Torelli groups and the complex of minimizing cycles

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joint work with Mladen Bestvina
and Kai-Uwe Bux

Informal Seminar

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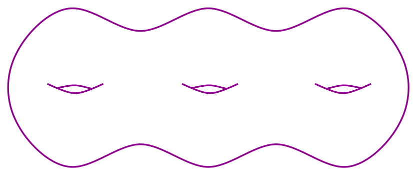
Torelli groups

S_g = surface of genus g



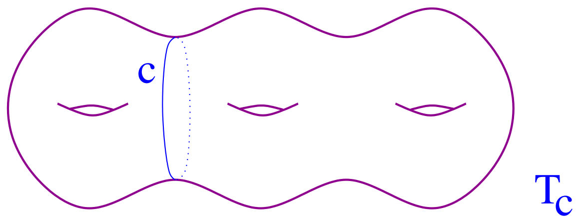
$\text{MCG}(S_g) = \pi_0(\text{Homeo}^+(S_g))$

Definition of the Torelli group $\mathcal{I}(S_g)$:

$$1 \to \mathcal{I}(S_g) \to \text{MCG}(S_g) \to \text{Sp}(2g, \mathbb{Z}) \to 1$$

Elements of the Torelli group

Dehn twists about separating curves



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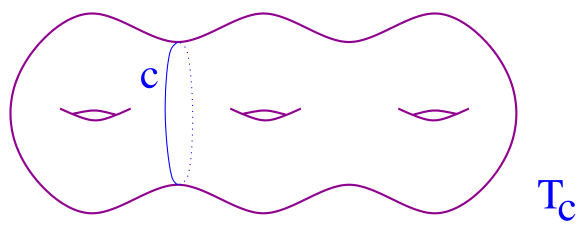
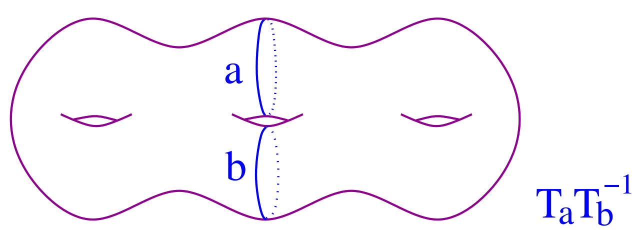


Bounding pair maps



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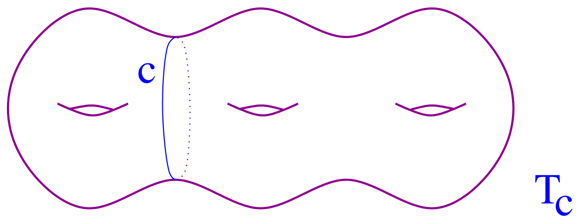

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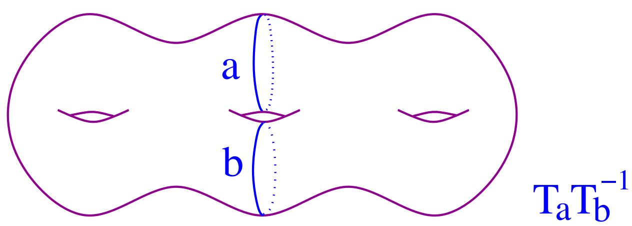

Theorem (Birman '71 + Powell '78, Putman '07)

These elements generate $\mathcal{I}(S_g)$.

Finiteness properties

Finite generation

Finite presentability

Finite generation of homology

Cohomological dimension

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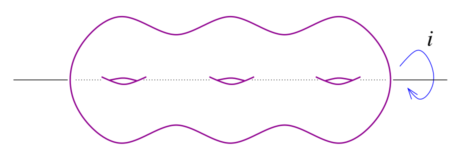

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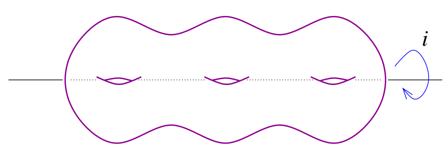

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$\{\mathcal{N}_k(S_g)\}$ =the Johnson filtration. For $g, k \geq 2$, we have

$$g-1 \leq \mathrm{cd}(\mathcal{N}_k(S_g)) \leq 2g-3 \quad \text{(Farb, BBM)}$$

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Then

1. $\mathrm{cd}(G) \leq D$ (Quillen)
2. $\oplus H_D(\mathrm{Stab}_G(v)) \hookrightarrow H_D(G)$
   where the sum is over a set of reps of vertices of X/G.

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Informal definition

Fix any nonzero $x \in H_1(S, \mathbb{Z})$.

