Differential Topology

Math 6452

Fall 2019

GaTech

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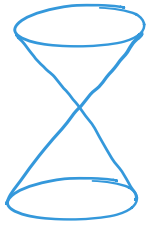
What is Diff. Top.?

Essentially, Calculus N+1: calculus on manifolds.

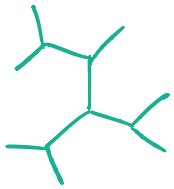
A manifold is a space (e.g. subset of $\mathbb{R}^n$) that locally looks like $\mathbb{R}^m$.

First examples: $\mathbb{R}^n$
unit sphere
graphs of fns on $\mathbb{R}^{n-1}$
tori, other surfaces
Knots
discrete set of points

First non-examples: solns to $x^2+y^2=z^2$



trees



Cantor set

Why are Manifolds Important?

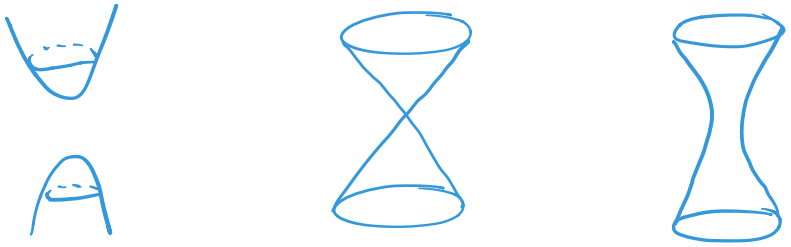
Answer: they are ubiquitous.

As we will see, if you write down a random set of equations in n variables, the set of solutions will be a manifold, generically.

So manifolds are a natural thing to study after linear algebra.

What about $x^2+y^2=z^2$?

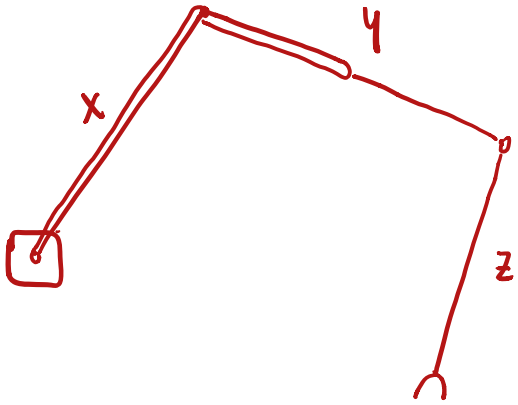
Consider $x^2+y^2=z^2+t$



$t<0$ $\quad$ $t=0$ $\quad$ $t>0$

Exactly <u>one</u> value of t gives a non-manifold.

Manifolds in the Wild: A Robot Arm



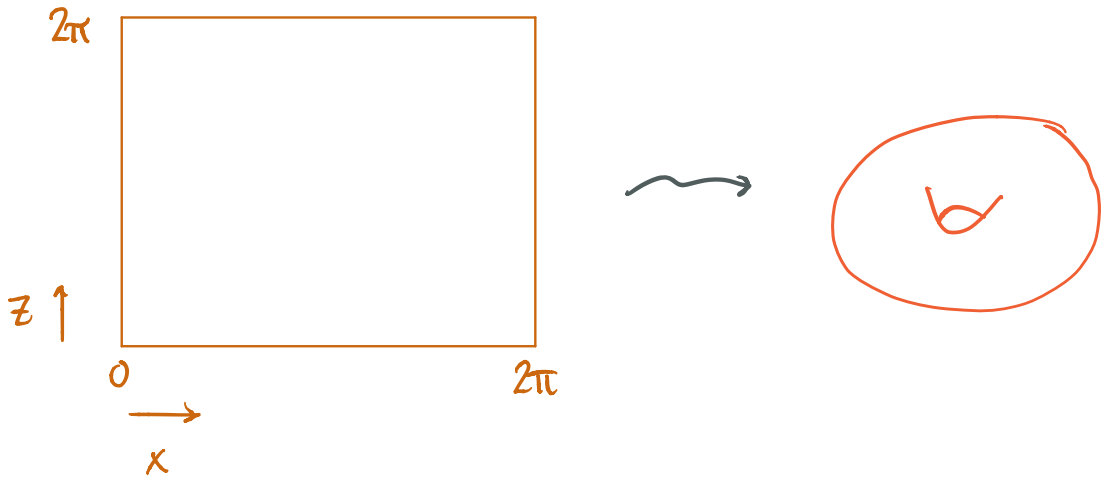
With: $x, y, z \in \mathbb{R}^2$, $|x| = |z| = 1$, $1 \leq |y| \leq 5$

