk with antipodal points on the boundary circle identified. Pushing in the disk so that we get a rectangle we get the following equivalent picture (disregard the lines in the interior of the rectangle for now)



The two dotted diagonal lines in the picture above represents a circle. Cut $\mathbf{R}\mathrm{P}^2$ along this circle yielding a Möbius strip



and two pieces



that glue together to a disk (the pieces have been straightened out at the angles of the rectangle, and one of the pieces has to be reflected before it can be glued to the other to form a disk).

Exercise 2.4.7

Do an internet search (check for instance the Wikipedia) to find the definition of the Euler characteristic. To calculate the Euler characteristic of surfaces you can simply use our flat representations as polygons, just remembering what points and edges really are identified.

Exercise 2.4.8

The beings could triangulate their universe, count the number of vertices V, edges E and surfaces F in this triangulation (this can be done in finite time). The Euler characteristic $V - E + F$ uniquely determines the surface.

Chapter 3

Exercise 3.1.5

The map $x^{k,i}$ is the restriction of the corresponding projection $\mathbf{R}^{n+1} \to \mathbf{R}^n$ which is continuous, and the inverse is the restriction of the continuous map $\mathbf{R}^n \to \mathbf{R}^{n+1}$ sending $p = (p_0, \dots, \widehat{p_k}, \dots, p_n) \in \mathbf{R}^n$ (note the smart indexing) to $(p_0, \dots, (-1)^i \sqrt{1 - |p|^2}, \dots, p_n)$.

Exercise 3.1.6

(Uses many results from the point set topology appendix). Assume there was a chart covering all of S^n. That would imply that we had a homeomorphism $x\colon S^n \to U'$ where U' is an open subset of $\mathbf{R}^n$. For $n = 0$, this clearly is impossible since S^0 consists of two points, whereas $\mathbf{R}^0$ is a single point. Also for $n > 0$ this is impossible since S^n is compact (it is a bounded and closed subset of $\mathbf{R}^{n+1}$), and so $U' = x(S^n)$ would be compact (and nonempty), but $\mathbf{R}^n$ does not contain any compact **and** open nonempty subsets.

Exercise 3.2.8

Draw the lines in the picture in example 3.2.7 and use high school mathematics to show that the formulae for $x^{\pm}$ are correct and define continuous functions (the formulae extend to functions defined on open sets in $\mathbf{R}^{n+1}$ where they are smooth and so continuous). Then invert x^- and x^+, which again become continuous functions (so that $x^{\pm}$ are homeomorphisms), and check the chart transformation formulae.

Exercise 3.2.11

To get a smooth atlas, repeat the discussion in Example 3.1.7 for the real projective space, exchanging $\mathbf{R}$ with $\mathbf{C}$ everywhere. To see that $\mathbf{C}\mathrm{P}^n$ is compact, it is convenient to notice that any $[p] \in \mathbf{C}\mathrm{P}^n$ can be represented by $p/|p| \in S^{2n+1} \subseteq \mathbf{C}^n$, so that $\mathbf{C}\mathrm{P}^n$ can alternatively be described as $S^{2n+1}/\sim$, where $p \sim q$ if there is a $z \in S^1$ such that $zp = zq$. Showing that $\mathbf{C}\mathrm{P}^n$ is Hausdorff and has a countable basis for its topology can be a bit more irritating, but a reference to Theorem 10.7.11 provides an easy fix.

Exercise 3.2.12

Transport the structure radially out from the unit circle (i.e., use the homeomorphism from the unit circle to the square gotten by blowing up a balloon in a square box in flatland). All charts can then be taken to be the charts on the circle composed with this homeomorphism.

Exercise 3.2.13

The only problem is in the origin. If you calculate (one side of the limit needed in the definition of the derivative at the origin),

$$\lim_{t\to 0^+}\frac{\lambda(t)-\lambda(0)}{t}=\lim_{t\to 0^+}\frac{e^{-1/t}}{t}=\lim_{s\to\infty}\frac{s}{e^s}=0,$$

you see that λ is once differentiable. It continues this way (you have to do a small induction showing that all derivatives at the origin involve limits of exponential times rational), proving that λ is smooth.

Exercise 3.3.4

If $\mathcal{B}$ is any smooth atlas containing $\mathcal{D}(\mathcal{A})$, then $\mathcal{D}(\mathcal{A})\subseteq\mathcal{B}\subseteq\mathcal{D}(\mathcal{D}(\mathcal{A}))$. Prove that $\mathcal{D}(\mathcal{D}(\mathcal{A}))=\mathcal{D}(\mathcal{A})$.

Exercise 3.3.9

It is enough to show that all the "mixed chart transformations" (like $x^{0,0}(x^+)^{-1}$) are smooth. Why?

Exercise 3.3.10

Because saying that "x is a diffeomorphism" is just a rephrasing of "$x = x(\mathrm{id})^{-1}$ and $x^{-1} = (\mathrm{id})x^{-1}$ are smooth". The charts in this structure are all diffeomorphisms $U\to U'$ where both U and U' are open subsets of $\mathbf{R}^n$.

Exercise 3.3.11

This will be discussed more closely in Lemma 9.1.1, but can be seen directly as follows: since M is a topological manifold it has a countable basis $\mathcal{B}$ for its topology. For each $(x,U)\in\mathcal{A}$ with $E^n\subseteq x(U)$ choose a $V\in\mathcal{B}$ such that $V\subseteq x^{-1}(E^n)$. The set $\mathcal{U}$ of such sets V is a countable subset of $\mathcal{B}$, and $\mathcal{U}$ covers M, since around any point p in M there is a chart sending p to 0 and containing E^n in its image (choose any chart (x,U) containing your point p. Then $x(U)$, being open, contains **some** small ball. Restrict to this, and reparametrize and translate so that it becomes the open unit ball). Now, for every $V\in\mathcal{U}$ choose **one** of the charts $(x,U)\in\mathcal{A}$ with $E^n\subseteq x(U)$ such that $V\subseteq x^{-1}(E^n)$. The resulting set $\mathcal{V}\subseteq\mathcal{U}$ is then a countable smooth atlas for $(M,\mathcal{A})$.

Exercise 3.4.4

Use the identity chart on $\mathbf{R}$. The standard atlas on $S^1\subseteq\mathbf{C}$ using projections is given simply by real and imaginary part. Hence the formulae you have to check are smooth are sin and cos. This we know! One comment on domains of definition: let $f\colon\mathbf{R}\to S^1$ be the map in question; if we use the chart $(x^{0,0},U^{0,0}$, then $f^{-1}(U^{0,0})$ is the union of all the intervals on the form $(-\pi/2+2\pi k,\pi/2+2\pi k)$ when k varies over the integers. Hence the function to check in this case is the function from this union to $(-1,1)$ sending θ to $\sin\theta$.

Exercise 3.4.5

First check that $\tilde{g}$ is well defined ($g(p)=g(-p)$ for all $p\in S^2$). Check that it is smooth using the standard charts on $\mathbf{R}\mathrm{P}^2$ and $\mathbf{R}^4$ (for instance: $\tilde{g}x^0(q_1,q_2)=\frac{1}{1+q_1^2+q_2^2}(q_1q_2,q_2,q_1,1+2q_1^2+3q_2^2))$. To show that $\tilde{g}$ is injective, show that $g(p)=g(q)$ implies that $p=\pm q$.

Exercise 3.4.6

One way follows since the composite of smooth maps is smooth. The other follows since smoothness is a local question, and the projection $g\colon S^n\to\mathbf{R}\mathrm{P}^n$ is a local diffeomorphism. More (or perhaps, too) precisely, if $f\colon\mathbf{R}\mathrm{P}^n\to M$ is a map, we have to show that for all charts (y,V) on M, the composites $yf(x^k)^{-1}$ (defined on $U^k\cap f^{-1}(V)$) are smooth. But $x^kg(x^{k,0})^{-1}\colon D^n\to\mathbf{R}^n$ is a diffeomorphism (given by sending $p\in D^n$ to $\frac{1}{\sqrt{1-|p|^2}}p\in\mathbf{R}^n$), and so claiming that $yf(x^k)^{-1}$ is smooth is the same as claiming that $y(fg)(x^{k,0})^{-1}=yf(x^k)^{-1}x^kg(x^{k,0})^{-1}$ is smooth.

Exercise 3.4.11

Consider the map $f\colon \mathbf{R}\mathrm{P}^1 \to S^1$ sending $[z]$ (with $z \in S^1 \subseteq \mathbf{C}$) to $z^2 \in S^1$, which is well defined since $(-z)^2 = z^2$. To see that f is smooth either consider the composite

$$S^1 \to \mathbf{R}\mathrm{P}^1 \stackrel{f}{\to} S^1 \subseteq \mathbf{C}$$

(where the first map is the projection $z \mapsto [z]$) using Exercise 3.4.6 and Exercise 3.5.15, or do it from the definition: consider the standard atlas for $\mathbf{R}\mathrm{P}^1$. In complex notation $U^0 = \{[z]|\mathrm{re}(z) \neq 0\}$ and $x^0([z]) = \mathrm{im}(z)/\mathrm{re}(z)$ with inverse $t \mapsto e^{i\tan^{-1}(t)}$. If $[z] \in U^0$, then $f([z]) = z^2 \in V = \{v \in S^1|v \neq -1\}$. On V we choose the convenient chart $y\colon V \to (-\pi, \pi)$ with inverse $\theta \mapsto e^{i\theta}$, and notice that the "up, over and across" $yf(x^0)^{-1}(t) = 2\tan^{-1}(t)$ obviously is smooth. Likewise we cover the case $[z] \in U^1$. Showing that the inverse is smooth is similar.

Exercise 3.4.12

Consider the map $\mathbf{C}\mathrm{P}^1 \to S^2 \subseteq \mathbf{R} \times \mathbf{C}$ sending $[z_0, z_1]$ to

$$\frac{1}{|z_0|^2 + |z_1|^2}\left(|z_1|^2 - |z_0|^2, 2z_1\overline{z_0}\right),$$

check that it is well defined and continuous (since the composite $\mathbf{C}^2 \to \mathbf{C}\mathrm{P}^1 \to S^2 \subseteq \mathbf{R}\times\mathbf{C}$ is), calculate the inverse (which is continuous by 10.7.8), and use the charts on S^2 from stereographic projection 3.2.7 to check that the map and its inverse are smooth. The reader may enjoy comparing the above with the discussion about qbits in 2.3.1.

Exercise 3.4.14

Given a chart (x, U) on M, define a chart $(xf^{-1}, f(U))$ on N.