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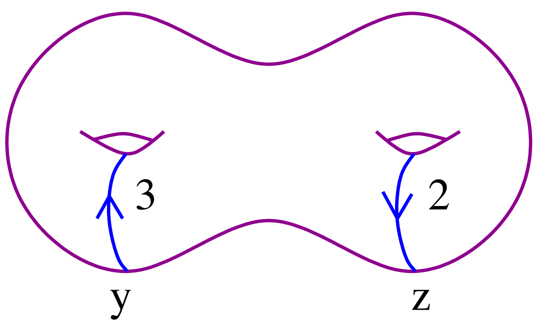

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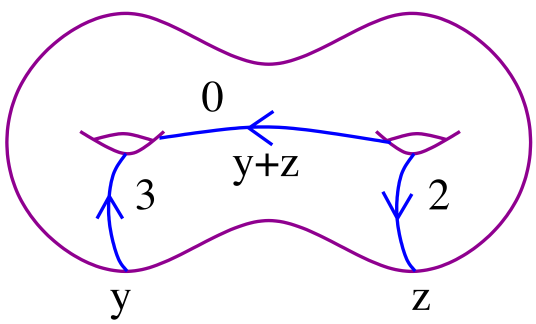

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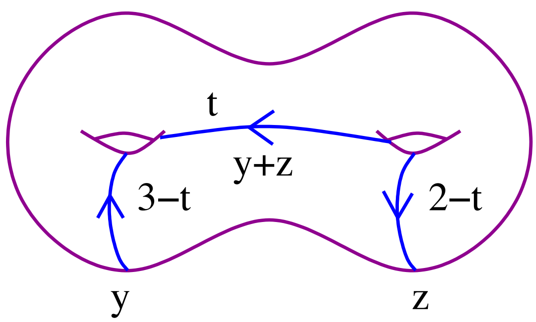

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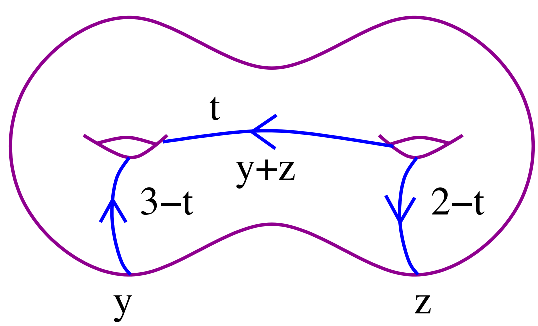

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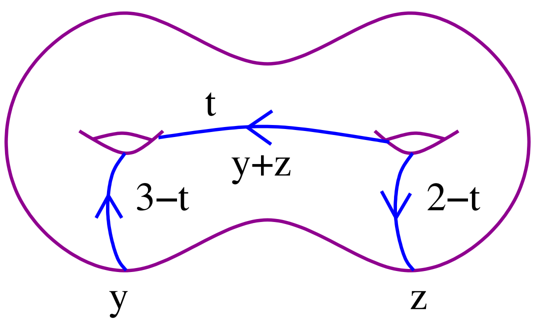

Nonnegativity $\rightsquigarrow 0 \leq t \leq 2$. Resulting cell:

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Fact: Cell(M) is a polytope.

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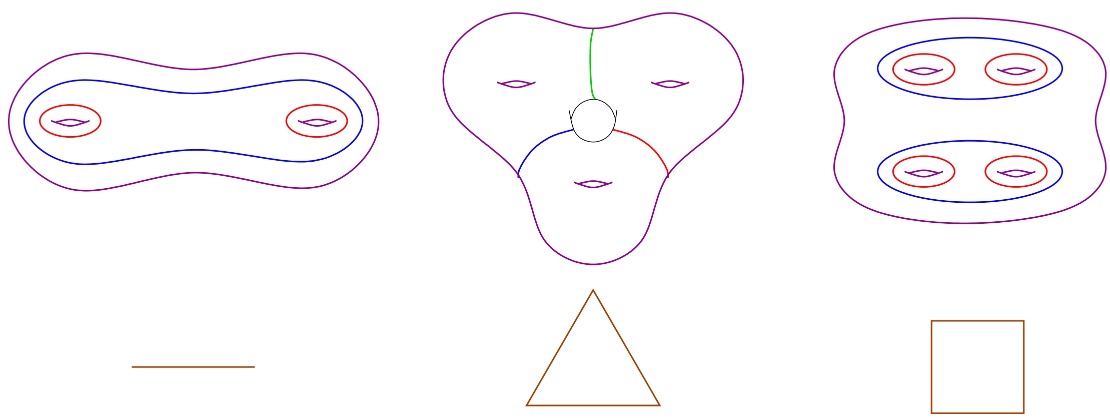