Location of robot hand is $x+y+z$.

Q. Which configurations reach $(3,0)$.

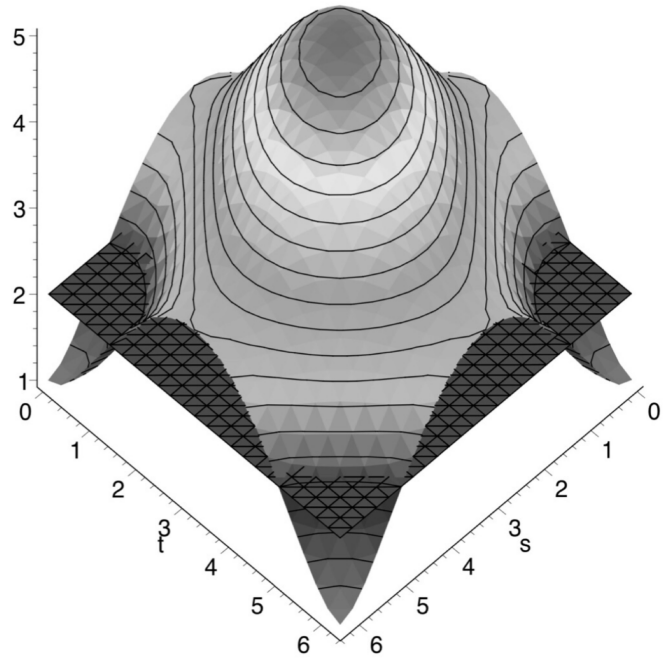
A. x & z are free, determine y.

$\rightsquigarrow$ set of possible configs:

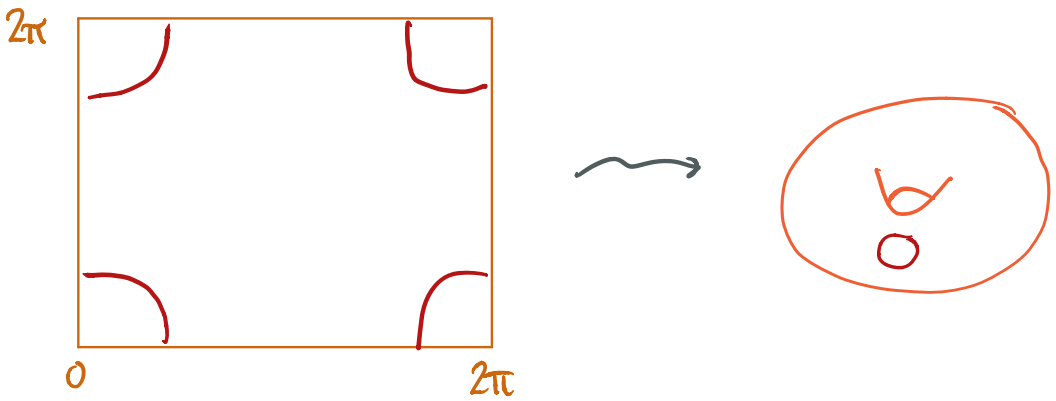


What if y gets stuck at length 2?

If we look at the graph of $|y|$ as a function of x, z get a slice at $|y|=2$.