Exercise 3.4.19

To see this, note that given any p, there are open sets U_1 and V_1 with $p \in U_1$ and $i(p) \in V_1$ and $U_1 \cap V_1 = \emptyset$ (since M is Hausdorff). Let $U = U_1 \cap i(V_1)$. Then U and $i(U) = i(U_1) \cap V_1$ do not intersect. As a matter of fact M has a basis for its topology consisting of these kinds of open sets.

By shrinking even further, we may assume that U is a chart domain for a chart $x\colon U \to U'$ on M.

We see that $f|_U$ is open (it sends open sets to open sets, since the inverse image is the union of two open sets).

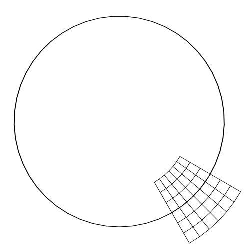
On U we see that f is injective, and so it induces a homeomorphism $f|_U\colon U \to f(U)$. We define the smooth structure on M/i by letting $x(f|_U)^{-1}$ be the charts for varying U. This is obviously a smooth structure, and f is a local diffeomorphism.

Exercise 3.4.20

Choose **any** chart $y\colon V \to V'$ with $p \in V$ in $\mathcal{U}$, choose a small open ball $B \subseteq V'$ around $y(p)$. There exists a diffeomorphism h of this ball with all of $\mathbf{R}^n$. Let $U = y^{-1}(B)$ and define x by setting $x(q) = hy(q) - hy(p)$.

Exercise 3.5.4

Use "polar coordinates".



Exercise 3.5.5

Let $f(a_0, \ldots, a_{n-1}, t) = t^t + a_{n-1}t^{n-1} + \cdots + a_0$ and consider the map $x\colon \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ given by

$$x(a_0, \ldots, a_n) = (a_1, \ldots, a_n, f(a_0, \ldots, a_n))$$

This is a smooth chart on $\mathbf{R}^{n+1}$ since x is a diffeomorphism with inverse given by sending $(b_1, \dots, b_{n+1})$ to $(b_{n+1} - f(0, b_1, \dots, b_n), b_1, \dots, b_n)$. We see that $x(C) = \mathbf{R}^n \times 0$, and we have shown that C is an n-dimensional submanifold. Notice that we have only used that $f\colon \mathbf{R}^{n+1} \to \mathbf{R}$ is smooth and that $f(a_0, \dots, a_n) = a_0 + f(0, a_1, a_n)$.

Exercise 3.5.6

Assume there is a chart $x\colon U \to U'$ with $(0,0) \in U$, $x(0,0) = (0,0)$ and $x(K \cap U) = (\mathbf{R} \times 0) \cap U'$.

Then the composite (V is a sufficiently small neighborhood of 0)

$$V \xrightarrow{q \mapsto (q,0)} U' \xrightarrow{x^{-1}} U$$

is smooth, and of the form $q \mapsto T(q) = (t(q), |t(q)|)$. But

$$T'(0) = \left(\lim_{h \to 0} \frac{t(h)}{h}, \lim_{h \to 0} \frac{|t(h)|}{h} \right),$$

and for this to exist, we must have $t'(0) = 0$.

On the other hand $x(p, |p|) = (s(p), 0)$, and we see that s and t are inverse functions. The directional derivative of $\mathrm{pr}_1 x$ at $(0,0)$ in the direction $(1,1)$ is equal

$$\lim_{h \to 0^+} \frac{s(h)}{h}$$

but this limit does not exist since $t'(0) = 0$, and so x can't be smooth, contradiction.

Exercise 3.5.9

Let $f_1, f_2\colon V \to \mathbf{R}^n$ be linear isomorphisms. Let G^1, G^2 be the two resulting two smooth manifolds with underlying set $\mathrm{GL}(V)$. Showing that $G^1 = G^2$ amounts to showing that the composite

$$\mathrm{GL}(f_1)\mathrm{GL}(f_2)\colon \mathrm{GL}(\mathbf{R}^n) \to \mathrm{GL}(\mathbf{R}^n)$$

is a diffeomorphism. Noting that $\mathrm{GL}(f_2)^{-1} = \mathrm{GL}(f_2^{-1})$ and $\mathrm{GL}(f_1)\mathrm{GL}(f_2^{-1}) = \mathrm{GL}(f_1 f_2^{-1})$, this amounts to showing that given a fixed invertible matrix A (representing $f_1 f_2^{-1}$ in the standard basis), then conjugation by A, i.e., $B \mapsto ABA^{-1}$ is a smooth map $M_n(\mathbf{R}) \to M_n(\mathbf{R})$. This is true since addition and multiplication are smooth.

That $\mathrm{GL}(h)$ is a diffeomorphism boils down to the fact that the composite

$$\mathrm{GL}(f)\mathrm{GL}(h)\mathrm{GL}(f)^{-1}\colon \mathrm{GL}(\mathbf{R}^n) \to \mathrm{GL}(\mathbf{R}^n)$$

is nothing but $\mathrm{GL}(fhf^{-1})$. If $fhf^{-1}\colon \mathbf{R}^n \to \mathbf{R}^n$ is represented by the matrix A, then $\mathrm{GL}(fhf^{-1})$ is represented by conjugation by A and so a diffeomorphism.

If $\alpha, \beta\colon V \cong V$ are two linear isomorphisms, we may compose them to get $\alpha\beta\colon V \to V$. That $\mathrm{GL}(h)$ respects composition follows, since $\mathrm{GL}(h)(\alpha\beta) = h(\alpha\beta)h^{-1} = h\alpha h^{-1} h\beta)h^{-1} = \mathrm{GL}(h)(\alpha)\mathrm{GL}(h)(\beta)$. Also, $\mathrm{GL}(h)$ preserves the identity element since $\mathrm{GL}(h)(\mathrm{id}_V) = h\mathrm{id}_V h^{-1} = hh^{-1} = \mathrm{id}_W$.

Exercise 3.5.12

The subset $f(\mathbf{R}\mathrm{P}^n) = \{[p, 0] \in \mathbf{R}\mathrm{P}^{n+1}\}$ is a submanifold by using all but the last of the standard charts on $\mathbf{R}\mathrm{P}^{n+1}$. Checking that $\mathbf{R}\mathrm{P}^n \to f(\mathbf{R}\mathrm{P}^n)$ is a diffeomorphism is now straightforward (the "ups, over and acrosses" correspond to the chart transformations in $\mathbf{R}\mathrm{P}^n$).

Exercise 3.5.15

Assume $i_j\colon N_j \to M_j$ are inclusions of submanifolds — the diffeomorphism part of "imbedding" being the trivial case — and let $x_j\colon U_j \to U_j'$ be charts such that

$$x_j(U_j \cap N_j) = U_j' \cap (\mathbf{R}^{n_j} \times \{0\}) \subseteq \mathbf{R}^{m_j}$$

for $j = 1, 2$. To check whether f is smooth at $p \in N_1$ it is enough to assert that $x_2 f x_1^{-1}|_{x_1(V)} = x_2 g x_1^{-1}|_{x_1(V)}$ is smooth at p where $V = U_1 \cap N_1 \cap g^{-1}(U_2)$ which is done by checking the higher order partial derivatives in the relevant coordinates.

Exercise 3.5.16

Let $f: X \to Y$ and $g: Y \to Z$ be imbeddings. Then the induced map $X \to gf(X)$ is a diffeomorphism. Hence it is enough to show that the composite of inclusions of submanifolds is an inclusion of a submanifold. Let $X \subseteq Y \subset Z$ be inclusions of submanifolds (of dimension n, $n+k$ and $n+k+l$). Given $p \in X$ let $z: U \to U'$ be a chart on Z such that $z(U \cap Y) = (\mathbf{R}^{n+k} \times \{0\}) \cap U'$ and let $y: V \to V'$ be a chart on Y such that $y(V \cap X) = (\mathbf{R}^n \times \{0\}) \cap V'$ with $p \in U \cap V$. We may even assume (by shrinking the domains) that $V = Y \cap U$. Then

$$\left(\left(yz^{-1}|_{z(V)} \times id_{\mathbf{R}^l}\right)|_{U'} \circ z, U\right)$$

is a chart displaying X as a submanifold of Z.

Exercise 3.6.2

Check that all chart transformations are smooth.

Exercise 3.6.5

Up over and across using appropriate charts on the product, reduces this to saying that the identity is smooth and that the inclusion of $\mathbf{R}^m$ in $\mathbf{R}^m \times \mathbf{R}^n$ is an imbedding.

Exercise 3.6.6

The heart of the matter is that $\mathbf{R}^k \to \mathbf{R}^m \times \mathbf{R}^n$ is smooth if and only if both the composites $\mathbf{R}^k \to \mathbf{R}^m$ and $\mathbf{R}^k \to \mathbf{R}^n$ are smooth.

Exercise 3.6.7

Consider the map $(t, z) \mapsto e^t z$.

Exercise 3.6.8

Reduce to the case where f and g are inclusions of submanifolds. Then rearrange some coordinates to show that case.

Exercise 3.6.9

Use the preceding exercises.

Exercise 3.6.10

Remember that $\mathrm{GL}_n(\mathbf{R})$ is an open subset of $M_n(\mathbf{R})$ and so this is in flatland 3.5.8). Multiplication of matrices is smooth since it is made out of addition and multiplication of real numbers.

Exercise 3.6.11

Use the fact that multiplication of complex numbers is smooth, plus Exercise 3.5.15).

Exercise 3.6.13

Check chart transformations.

Exercise 3.6.16

Using the "same" charts on both sides, this reduces to saying that the identity is smooth.

Exercise 3.6.17

A map from a disjoint union is smooth if and only if it is smooth on both summands since smoothness is measured locally.

Chapter 4

Exercise 4.1.5

The only thing that is slightly ticklish with the definition of germs is the transitivity of the equivalence relation: assume

$$f: U_f \to N, \qquad g: U_g \to N, \text{ and } h: U_h \to N$$

and $f \sim g$ and $g \sim h$. Writing out the definitions, we see that $f = g = h$ on the open set $V_{fg} \cap V_{gh}$, which contains p.

Exercise 4.1.6

Choosing other representatives changes nothing in the intersection of the domains of definition. Associativity and the behavior of identities follows from the corresponding properties for the composition of representatives.