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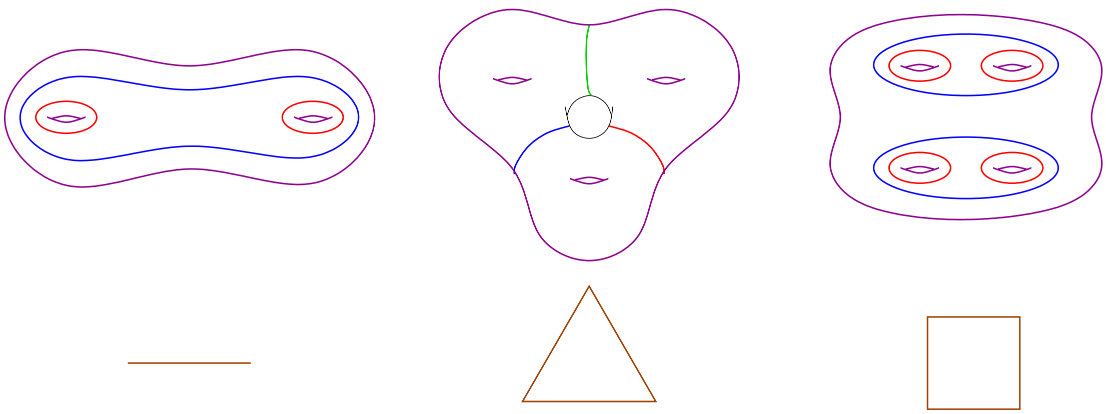

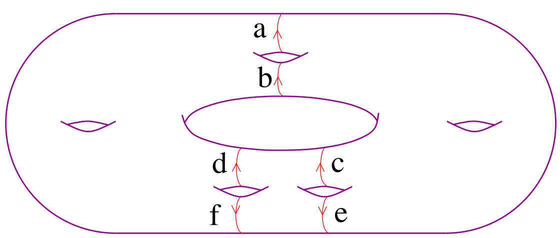

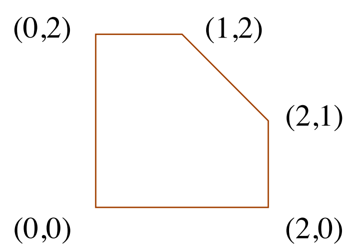

$$x = [d] + 2[e] + [f]$$

The complex of minimizing cycles
Definition

$\mathcal{B}(S) =$ the complex of minimizing cycles

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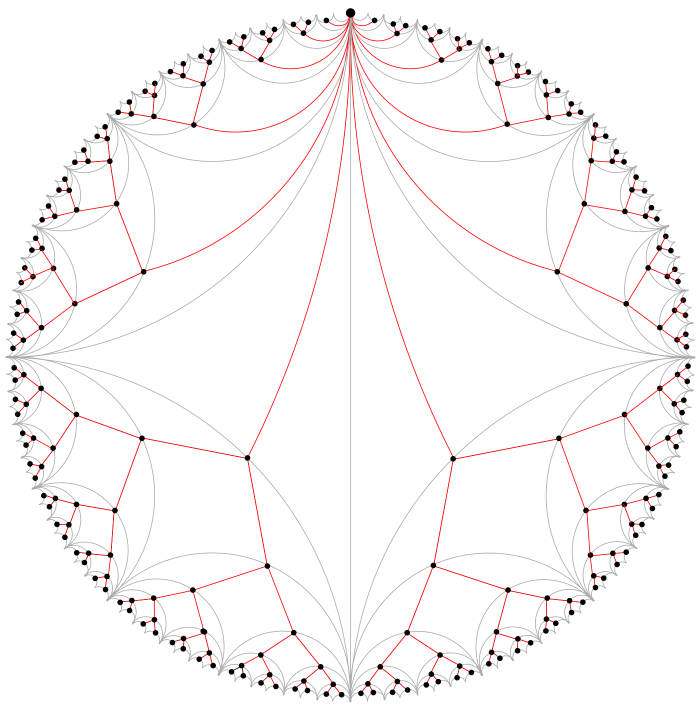
Theorem (BBM)

$\mathcal{B}(S_g)$ is contractible.