On the torus:

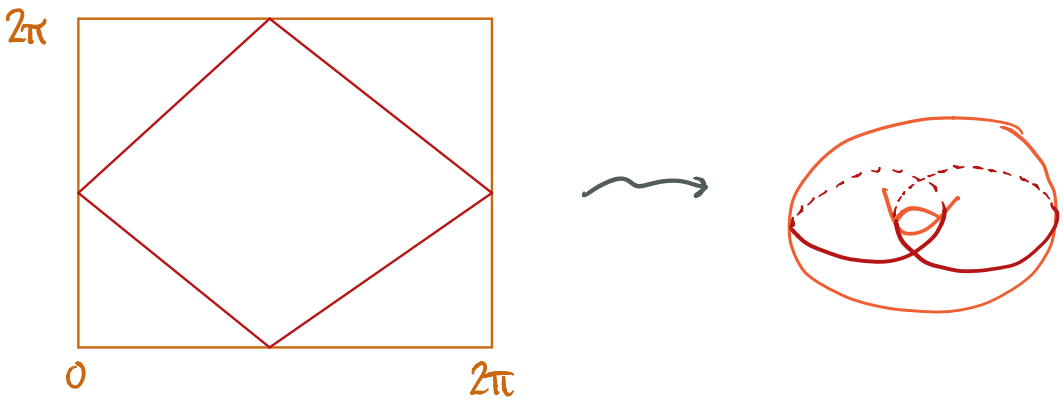


What is the interpretation for the robot arm?

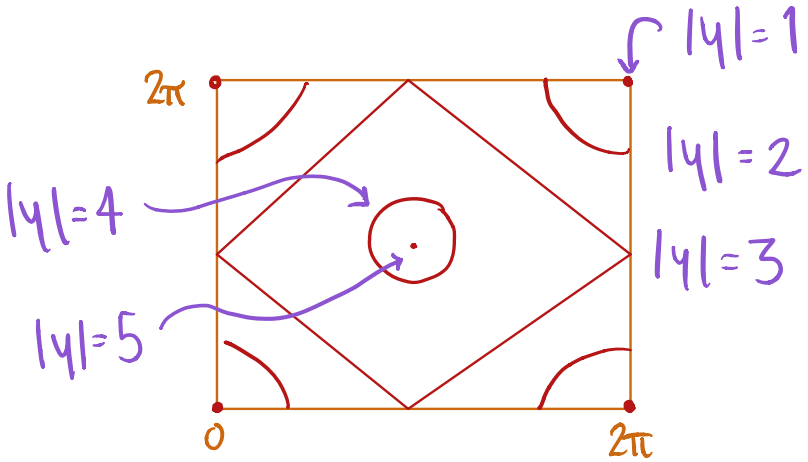
There are 3 critical values: $|y| = 1, 3, 5$

For $|y| = 1, 5$ there is <u>one</u> solution (why?)

For $|y| = 3$ the solutions are:

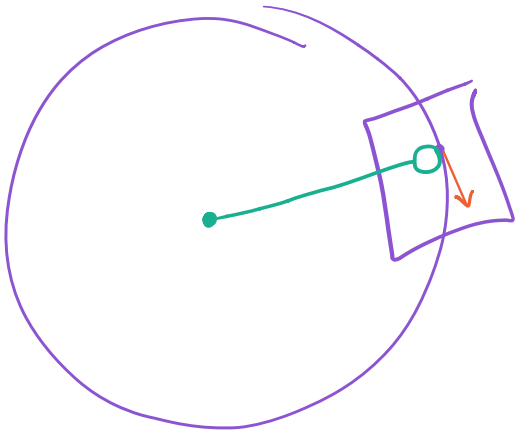


Putting it together:



Other Examples of Manifolds

① Tangent bundle on S^2
or: phase space of a pendulum



② Projective space = space of lines in $\mathbb{R}^3$
or: phase space of indistinguishable electrons rotating about a center of mass.
(ignoring their distance)

③ $GL_n\mathbb{R}$ = $n \times n$ matrices with $\det \neq 0$.

④ Solutions to a typical polynomial

$$y^2 = x(x-1)(x-2)(x-t) \rightsquigarrow$$