Exercise 4.1.18

We do it for $\epsilon = \pi/2$. Other ϵs are then obtained by scaling. Let

$$f(t) = \gamma_{(\pi/4,\pi/4)}(t)\cdot t + (1-\gamma_{(\pi/4,\pi/4)}(t))\cdot\tan(t).$$

As to the last part, if $\bar{\gamma}\colon(\mathbf{R},0)\to(M,p)$ is represented by $\gamma_1\colon(-\epsilon,\epsilon)\to\mathbf{R}$, we let $\gamma=\gamma_1 f^{-1}$ where f is a diffeomorphism $(-\epsilon,\epsilon)\to\mathbf{R}$ with $f(t)=t$ for $|t|$ small.

Exercise 4.1.19

Let $\phi\colon U_\phi\to\mathbf{R}$ be a representative for $\bar{\phi}$, and let (x,U) be any chart around p such that $x(p)=0$. Choose an $\epsilon>0$ such that $x(U\cap U_\phi)$ contains the open ball of radius ϵ. Then the germ represented by ϕ is equal to the germ represented by the map defined on all of M given by

$$q\mapsto\begin{cases}\gamma_{(\epsilon/3,\epsilon/3)}(x(q))\phi(q) & \text{for } q\in U\cap U_\phi\\ 0 & \text{otherwise.}\end{cases}$$

Exercise 4.1.20

You can extend any chart to a function defined on the entire manifold.

Exercise 4.2.6

This is immediate from the chain rule (or, for that matter, from the definition).

Exercise 4.2.9

Let $\gamma,\gamma_1\in W_p$. If (x,U) is a chart with $p\in U$ and if for all function germs $\bar{\phi}\in\mathcal{O}_p$ $(\phi\gamma)'(0)=(\phi\gamma_1)'(0)$, then letting $\phi=x_k$ be the kth coordinate of x for $k=1,\dots,n$ we get that $(x\gamma)'(0)=(x\gamma_1)'(0)$. Conversely, assume $\bar{\phi}\in\mathcal{O}_p$ and $(x\gamma)'(0)=(x\gamma_1)'(0)$ for all charts (x,U). Then $(\phi\gamma)'(0)=(\phi x^{-1}x\gamma)'(0)=D(\phi x^{-1})(x(p))\cdot(x\gamma)'(0)$ by the flat chain rule 4.2.8 and we are done.

Exercise 4.2.10

If (y,V) is some other chart with $p\in V$, then the flat chain rule 4.2.8 gives that

$$\begin{aligned}(y\gamma)'(0) &= (yx^{-1}x\gamma)'(0)\\ &= D(yx^{-1})(x(p))\cdot(x\gamma)'(0)\\ &= D(yx^{-1})(x(p))\cdot(x\gamma_1)'(0)\\ &= (yx^{-1}x\gamma_1)'(0)=(y\gamma_1)'(0),\end{aligned}$$

where $D(yx^{-1})(x(p))$ is the Jacobi matrix of the function yx^{-1} at the point $x(p)$.

Exercise 4.2.3

It depends neither on the representation of the tangent vector nor on the representation of the germ, because if $[\gamma]=[\nu]$ and $\bar{f}=\bar{g}$, then $(\phi f\gamma)'(0)=(\phi f\nu)'(0)=(\phi g\nu)'(0)$ (partially by definition).

Exercise 4.2.17

Expanding along the ith row, we see that the partial differential of det with respect to the i,j-entry is equal to the determinant of the matrix you get by deleting the ith row and the jth column. Hence, the Jacobian of det is the $1\times n^2$-matrix consisting of these determinants (in some order), and is zero if and only if all of them vanish, which is the same as saying that A has rank less than $n-1$.

Exercise 4.2.18

Either by sitting down and calculating partial derivatives or arguing abstractly: Since the Jacobian $DL(p)$ represents the unique linear map

K such that $\lim h \to 0 \frac{1}{h}(L(p+h) - L(p) - K(h)) = 0$ and L is linear we get that $K = L$.

Exercise 4.3.3

Directly from the definition, or by $f^*g^* = (gf)^*$ and the fact that if $id\colon M \to M$ is the identity, then $id^*\colon \mathcal{O}_{M,p} \to \mathcal{O}_{M,p}$ is the identity too.

Exercise 4.3.6

Both ways around the square sends $\bar{\phi} \in \mathcal{O}_{M,p}$ to $d(\phi f)$.

Exercise 4.3.16

Use the definitions.

Exercise 4.3.14

If V has basis $\{v_1, \dots, v_n\}$, W has basis $\{w_1, \dots, w_m\}$, then $f(v_i) = \sum_{j=1}^m a_{ij} w_j$ means that $A = (a_{ij})$ represents f in the given basis. Then $f^*(w_j^*) = w_j^* f = \sum_{i=1}^n a_{ij} v_i^*$ as can be checked by evaluating at v_i: $w_j^* f(v_i) = w_j(\sum_{k=1}^m a_{ik} w_k) = a_{ij}$.

Exercise 4.3.18

The two Jacobi matrices in question are given by

$$D(xy^{-1})(y(p))^t = \begin{bmatrix} p_1/|p| & p_2/|p| \\ -p_2 & p_1 \end{bmatrix}.$$

and

$$D(yx^{-1})(x(p)) = \begin{bmatrix} p_1/|p| & -p_2/|p|^2 \\ p_2/|p| & p_1/|p|^2 \end{bmatrix}.$$

Exercise 4.4.12

Assume

$$X = \sum_{j=1}^n v_j \, D_j|_0 = 0$$

Then

$$0 = X(\overline{\mathrm{pr}_i}) = \sum_{j=1}^n v_j D_j(\mathrm{pr}_i)(0) = \begin{cases} 0 & \text{if } i \neq j \\ v_i & \text{if } i = j \end{cases}$$

Hence $v_i = 0$ for all i and we have linear independence.

If $X \in D|_0\mathbf{R}^n$ is any derivation, let $v_i = X(\overline{\mathrm{pr}_i})$. If $\bar{\phi}$ is any function germ, we have by lemma 4.3.8 that

$$\bar{\phi} = \phi(0) + \sum_{i=1}^n \overline{\mathrm{pr}_i} \cdot \overline{\phi_i}, \qquad \phi_i(p) = \int_0^1 D_i\phi(t \cdot p)\, dt,$$

and so

$$\begin{aligned} X(\bar{\phi}) =& X(\phi(0)) + \sum_{i=1}^n X(\overline{\mathrm{pr}_i} \cdot \overline{\phi_i}) \\ =& 0 + \sum_{i=1}^n \Big(X(\overline{\mathrm{pr}_i}) \cdot \phi_i(0) + \mathrm{pr}_i(0) \cdot X(\overline{\phi_i}) \Big) \\ =& \sum_{i=1}^n \Big(v_i \cdot \phi_i(0) + 0 \cdot X(\overline{\phi_i})) \Big) \\ =& \sum_{i=1}^n v_i D_i \phi(0) \end{aligned}$$

where the identity $\phi_i(0) = D_i\phi(0)$ was used in the last equality.

Exercise 4.4.16

If $[\gamma] = [\nu]$, then $(\phi f \gamma)'(0) = (\phi f \nu)'(0)$.

Exercise 4.4.19

The tangent vector $[\gamma]$ is sent to $X_\gamma f^*$ one way, and $X_{f\gamma}$ the other, and if we apply this to a function germ $\bar{\phi}$ we get

$$X_\gamma f^*(\bar{\phi}) = X_\gamma(\bar{\phi}\bar{f}) = (\phi f \gamma)'(0) = X_{f\gamma}(\bar{\phi}).$$

If you find such arguments hard: notice that $\phi f \gamma$ is the **only possible composition of these functions**, and so either side better relate to this!

Chapter 5

Exercise 5.1.11

Observe that the function in question is

$$f(e^{i\theta}, e^{i\phi}) = \sqrt{(3 - \cos\theta - \cos\phi)^2 + (\sin\theta + \sin\phi)^2},$$

giving the claimed Jacobi matrix. Then solve the system of equations

$$\begin{aligned} 3\sin\theta - \cos\phi\sin\theta + \sin\phi\cos\theta &= 0 \\ 3\sin\phi - \cos\theta\sin\phi + \sin\theta\cos\phi &= 0 \end{aligned}$$

Adding the two equations we get that $\sin\theta = \sin\phi$, but then the upper equation claims that $\sin\phi = 0$ or $3 - \cos\phi + \cos\theta = 0$. The latter is clearly impossible.

Exercise 5.2.4

Consider the smooth map

$$\begin{aligned} f\colon G \times G &\to G \times G \\ (g, h) &\mapsto (gh, h) \end{aligned}$$

with inverse $(g, h) \mapsto (gh^{-1}, h)$. Use that, for a given $h \in G$, the map $L_h\colon G \to G$ sending g to $L_h(g) = gh$ is a diffeomorphism, and that

$$\begin{array}{ccc} T_gG \times T_hG & \xrightarrow{\begin{bmatrix} T_gL_h & T_hR_g \\ 0 & 1 \end{bmatrix}} & T_{gh}G \times T_hG \\ \cong\downarrow & & \cong\downarrow \\ T_{(g,h)}(G \times G) & \xrightarrow{T_{(g,h)}f} & T_{(gh,h)}(G \times G) \end{array}$$

commutes (where the vertical isomorphisms are the "obvious" ones and $R_g(h) = gh$) to conclude that f has maximal rank, and is a diffeomorphism. Then consider a composite

$$G \xrightarrow{g \mapsto (1,g)} G \times G \xrightarrow{f^{-1}} G \times G \xrightarrow{(g,h) \mapsto g} G.$$

Perhaps a word about the commutativity of the above square is desirable. Starting with a pair $([\gamma], [\eta])$ in the upper left hand corner, going down, right and up you get $([\gamma \cdot \eta], [\eta])$. However, if go along the upper map you get $([\gamma \cdot h] + [g \cdot \eta], [\eta])$, so we need to prove that $[\gamma \cdot \eta] = [\gamma \cdot h] + [g \cdot \eta]$.

Choose a chart (z, U) around $g \cdot h$, let $\Delta\colon \mathbf{R} \to \mathbf{R} \times \mathbf{R}$ be given by $\Delta(t) = (t, t)$ and let $\mu\colon G \times G \to G$ be the multiplication in G. Then the chain rule, as applied to the function $z(\gamma \cdot \eta) = z\mu(\gamma, \eta)\Delta\colon \mathbf{R} \to \mathbf{R}^n$ gives that

$$\begin{aligned} &(z(\gamma \cdot \eta))'(0) = D(z\mu(\gamma, \eta))(0, 0) \cdot \Delta'(0) \\ =& \begin{bmatrix} D_1(z\mu(\gamma, \eta))(0, 0) & D_2(z\mu(\gamma, \eta))(0, 0) \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ =& (z(\gamma \cdot h))'(0) + (z(g \cdot \eta))'(0). \end{aligned}$$

Exercise 5.3.2

The rank theorem says that around any regular point there is a neighborhood on which f is a diffeomorphism. Hence $f^{-1}(q)$ is discrete, and, since M is compact, finite. Choose neighborhoods U_x around each element in $x \in f^{-1}(q)$ s.t. f defines a diffeomorphism from U_x to an open neighborhood $f(U_x)$ of q. Let the promised neighborhood around q be

$$\bigcap_{x \in f^{-1}(q)} f(U_x) - f\Big(M - \bigcup_{x \in f^{-1}(q)} U_x\Big)$$

(remember that f takes closed sets to closed sets since M is compact and N Hausdorff).

Exercise 5.3.3

(after [8, p. 8]). Extend P to a smooth map $f\colon S^2 \to S^2$ by stereographic projection (check that f is smooth at the North pole). Assume 0 is a regular value (if it is a critical value we are done!). The critical points of f correspond to the zeros of the derivative P', of which there are only finitely many. Hence the regular values of f cover all but finitely many points of of S^2, and so give a connected space. Since by Exercise 5.3.2 $q \mapsto |f^{-1}(q)|$ is a locally constant function of the regular values, we get that there is an n such that $n = |f^{-1}(q)|$ for all regular values q. Since n can't be zero (P was not constant) we are done.

Exercise 5.3.4

(after [3]). Use the rank theorem, which gives the result immediately if we can prove that the rank of f is constant (spell this out). To prove that the rank of f is constant, we first prove it for all points in $f(M)$ and then extend it to some neighborhood using the chain rule.

The chain rule gives that

$$T_p f = T_p(ff) = T_{f(p)} f T_p f.$$

If $p \in f(M)$, then $f(p) = p$, so we get that $T_p f = T_p f T_p f$ and

$$T_p f(T_p M) = \{v \in T_p M \mid T_p f(v) = v\} = \ker\{1 - T_p f\}.$$

By the dimension theorem in linear algebra we get that

$$rk\,(T_p f) + rk\,(1 - T_p f) = \dim(M),$$

and since both ranks only can increase locally, they must be locally constant, and so constant, say r, since M was supposed to be connected.

Hence there is an open neighborhood U of $p \in f(M)$ such that $rkT_q f \geq r$ for all $q \in U$, but since $T_q f = T_{f(q)} f T_q f$ we must have $rkT_q f \leq T_{f(p)} f = r$, and so $rkT_q f = r$ too.

That $f(M) = \{p \in M | f(p) = p\}$ is closed in M follows since the complement is open: if $p \neq f(p)$ choose disjoint open sets U and V around p and $f(p)$. Then $U \cap f^{-1}(V)$ is an open set disjoint from $f(M)$ (since $U \cap f^{-1}(V) \subseteq U$ and $f(U \cap f^{-1}(V)) \subseteq V$) containing p.

Exercise 5.4.4

Prove that 1 is a regular value for the function $\mathbf{R}^{n+1} \to \mathbf{R}$ sending p to $|p|^2$.

Exercise 5.4.7

Show that the map

$$SL_2(\mathbf{R}) \to (\mathbf{C} \setminus \{0\}) \times \mathbf{R}$$
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto (a + ic, ab + cd)$$

is a diffeomorphism, and that $S^1 \times \mathbf{R}$ is diffeomorphic to $\mathbf{C} \setminus \{0\}$.

Exercise 5.4.8

Calculate the Jacobi matrix of the determinant function. With some choice of indices you should get

$$D_{ij}(\det)(A) = (-1)^{i+j} \det(A_{ij})$$

where A_{ij} is the matrix you get by deleting the ith row and the jth column. If the determinant is to be one, some of the entries in the Jacobi matrix then has got to be nonzero.

Exercise 5.4.10

By Corollary 6.5.12 we identify $TO(n)$ with

$$E = \left\{ (g, A) \in O(n) \times M_n(\mathbf{R}) \;\middle|\; \begin{array}{c} g = \gamma(0) \\ A = \gamma'(0) \\ \text{for some curve} \\ \gamma\colon (-\epsilon, \epsilon) \to O(n) \end{array} \right\}$$

That $\gamma(s) \in O(n)$ is equivalent to saying that $I = \gamma(s)^t \gamma(s)$. This holds for all $s \in (-\epsilon, \epsilon)$, so we may derive this equation and get

$$\begin{aligned} 0 &= \left.\frac{d}{ds}\right|_{s=0} \left(\gamma(s)^t \gamma(s)\right) \\ &= \gamma'(0)^t \gamma(0) + \gamma(0)^t \gamma'(0) \\ &= A^t g + g^t A \end{aligned}$$

Exercise 5.4.12

Consider the chart $x\colon M_2(\mathbf{R}) \to \mathbf{R}^4$ given by

$$x\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a, b, a - d, b + c).$$

Exercise 5.4.13

Copy one of the proofs for the orthogonal group, replacing the symmetric matrices with Hermitian matrices.

Exercise 5.4.14

The space of orthogonal matrices is compact since it is a closed subset of $[-1,1]^{n^2}$. It has at least two components since the sets of matrices with determinant 1 is closed, as is the complement: the set with determinant -1.

Each of these are connected since you can get from any rotation to the identity through a path of rotations. One way to see this is to use the fact from linear algebra which says that any element $A \in SO(n)$ can be written in the form $A = BTB^{-1}$ where B and T are orthogonal, and furthermore T is a block diagonal matrix where the block matrices are either a single 1 on the diagonal, or of the form

$$T(\theta_k) = \begin{bmatrix} \cos\theta_k & -\sin\theta_k \\ \sin\theta_k & \cos\theta_k \end{bmatrix}.$$

So we see that replacing all the θ_k's by $s\theta_k$ and letting s vary from 1 to 0 we get a path from A to the identity matrix.

Exercise 5.4.18

Consider a k-frame as a matrix A with the property that $A^tA = I$, and proceed as for the orthogonal group.

Exercise 5.4.20

Either just solve the equation or consider the map

$$f\colon P_3 \to P_2$$

sending $y \in P_3$ to $f(y) = (y'')^2 - y' + y(0) + xy'(0) \in P_2$. If you calculate the Jacobian in obvious coordinates you get that

$$Df(a_0 + a_1x + a_2x^2 + a_3x^3) = \begin{bmatrix} 1 & -1 & 8a_2 & 0 \\ 0 & 1 & 24a_3 - 2 & 24a_2 \\ 0 & 0 & 0 & 72a_3 - 3 \end{bmatrix}$$

The only way this matrix can be singular is if $a_3 = 1/24$, but the top coefficient in $f(a_0+a_1x+a_2x^2+a_3x^3)$ is $36a_3^2 - 3a_3$ which won't be zero if $a_3 = 1/24$. By the way, if I did not calculate something wrong, the solution is the disjoint union of two manifolds $M_1 = \{2t(1-2t)+2tx+tx^2 | t \in \mathbf{R}\}$ and $M_2 = \{-24t^2 + tx^2 + x^3/12 | t \in \mathbf{R}\}$, both diffeomorphic to $\mathbf{R}$.

Exercise 5.4.21

Yeah.

Exercise 5.4.22

Consider the function

$$f\colon \mathbf{R}^n \to \mathbf{R}$$

given by

$$f(p) = p^tAp.$$

The Jacobi matrix is easily calculated, and using that A is symmetric we get that $Df(p) = 2p^tA$. But given that $f(p) = b$ we get that $Df(p)\cdot p = p^tAp = b$, and so $Df(p) \neq 0$ if $b \neq 0$. Hence all values but $b = 0$ are regular. The value $b = 0$ is critical since $0 \in f^{-1}(0)$ and $Df(0) = 0$.

Exercise 5.4.23

You don't actually need theorem 5.4.3 to prove this since you can isolate T in this equation, and show directly that you get a submanifold diffeomorphic to $\mathbf{R}^2$, but still, as an exercise you should do it by using theorem 5.4.3.

Exercise 5.4.24

Code a flexible n-gon by means of a vector $x^0 \in \mathbf{R}^2$ giving the coordinates of the first point, and vectors $x^i \in S^1$ going from point i to point $i+1$ for $i = 1, \dots, n-1$ (the vector from point n to point 1 is not needed, since it will be given by the requirement that the curve is closed). The set $\mathbf{R}^2 \times (S^1)^{n-1}$ will give a flexible n-gon, except, that the last line may not be of length 1. To ensure this, look at the map

$$f \colon \mathbf{R}^k \times (S^{k-1})^{n-1} \to \mathbf{R}$$
$$(x^0, (x^1, \dots, x^{n-1})) \mapsto \left|\sum_{i=1}^{n-1} x^i\right|^2$$

and show that 1 is a regular value. If you let $x^j = e^{i\phi_j}$ and $x = (x^0, (x^1, \dots, x^{n-1}))$, you get that

$$\begin{aligned} D_j f(x) &= D_j \left((\sum_{k=1}^{n-1} e^{i\phi_k})(\sum_{k=1}^{n-1} e^{-i\phi_k}) \right) \\ &= ie^{i\phi_j}(\sum_{k=1}^{n-1} e^{-i\phi_k}) + (\sum_{k=1}^{n-1} e^{i\phi_k})(-ie^{-i\phi_j}) \\ &= -2Im\left(e^{i\phi_j}(\sum_{k=1}^{n-1} e^{-i\phi_k}) \right) \end{aligned}$$

That the rank is not 1 is equivalent to $D_j f(x) = 0$ for all j. Analyzing this, we get that $x_1, \dots, x_{n-1}$ must then all be parallel. But this is impossible if n is odd and $\left|\sum_{i=1}^{n-1} x^i\right|^2 = 1$. (Note that this argument fails for n even. If $n = 4$ $LF_{4,2}$ is not a manifold: given x^1 and x^2 there are two choices for x^3 and x^4: (either $x^3 = -x^2$ and $x^4 = -x^1$ or $x^3 = -x^1$ and $x^4 = -x^2$), but when $x^1 = x^2$ we get a crossing of these two choices).

Exercise 5.4.25

The non-self-intersecting flexible n-gons form an open subset.

Exercise 5.6.2

It is enough to prove that for any point $p \in U$ there is a point $q \in U$ with all coordinates rational and a rational r such that $p \in C \subseteq U$ with C the closed ball with center q and radius r. Since U is open, there is an $\epsilon > 0$ such that the open ball with center p and radius ϵ is within U. Since $\mathbf{Q}^n \subseteq \mathbf{R}^n$ is dense we may choose $r \in \mathbf{Q}$ and $q \in \mathbf{Q}^n$ such that $|q - p| < r < \epsilon/2$.

Exercise 5.6.3

Let $\{C_i\}_{i\in\mathbf{N}}$ be a countable collection of measure zero sets, let $\epsilon < 0$ and for each $i \in \mathbf{N}$ choose a sequence of cubes $\{C_{ij}\}_{j\in\mathbf{N}}$ with $C_i \subseteq \bigcup_{j\in\mathbf{N}} C_{ij}$ and $\sum_{j\in\mathbf{N}} \text{volume}(C_{ij}) < \epsilon/2^i$.

Exercise 5.6.4

By Exercise 5.6.2 we may assume that C is contained in a closed ball contained in U. Choosing $\epsilon > 0$ small enough, a covering of C by closed balls $\{C_i\}$ whose sum of volumes is less than ϵ will also be contained in a closed ball K contained in U. Now, the mean value theorem assures that there is a positive number M such that $|f(a) - f(b)| \leq M|a - b|$ for $a, b \in K$. Hence, f sends closed balls of radius r into closed balls of radius Mr, and $f(C)$ is covered by closed balls whose sum of volumes is less than $M\epsilon$.

Note the crucial importance of the mean value theorem. The corresponding statements are false if we just assume our maps are continuous.

Exercise 5.6.6

Since $[0, 1]$ is compact, we may choose a finite subcover. Excluding all subintervals contained in another subinterval we can assure that no point in $[0, 1]$ lies in more than two subintervals (the open cover $\{[0, 1), (0, 1]\}$ shows that 2 is attainable).

Exercise 5.6.7

It is enough to do the case where C is compact, and we may assume that $C \subseteq [0,1]^n$. Let $\epsilon > 0$. Given $t \in [0,1]$, let $d_t\colon C \to \mathbf{R}$ be given by $d_t(t_1,\dots,t_n) = |t_n - t|$ and let $C^t = d_t^{-1}(0)$. Choose a cover $\{B_i^t\}$ of C^t by open cubes whose sum of volumes is less than $\epsilon/2$. Let $J_t\colon \mathbf{R}^{n-1} \to \mathbf{R}^n$ be given by $J_t(t_1,\dots,t_{n-1}) = (t_1,\dots,t_{n-1},t)$ and $B^t = J_t^{-1}(\bigcup_i B_i^t)$. Since C is compact and B^t is open, d_t attains a minimum value $m_t > 0$ outside $B^t \times \mathbf{R}$, and so $d_t^{-1}(-m_t, m_t) \subseteq B^t \times I_t$, where $I_t = (t - m_t, t + m_t) \cap [0,1]$. By Exercise 5.6.6, there is a is a finite collection $\{t_1,\dots,t_k\} \in [0,1]$ such that the I_{t_j} cover $[0,1]$ and such that the sum of the diameters in less than 2. From this we get the cover of C by rectangles $\{B_i^{t_j} \times I_{t_j}\}_{j=1,\dots,k,i\in\mathbf{N}}$ whose sum of volumes is less than $\epsilon = 2\epsilon/2$.

Exercise 5.6.8

Use the preceding string of exercises.

Exercise 5.6.9

Let $C' = C_0 - C_1$. We may assume that $C' \neq \emptyset$ (excluding the case $m \leq 1$). If $p \in C'$, there is a nonzero partial derivative of f at p, and by permuting the coordinates, we may just as well assume that $D_1 f(p) \neq 0$. By the inverse function theorem, the formula

$$x(q) = (f_1(q), q_2, \dots, q_m)$$

defines a chart $x\colon V \to V'$ in a neighborhood V of p, and it suffices to prove that $g(K)$ has measure zero, where $g = fx^{-1}\colon V' \to \mathbf{R}^n$ and K is the set of critical points for g. Now, $g(q) = (q_1, g_2(q), \dots, g_n(q))$, and writing $g_k^{q_1}(q_2,\dots,q_m) = g_k(q_1,\dots,q_m)$ for $k = 1,\dots,n$, we see that since

$$Dg(q_1,\dots,q_m) = \begin{bmatrix} 1 & 0 \\ ? & Dg^{q_1}(q_2,\dots,q_m) \end{bmatrix}$$

the point $(q_1,\dots,q_m)$ is a critical point for g if and only if $(q_2,\dots,q_m)$ is a critical point for g^{q_1}. By the induction hypothesis, for each q_1 the set of critical values for g^{q_1} has measure zero. By Fubini's theorem 5.6.7, we are done.

Exercise 5.6.10

The proof is similar to that of Exercise 5.6.9, except that the chart x is defined by

$$x(q) = (D_{k_1} \dots D_{k_i} f(q), q_2, \dots, q_m),$$

where we have assumed that $D_1 D_{k_1} \dots D_{k_i} f(p) \neq 0$ (but, of course $D_{k_1} \dots D_{k_i} f(p) = 0$).

Exercise 5.6.11

By Exercise 5.6.2 U is a countable union of closed cubes (balls or cubes have the same proof), so it is enough to show that $f(K \cap C_k)$ has measure zero, where K is a closed cube with side s. Since all partial derivatives of order less than or equal to k vanish on C_k, Taylor expansion gives that there is a number M such that

$$|f(a) - f(b)| \leq M|a-b|^{k+1}$$

for all $a \in K \cap C_k$ and $b \in K$. Subdivide K into N^m cubes $\{K_{ij}\}_{i,j=1,\dots,N}$ with sides s/N for some positive integer. If $a \in C_k \cap K_{ij}$, then $f(K_{ij})$ lies in a closed ball centered at $f(a)$ with radius $M(\sqrt{m} \cdot s/N)^{k+1}$. Consequently, $f(K \cap C_k)$ lies in a union of closed balls with volume sum less than or equal to $N^m \cdot \frac{4\pi\left(M(\sqrt{m}\cdot s/N)^{k+1}\right)^n}{3} = \frac{4\pi\left(M(\sqrt{m}\cdot s)^{k+1}\right)^n}{3} N^{m-n(k+1)}$. If $nk \geq m$, this tends to zero as N tends to infinity, and we are done.

Exercise 5.6.12

By induction on m. When $m = 0$, $\mathbf{R}^m$ is a point and the result follows. Assume Sard's theorem is proven in dimension less than $m > 0$. Then the Exercises 5.6.8, 5.6.9, 5.6.10 and 5.6.11 together prove Sard's theorem in dimension m.

Exercise 5.7.2

It is clearly injective, and an immersion since it has rank 1 everywhere. It is not an imbedding since $\mathbf{R} \coprod \mathbf{R}$ is disconnected, whereas the image is connected.

Exercise 5.7.3

It is clearly injective, and immersion since it has rank 1 everywhere. It is not an imbedding since an open set containing a point z with $|z| = 1$ in the image must contain elements in the image of the first summand.

Exercise 5.7.6

If a/b is irrational then the image of $f_{a,b}$ is dense: that is any open set on $S^1 \times S^1$ intersects the image of $f_{a,b}$.

Exercise 5.7.7

Show that it is an injective immersion homeomorphic to its image. The last property follows since both the maps in

$$M \longrightarrow i(M) \longrightarrow ji(M)$$

are continuous and bijective and the composite is a homeomorphism.

Exercise 9.4.10

Prove that the diagonal $M \to M \times M$ is an imbedding by proving that it is an immersion inducing a homeomorphism onto its image. The tangent space of the diagonal at (p, p) is exactly the diagonal of $T_{(p,p)}(M \times M) \cong T_pM \times T_pM$. For any vector space V, the quotient space $V \times V/$diagonal is canonically isomorphic to V via the map given by sending $(v_1, v_2) \in V \times V$ to $v_1 - v_2 \in V$.

Chapter 6

Exercise 6.1.3

For the first case, you may assume that the regular value in question is 0. Since zero is a regular value, the derivative in the "fiber direction" has got to be nonzero, and so the values of f are positive on one side of the zero section... but there IS no "one side" of the zero section! This takes care of all one-dimensional cases, and higher dimensional examples are excluded since the map won't be regular if the dimension increases.

Exercise 6.2.3

See the next exercise. This refers the problem away, but the same reference helps you out on this one too!

Exercise 6.2.4

This exercise is solved in the smooth case in exercise 6.3.15. The only difference in the continuous case is that you delete every occurrence of "smooth" in the solution. In particular, the solution refers to a "smooth bump function $\phi\colon U_2 \to \mathbf{R}$ such that ϕ is one on (a, c) and zero on $U_2 \setminus (a, d)$". This can in our case be chosen to be the (non smooth) map $\phi\colon U_2 \to \mathbf{R}$ given by

$$\phi(t) = \begin{cases} 1 & \text{if } t \le c \\ \frac{d-t}{d-c} & \text{if } c \le t \le d \\ 0 & \text{if } t \ge d \end{cases}$$

Exercise 6.4.4

Use the chart domains on $\mathbf{R}\mathrm{P}^n$ from the manifold section:

$$U^k = \{[p] \in \mathbf{R}\mathrm{P}^n | p_k \neq 0\}$$

and construct bundle charts $\pi^{-1}(U^k) \to U^k \times \mathbf{R}$ sending $([p], \lambda p)$ to $([p], \lambda p_k)$. The chart transformations then should look something like

$$([p], \lambda) \mapsto ([p], \lambda \frac{p_l}{p_k})$$

If the bundle were trivial, then $\eta_n \setminus \sigma_0(\mathbf{R}\mathrm{P}^n)$ would be disconnected. In particular $([e_1], e_1)$ and $([e_1], -e_1)$ would be in different components. But $\gamma\colon [0,\pi] \to \eta_n \setminus \sigma_0(\mathbf{R}\mathrm{P}^n)$ given by

$$\gamma(t) = ([\cos(t)e_1 + \sin(t)e_2], \cos(t)e_1 + \sin(t)e_2)$$

is a path connecting them.

Exercise 6.3.13

Check locally by using charts: if (h, U) is a bundle chart, then the resulting square

$$\begin{array}{ccc} E|U & \xrightarrow[\cong]{h} & U \times \mathbf{R}^k \\ {\scriptstyle a_E}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_U \times a\cdot} \\ E|U & \xrightarrow[\cong]{h} & U \times \mathbf{R}^k \end{array}$$

commutes.

Exercise 6.3.14

Smoothen up the proof you gave for the same question in the vector bundle chapter, or use parts of the solution of exercise 6.3.15.

Exercise 6.3.15

Let $\pi\colon E \to S^1$ be a one-dimensional smooth vector bundle (one-dimensional smooth vector bundles are frequently called *line bundles*). Since S^1 is compact we may choose a finite bundle atlas, and we may remove superfluous bundle charts, so that no domain is included in another. We may also assume that all chart domains are connected. If there is just one bundle chart, we are finished, otherwise we proceed as follows. If we start at some point, we may order the charts, so that they intersect in a nonempty interval (or a disjoint union of two intervals if there are exactly two charts). Consider two consecutive charts (h_1, U_1) and (h_2, U_2) and let (a, b) be (one of the components of) their intersection. The transition function

$$h_{12}\colon (a,b) \to \mathbf{R} \setminus \{0\} \cong \mathrm{GL}_1(\mathbf{R})$$

must take either just negative or just positive values. Multiplying h_2 by the sign of h_{12} we get a situation where we may assume that h_{12} always is positive. Let $a < c < d < b$, and choose a smooth bump function $\phi\colon U_2 \to \mathbf{R}$ such that ϕ is one on (a, c) and zero on $U_2 \setminus (a, d)$. Define a new chart (h_2', U_2) by letting

$$h_2'(t) = \left(\frac{\phi(t)}{h_{12}(t)} + 1 - \phi(t)\right) h_2(t)$$

(since $h_{12}(t) > 0$, the factor by which we multiply $h_2(t)$ is never zero). On (a, c) the transition function is now constantly equal to one, so if there were more than two charts we could merge our two charts into a chart with chart domain $U_1 \cup U_2$.

So we may assume that there are just two charts. Then we may proceed as above on one of the components of the intersection between the two charts, and get the transition function to be the identity. But then we would not be left with the option of multiplying with the sign of the transition function on the other component. However, by the same method, we could only make it plus or minus one, which exactly correspond to the trivial bundle and the unbounded Möbius band.

Just the same argument shows that there are exactly two isomorphism types of n-dimensional smooth vector bundles over S^1 (using that $\mathrm{GL}_n(\mathbf{R})$ has exactly two components). The same argument also gives the corresponding topological fact.

Exercise 6.4.5

You may assume that $p = [0, \dots, 0, 1]$. Then any point $[x_0, \dots, x_{n-1}, x_n] \in X$ equals $\left[\frac{x}{|x|}, \frac{x_n}{|x|}\right]$ since $x = (x_0, \dots, x_{n-1})$ must be different from 0. Consider the map

$$\begin{aligned} X &\to \eta_{n-1} \\ [x, x_n] &\mapsto \left(\left[\frac{x}{|x|}\right], \frac{x_n x}{|x|^2}\right) \end{aligned}$$

with inverse $([x], \lambda x) \mapsto [x, \lambda]$.

Exercise 6.5.8

View S^3 as the unit quaternions, and copy the argument for S^1.

Exercise 6.5.9

Lie group is a smooth manifold equipped with a smooth associative multiplication, having a unit and possessing all inverses, so the proof for S^1 will work.

Exercise 6.5.13

If we set $z_j = x_j + iy_j$, $x = (x_0, \dots, x_n)$ and $y = (y_0, \dots, y_n)$, then $\sum_{i=0}^n z^2 = 1$ is equivalent to $x \cdot y = 0$ and $|x|^2 - |y|^2 = 1$. Use this to make an isomorphism to the bundle in example 6.5.10 sending the point (x, y) to $(p, v) = (\frac{x}{|x|}, y)$ (with inverse sending (p, v) to $(x, y) = (\sqrt{1 + |v|^2}p, v)$).

Exercise 6.5.15

Consider the isomorphism

$$TS^n \cong \{(p, v) \in S^n \times \mathbf{R}^{n+1} \mid p \cdot v = 0\}.$$

Any path $\bar{\gamma}$ in $\mathbf{R}\mathrm{P}^n$ through $[p]$ lifts uniquely to a path γ trough p and to the corresponding path $-\gamma$ through $-p$.

Exercise 6.5.17

Use the trivialization to pass the obvious solution on the product bundle to the tangent bundle.

Exercise 6.5.18

Any curve to a product is given uniquely by its projections to the factors.

Exercise 6.5.20

Let $x\colon U \to U'$ be a chart for M. Show that the assignment sending an element $(q, v_1, v_2, v_{12}) \in U' \times \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^n$ to

$$[t \mapsto [s \mapsto x^{-1}(q + tv_1 + sv_2 + stv_{12})]] \in T(TU)$$

gives an isomorphism

$$U' \times \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^n \cong T(TU),$$

so that all elements in $T(TM)$ are represented by germs of surfaces. Check that the equivalence relation is the one given in the exercise so that the resulting isomorphisms $T(TU) \cong E|_U$ give smooth bundle charts for E.

Exercise 6.6.1

Check that each of the pieces of the definition of a pre-vector bundle are accounted for.

Exercise 6.6.7

Choose a chart (x, U) with $p \in U$ and write out the corresponding charts on T^*M and $T^*(T^*M)$ to check smoothness. It may be that you will find it easier to think in terms of the "dual bundle" $(TM)^*$ rather than the isomorphic cotangent bundle T^*M and avoid the multiple occurrences of the isomorphism α, but strictly speaking the dual bundle will not be introduced till example **??**.

Exercise 5.7.9

Prove that the diagonal $M \to M \times M$ is an imbedding by proving that it is an immersion inducing a homeomorphism onto its image. The tangent space of the diagonal at (p, p) is exactly the diagonal of $T_{(p,p)}(M \times M) \cong T_pM \times T_pM$.

Exercise 5.7.10

Show that the map

$$f \times g\colon M \times L \to N \times N$$

is transverse to the diagonal (which is discussed in exercise 5.7.9), and that the inverse image of the diagonal is exactly $M \times_N L$.

Exercise 5.7.11

Since π is a submersion, Exercise 5.7.10 shows that $E \times_M N \to N$ is smooth. If $(e,n) \in E \times_M N$, then a tangent vector in $T_{(e,n)} E \times_M N$ is represented by a curve $\gamma = (\gamma_E, \gamma_N)\colon J \to E \times_M N$ with $\pi\gamma_E = f\gamma_N$ with $\gamma_E(0) = e$ and $\gamma_N(0) = n$. In effect, this shows that the obvious map $T_{(e,p)}(E \times_M N) \to T_e E \times_{T_{\pi(e))}M} T_n N$ is an isomorphism. Furthermore, since $T_e E \to T_{\pi(e)} M$ is a surjection, so is the projection $T_e E \times_{T_{\pi(e)}M} T_n N \to T_n N$.

Chapter 7

Exercise 7.1.4

As an example, consider the open subset $U^{0,0} = \{e^{i\theta} \in S^1 | \cos\theta > 0\}$. The bundle chart $h\colon U^{0,0} \times \mathbf{C} \to U^{0,0} \times \mathbf{C}$ given by sending $(e^{i\theta}, z)$ to $(e^{i\theta}, e^{-i\theta/2} z)$. Then $h((U^{0,0} \times \mathbf{C}) \cap \eta_1) = U^{0,0} \times \mathbf{R}$. Continue this way all around the circle. The idea is the same for higher dimensions: locally you can pick the first coordinate to be on the line $[p]$.

Exercise 7.1.11

Let $X_k = \{p \in X | rk_p f = k\}$. We want to show that X_k is both open and closed, and hence either empty or all of X since X is connected.

Let $P = \{A \in M_m(A) | A = A^2\}$, then $P_k = \{A \in P | rk(A) = k\} \subseteq P$ is open. To see this, write P_k as the intersection of P with the two open sets

$$\{A \in M_n(\mathbf{R}) | rkA \geq k\}$$

and

$$\left\{ A \in M_n(\mathbf{R}) \;\middle|\; \begin{matrix} A \text{ has less than} \\ \text{or equal to } k \\ \text{linearly independent} \\ \text{eigen vectors with} \\ \text{eigenvalue } 1 \end{matrix} \right\}.$$

But, given a bundle chart (h, U), then the map

$$U \xrightarrow{p \mapsto h_p f_p h_p^{-1}} P$$

is continuous, and hence $U \cap X_k$ is open in U. Varying (h, U) we get that X_k is open, and hence also closed since $X_k = X \setminus \bigcup_{i \neq k} X_i$.

Exercise 7.1.12

Use Exercise 7.1.11 to show that the bundle map $\frac{1}{2}(id_E - f)$ has constant rank (here we use that the set of bundle morphisms is in an obvious way a vector space).

Exercise 7.1.13

Identifying $T\mathbf{R}$ with $\mathbf{R} \times \mathbf{R}$ in the usual way, we see that f corresponds to $(p, v) \mapsto (p, p)$ which is a nice bundle morphism, but $\coprod_p \ker\{v \mapsto pv\} = \{(p, v) \,|\, p \cdot v = 0\}$ and $\coprod_p \operatorname{Im}\{v \mapsto pv\} = \{(p, pv)\}$ are not bundles.

Exercise 7.1.14

The tangent map $Tf\colon TE \to TM$ is locally the projection $U \times \mathbf{R}^k \times \mathbf{R}^n \times \mathbf{R}^k \to U \times \mathbf{R}^n$ sending (p, u, v, w) to (p, v), and so has constant rank. Hence Corollary 7.1.10 gives that $V = \ker\{Tf\}$ is a subbundle of $TE \to E$.

Exercise 7.2.5

$A \times_X E = \pi^{-1}(A)$.

Exercise 7.2.6

This is not as complex as it seems. For instance, the map $\tilde{E} \to f^* E = X' \times_X E$ must send e to $(\tilde{\pi}(e), g(e))$ for the diagrams to commute.

Exercise 7.2.7

If $h\colon E \to X \times \mathbf{R}^n$ is a trivialization, then the map $f^* E = Y \times_X E \to Y \times_X (X \times \mathbf{R}^n)$ induced by h is a trivialization, since $Y \times_X (X \times \mathbf{R}^n) \to Y \times \mathbf{R}^n$ sending $(y, (x, v))$ to (y, v) is a homeomorphism.

Exercise 7.2.8

$X \times_Y (Y \times_Z E) \cong X \times_Z E$.

Exercise 7.2.9

The map

$$E \setminus \sigma_0(X) \to \pi_0^* E = (E \setminus \sigma_0(X)) \times_X E$$

sending v to (v, v) is a nonvanishing section.

Exercise 7.3.3

The transition functions will be of the type $U \mapsto \mathrm{GL}_{n_1+n_2}(\mathbf{R})$, which sends $p \in U$ to the block matrix

$$\begin{bmatrix} (h_1)_p(g_1)_p^{-1} & 0 \\ 0 & (h_2)_p(g_2)_p^{-1} \end{bmatrix}$$

which is smooth if each of the blocks are smooth. More precisely, the transition function is a composite of three smooth maps

1. the diagonal $U \to U \times U$,
2. the product

   $$U \times U \longrightarrow \mathrm{GL}_{n_1}(\mathbf{R}) \times \mathrm{GL}_{n_2}(\mathbf{R})$$

   of the transition functions, and
3. the block sum $(A, B) \mapsto \left[\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right]$

   $$\mathrm{GL}_{n_1}(\mathbf{R}) \times \mathrm{GL}_{n_2}(\mathbf{R}) \to \mathrm{GL}_{n_1+n_2}(\mathbf{R})$$

Similarly for the morphisms.

Exercise 7.3.5

Use the map $\epsilon \to S^n \times \mathbf{R}$ sending $(p, \lambda p)$ to (p, λ).

Exercise 7.3.7

Consider $TS^n \oplus \epsilon$, where ϵ is gotten from Exercise 7.3.5. Construct a trivialization $TS^n \oplus \epsilon \to S^n \times \mathbf{R}^{n+1}$.

Exercise 7.3.8

$$\epsilon_1 \oplus \epsilon_2 \xrightarrow[\cong]{h_1 \oplus h_2} X \times (\mathbf{R}^{n_1} \oplus \mathbf{R}^{n_2})$$

$$(E_1 \oplus E_2) \oplus (\epsilon_1 \oplus \epsilon_2) \cong (E_1 \oplus \epsilon_1) \oplus (E_2 \oplus \epsilon_2)$$

Exercise 7.3.9

Given f_1 and f_2, let $f\colon E_1 \oplus E_2 \to E_3$ be given by sending $(v, w) \in \pi_1^{-1}(p) \oplus \pi_2^{-1}(p)$ to $f_1(v) + f_2(w) \in \pi_3^{-1}(p)$. Given f let $f_1(v) = f(v, 0)$ and $f_2(w) = f(0, w)$.

Exercise 7.4.4

Send the bundle morphism f to the section which to any $p \in X$ assigns the linear map $f_p\colon E_p \to E'_p$.

Exercise 7.4.5

For the bundle morphisms, you need to extend the discussion in Example 7.4.3 slightly and consider the map $\mathrm{Hom}(V_1, V_2) \times \mathrm{Hom}(V_3, V_4) \to \mathrm{Hom}(\mathrm{Hom}(V_2, V_3), \mathrm{Hom}(V_1, V_4))$ obtained by composition.

Exercise 7.4.6

Let $F \subseteq E$ be a k-dimensional subbundle of the n-dimensional vector bundle $\pi\colon E \to X$. Define as a set

$$E/F = \coprod_{p \in X} E_p/F_p$$

with the obvious projection $\bar{\pi}\colon E/F \to X$. The bundle atlas is given as follows. For $p \in X$ choose bundle chart $h\colon \pi^{-1}(U) \to U \times \mathbf{R}^n$ such that $h(\pi^{-1}(U) \cap F) = U \times \mathbf{R}^k \times \{0\}$. On each fiber this gives a linear map on the quotient $\bar{h}_p\colon E_p/F_p \to \mathbf{R}^n/\mathbf{R}^k \times \{0\}$ via the formula $\bar{h}_p(\bar{v}) = \overline{h_p(v)}$ as in 2. This gives a function

$$\begin{aligned} \bar{h}\colon (\bar{\pi})^{-1}(U) = \coprod_{p \in U} E_p/F_p \\ \to \coprod_{p \in U} \mathbf{R}^n/\mathbf{R}^k \times \{0\} \\ \cong U \times \mathbf{R}^n/\mathbf{R}^k \times \{0\} \\ \cong U \times \mathbf{R}^{n-k}. \end{aligned}$$

You then have to check that the transition functions $p \mapsto \bar{g}_p \bar{h}_p^{-1} = \overline{g_p h_p^{-1}}$ are continuous (or smooth).

As for the map of quotient bundles, this follows similarly: define it on each fiber and check continuity of "up, over and down".

Exercise 7.4.8

Just write out the definition.

Exercise 7.4.9

First recall the trivializations we used to define the tangent and cotangent bundles. Given a chart (x, U) for M we have trivializations

$$(TM)_U \cong U \times \mathbf{R}^n$$

sending $[\gamma] \in T_pM$ to $(\gamma(0), (x\gamma)'(0))$ and

$$(T^*M)_U \cong U \times (\mathbf{R}^n)^*$$

sending $d\phi \in T_p^*M$ to $(p, D(\phi x^{-1})(x(p))\cdot) \in U \times (\mathbf{R}^n)^*$. The bundle chart for the tangent bundle has inverse

$$U \times \mathbf{R}^n \cong (TM)_U$$

given by sending (p, v) to $[t \mapsto x^{-1}(x(p)+vt)] \in T_pM$ which gives rise to the bundle chart

$$(TM)^*_U \cong U \times (\mathbf{R}^n)^*$$

on the dual, sending $f \in (T_pM)^*$ to

$$(p, v \mapsto f([t \mapsto x^{-1}(x(p) + vt)]).$$

The exercise is (more than) done if we show that the diagram

$$\begin{array}{ccc}
(T^*M)_U & \xrightarrow{d\phi \mapsto \{[\gamma] \mapsto (\phi\gamma)'(0)\}} & (TM)^*_U \\
\cong\downarrow & & \cong\downarrow \\
U \times (\mathbf{R}^n)^* & = & U \times (\mathbf{R}^n)^*
\end{array}$$

commutes, which it does since if we start with $d\phi \in T_p^*M$ in the upper left hand corner, we end up with $D(\phi x^{-1})(x(p))\cdot$ either way (check that the derivative at $t = 0$ of $\phi x^{-1}(x(p) + vt)$ is $D(\phi x^{-1})(x(p)) \cdot v$).

Exercise 7.4.10

The procedure is just as for the other cases. Let $SB(E) = \coprod_{p \in X} SB(E_p)$. If (h, U) is a bundle chart for $E \to X$ define a bundle chart $SB(E)_U \to U \times SB(\mathbf{R}^k)$ by means of the composite

$$\begin{array}{ccc}
SB(E)_U & = & \coprod_{p \in U} SB(E_p) \\
 & & \downarrow \coprod SB(h_p^{-1}) \\
U \times SB(\mathbf{R}^k) & = & \coprod_{p \in U} SB(\mathbf{R}^k).
\end{array}$$

Exercise 7.4.12

$\mathrm{Alt}^k(E) = \coprod_{p \in X} \mathrm{Alt}^k E_p$ and so on.

Exercise 7.4.13

The transition functions on $L \to M$ are maps into nonzero real numbers, and on the tensor product this number is squared, and so all transition functions on $L \otimes L \to M$ map into positive real numbers.

Exercise 7.6.1

The conditions you need are exactly the ones fulfilled by the elementary definition of the determinant: check your freshman introduction.

Exercise 7.7.1

Check out e.g., [9] page 59.

Exercise 7.7.3

Check out e.g., [9] page 60.

Chapter 8

Exercise 8.1.4

Check the two defining properties of a flow. As an aside: this flow could be thought of as the flow $\mathbf{R} \times \mathbf{C} \to \mathbf{C}$ sending (t, z) to $e^{-z-t/2}$, which obviously satisfies the two conditions.

Exercise 8.1.9

Symmetry ($\Phi(0,p) = p$) and reflexivity ($\Phi(-t,\Phi(t,p)) = p$) are obvious, and transitivity follows since if

$$p_{i+1} = \Phi(t_i, p_i), \qquad i = 0, 1$$

then

$$p_2 = \Phi(t_1, p_1) = \Phi(t_1, \Phi(t_0, p_0)) = \Phi(t_1 + t_0, p_0)$$

Exercise 8.1.11

i) Flow lines are constant. ii) All flow lines outside the origin are circles. iii) All flow lines outside the origin are rays flowing towards the origin.

Exercise 8.1.20

Consider one of the "bad" injective immersions that fail to be imbeddings, and force a discontinuity on the velocity field.

Exercise 8.2.4

Consider a bump function ϕ on the sphere which is 1 near the North pole and 0 near the South pole. Consider the vector field $\vec{\Phi} = \phi\vec{\Phi}_N + (1-\phi)\vec{\Phi}_S$. Near the North pole $\vec{\Phi} = \vec{\Phi}_N$ and near the South pole $\vec{\Phi} = \vec{\Phi}$, and so the flow associated with $\vec{\Phi}$ has the desired properties (that t is required to be small secures that we do not flow from one one pole to another).

Exercise 8.2.5

The vector field associated with the flow

$$\Phi\colon \mathbf{R} \times (S^1 \times S^1) \to (S^1 \times S^1)$$

given by $\Phi(t,(z_1,z_2)) = (e^{iat}z_1, e^{ibt}z_2)$ exhibits the desired phenomena when varying the real numbers a and b.

Exercise 8.2.6

All we have to show is that X is the velocity field of Φ. Under the diffeomorphism

$$\begin{aligned} TO(n) &\to E \\ [\gamma] &\to (\gamma(0), \gamma'(0)) \end{aligned}$$

this corresponds to the observation that

$$\left.\frac{\partial}{\partial s}\right|_{s=0} \Phi(s,g) = gA.$$

Exercise 8.4.3

Do a variation of example 8.3.5.

Exercise 8.5.3

There is no hint beyond: use the definitions!

Exercise 8.5.4

Use the preceding exercise: notice that $T\pi_M\xi = \pi_{TM}\xi$ is necessary for things to make sense since $\ddot{\gamma}$ had two repeated coordinates.

Chapter 9

Exercise 9.2.8

The conditions the sections have to satisfy are "convex", see the proof of existence of fiber metrics Theorem 9.3.5 in the next section.

Exercise 9.2.9

The thing to check is that $\mathcal{T}$ is an open neighborhood of the zero section.

Exercise 9.2.3

Consider a partition of unity $\{\phi_i\}_{i\in\mathbf{N}}$ as displayed in the proof of theorem 9.2.2 where $\mathrm{supp}(\phi) = x_i^{-1}E(2)$ for a chart (x_i, U_i) for each i. Let f_i be the composite

$$E(2) \overset{\cong}{\to} x_i^{-1}(E(2)) = \mathrm{supp}(\phi_i) \xrightarrow{f|_{\mathrm{supp}(\phi_i)}} \mathbf{R}$$

and choose a polynomial g_i such that $|f_i(x) - g_i(x)| < \epsilon$ for all $x \in E(2)$. Let $g(p) = \sum_i \phi_i(p) g_i(x_i(p))$ which is gives a well defined and smooth map. Then

$$\begin{aligned}
|f(p) - g(p)| &= |\sum_i \phi_i(p)(f(p) - g_i(x_i(p)))| \\
&= |\sum_i \phi_i(p)(f_i(x_i(p)) - g_i(x_i(p)))| \\
&\leq \sum_i \phi_i(p)|f_i(x_i(p)) - g_i(x_i(p))| \\
&< \sum_i \phi_i(p)\epsilon = \epsilon.
\end{aligned}$$

Exercise 9.2.4

By Exercise 7.4.13, all the transition functions $U \cap V \to \mathrm{GL}_1(\mathbf{R})$ in the associated bundle atlas on $L \otimes L \to M$ have values in the positive real numbers. Use partition of unity to glue these together and scale them to be the constantly 1.

Exercise 9.4.8

Use lemma 9.4.2 to show that the bundle in question is isomorphic to $(T\mathbf{R}^n)|_M \to M$.

Exercise 9.4.9

You have done this exercise before!

Exercise 9.5.10

Since $f\Phi_i(t, q') = f(q') + e_i t$ for all $q' \in E$ we get that $f\phi(t, q) = f(q) + t = r_0 + t$ for $q \in f^{-1}(r_0)$. This gives that the first coordinate of $\phi^{-1}\phi(t, q)$ is t, and that the second coordinate is q follows since $\Phi_i(-t_i, \Phi_i(t_i, q')) = q'$. Similarly for the other composite.

Exercise 9.5.11

Concerning the map $\ell\colon S^1 \to \mathbf{C}\mathrm{P}^1$: note that it maps into a chart domain on which Lemma 9.5.8 tells us that the projection is trivial.

Exercise 9.4.11

Analyzing $\mathrm{Hom}(\eta_n, \eta_n^\perp)$ we see that we may identify it with the set of pairs $(L, \alpha\colon L \to L^\perp)$, where $L \in \mathbf{R}\mathrm{P}^n$ and α a linear map. On the other hand, Exercise 6.5.15 identifies $T\mathbf{R}\mathrm{P}^n$ with $\{(p, v) \in S^n \times \mathbf{R}^{n+1} \mid p \cdot v = 0\}/(p, v) \sim (-p, -v)$. This means that we may consider the bijection $\mathrm{Hom}(\eta_n, \eta_n^\perp) \to T\mathbf{R}\mathrm{P}^n$ given by sending $(L, \alpha\colon L \to L^\perp)$ to $\pm(p, \alpha(p))$ where $\pm p = L \cap S^n$. This bijection is linear on each fiber. Check that it defines a bundle morphism by considering trivializations over the standard atlas for $\mathbf{R}\mathrm{P}^n$.

Exercise 9.5.3

If $s\colon M \to E$ is a nonvanishing vector field, then $m \mapsto \frac{s(m)}{|s(m)|}$ is a section of $S(E) \to M$.

Exercise 9.5.4

Let $\pi\colon E \to M$ be a locally trivial smooth fibration with M a connected non-empty smooth manifold. Choose a $p \in M$ and let $F = \pi^{-1}(p)$. Consider the set

$$U = \{x \in M \mid \pi^{-1}(x) \cong F\},$$

and let V be the complement. We will show that both U and V are open, and so $U = M$ since $p \in U$ and $M$is connected. If $x \in U$ choose a trivializing neighborhood $x \in W$,

$$h\colon \pi^{-1}(W) \to W \times \pi^{-1}(x).$$

Now, if $y \in W$, then h induces a diffeomorphism between $\pi^{-1}(y)$ and $\pi^{-1}(x) \cong F$, so U is open. Likewise for V.

Exercise 9.5.12

Write $\mathbf{R}$ as a union of intervals J_j so that for each j, $\gamma(U_j)$ is contained within one of the open subsets of M so that the fibration trivializes. On each of these intervals the curve lifts, and you may glue the liftings using bump functions.

Chapter 10

Exercise 10.1.5

Consider the union of the closed intervals $[1/n, 1]$ for $n \geq 1$)

Exercise 10.1.7

Consider the set of all open subsets of X contained in U. Its union is open.

Exercise 10.1.9

By the union axiom for open sets, int A is open and contains all open subsets of A.

Exercise 10.1.10

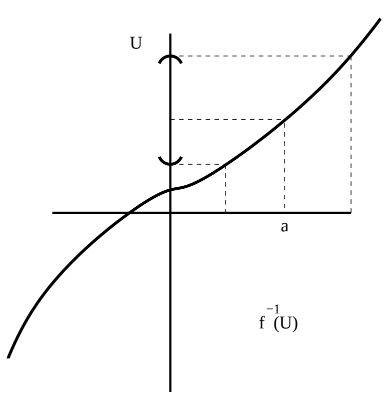
The intersection of two open balls is the union of all open balls contained in the intersection.

Exercise 10.1.11

All open intervals are open balls!

Exercise 10.2.2

Hint one way: the "existence of the δ" assures you that every point in the inverse image has a small interval around it inside the inverse image of the ϵ ball.



Exercise 10.2.3

$f^{-1}(g^{-1}(U)) = (gf)^{-1}(U)$.

Exercise 10.2.6

Use first year calculus.

Exercise 10.3.3

Can you prove that the set containing only the intervals (a, b) when a and b varies over the rational numbers is a basis for the usual topology on the real numbers?

Exercise 10.3.4

Show that given a point and an open ball containing the point there is a "rational" ball in between.

Exercise 10.3.5

Use note 10.3.2.

Exercise 10.3.6

$f^{-1}(\bigcup_\alpha V_\alpha) = \bigcup_\alpha f^{-1}(V_\alpha)$.

Exercise 10.5.2

Use that $(\bigcup_\alpha U_\alpha) \cap A = \bigcup_\alpha (U_\alpha \cap A)$ and $(\bigcap_\alpha U_\alpha) \cap A = \bigcap_\alpha (U_\alpha \cap A)$.

Exercise 10.5.3

Use 10.2.3 one way, and that if $f^{-1}(U \cap A) = f^{-1}(U)$ the other.

Exercise 10.5.4

The intersections of A with the basis elements of the topology on X will form a basis for the subspace topology on A.

Exercise 10.5.5

Separate points in A by means of disjoint neighborhoods in X, and intersect with A.

Exercise 10.6.4

Inverse image commutes with union and intersection.

Exercise 10.6.5

Use 10.2.3 one way, and the characterization of open sets in $X/\sim$ for the other.

Exercise 10.6.6

Show that open sets in one topology are open in the other.

Exercise 10.7.2

Cover $f(X) \subseteq Y$ by open sets i.e., by sets of the form $V \cap f(X)$ where V is open in Y. Since $f{-}1(V \cap f(X)) = f{-}1(V)$ is open in X, this gives an open cover of X. Choose a finite subcover, and select the associated V's to cover $f(X)$.

Exercise 10.7.5

The real projective space is compact by 10.7.2. The rest of the claims follows by 10.7.11, but you can give a direct proof by following the outline below.

For $p \in S^n$ let $[p]$ be the equivalence class of p considered as an element of $\mathbf{R}\mathrm{P}^n$. Let $[p]$ and $[q]$ be two different points. Choose an ϵ such that ϵ is less than both $|p-q|/2$ and $|p+q|/2$. Then the ϵ balls around p and $-p$ do not intersect the ϵ balls around q and $-q$, and their image define disjoint open sets separating $[p]$ and $[q]$.

Notice that the projection $p\colon S^n \to \mathbf{R}\mathrm{P}^n$ sends open sets to open sets, and that if $V \subseteq \mathbf{R}\mathrm{P}^n$, then $V = pp^{-1}(V)$. This implies that the countable basis on S^n inherited as a subspace of $\mathbf{R}^{n+1}$ maps to a countable basis for the topology on $\mathbf{R}\mathrm{P}^n$.

Exercise 10.7.9

You must show that if $K \subseteq C$ is closed, then $(f^{-1})^{-1}(K) = f(K)$ is closed.

Exercise 10.7.10

Use Heine-Borel 10.7.3 and exercise 10.7.2.

Exercise 10.8.2

One way follows by Exercise 10.2.3. For the other, observe that by Exercise 10.3.6 it is enough to show that if $U \subseteq X$ and $V \subseteq Y$ are open sets, then the inverse image of $U \times V$ is open in Z.

Exercise 10.8.3

Show that a square around any point contains a circle around the point and vice versa.

Exercise 10.8.4

If $\mathcal{B}$ is a basis for the topology on X and $\mathcal{C}$ is a basis for the topology on Y, then

$$\{U \times V | U \in \mathcal{B}, V \in \mathcal{C}\}$$

is a basis for $X \times Y$.

Exercise 10.8.5

If $(p_1, q_1) \neq (p_2, q_2) \in X \times Y$, then either $p_1 \neq p_2$ or $q_1 \neq q_2$. Assume the former, and let U_1 and U_2 be two open sets in X separating p_1 and p_2. Then $U_1 \times Y$ and $U_2 \times Y$ are...

Exercise 10.9.2

The inverse image of a set that is both open and closed is both open and closed.

Exercise 10.9.4

Both X_1 and X_2 are open sets.

Exercise 10.9.5

One way follows by Exercise 10.2.3. The other follows since an open subset of $X_1 \coprod X_2$ is the (disjoint) union of an open subset of X_1 with an open subset of X_2.

Exercise 10.10.4

If $p \in f(f^{-1}(B))$ then $p = f(q)$ for a $q \in f^{-1}(B)$. But that $q \in f^{-1}(B)$ means simply that $f(q) \in B$!

Exercise 10.10.5

These are just rewritings.

Exercise 10.10.7

We have that $p \in f^{-1}(B_1 \cap B_2)$ iff $f(p) \in B_1 \cap B_2$ iff $f(p)$ is in **both** B_1 and B_2 iff p is in **both** $f^{-1}(B_1)$ and $f^{-1}(B_2)$ iff $p \in f^{-1}(B_1) \cap f^{-1}(B_2)$. The others are equally fun.

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