Genus 2

The quotient tree $\mathcal{B}(S_2)/\mathcal{I}(S_2)$ is infinitely many copies of



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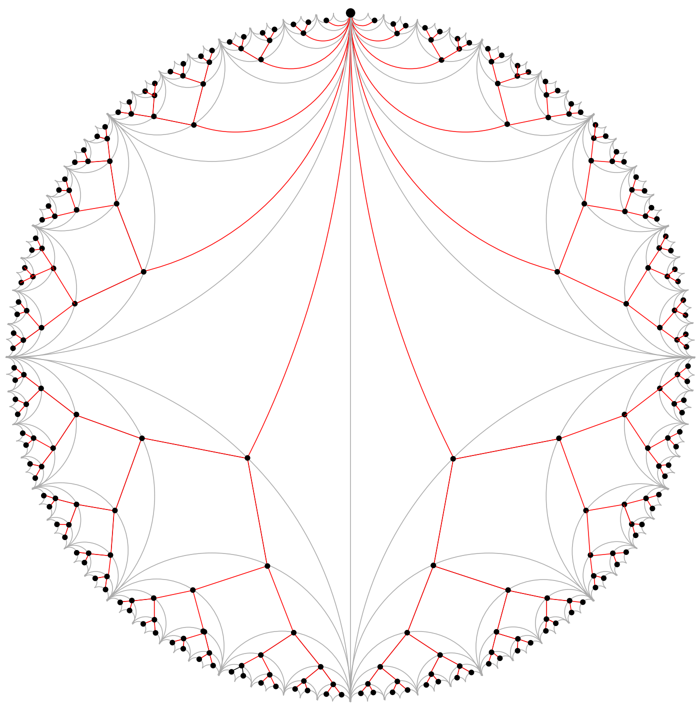


glued along their distinguished vertices (one for each isotropic subspace of $H_1(S_2, \mathbb{Z})$ containing x).

Two proofs of contractibility

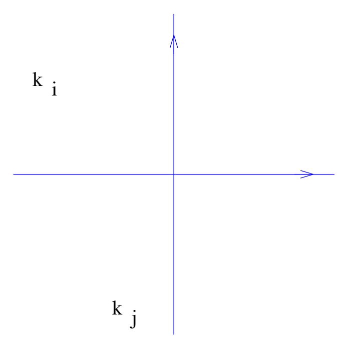
The Complex of Minimizing Cycles

Surgery proof of contractibility

Surgery on 1-cycles

Let c be a nonsimple 1-cycle representing x.

$$c = \sum k_i c_i$$



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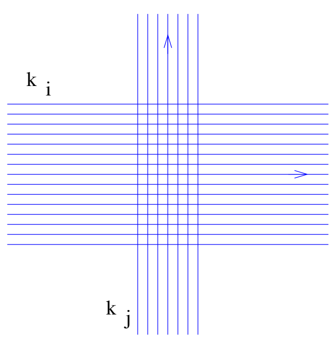

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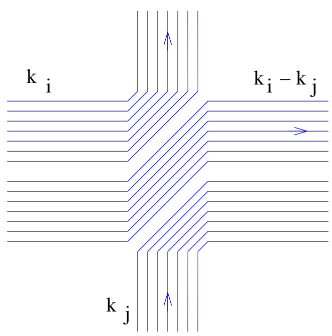

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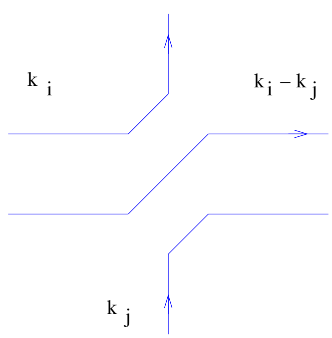

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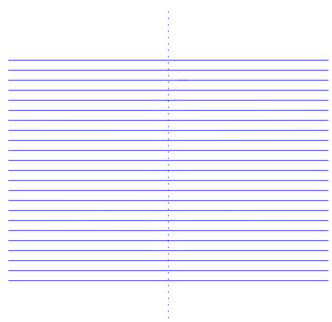

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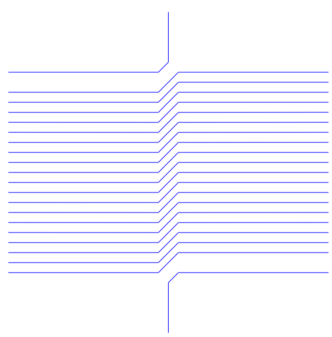

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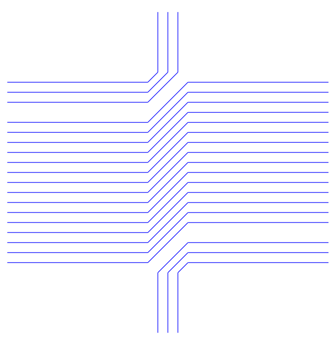

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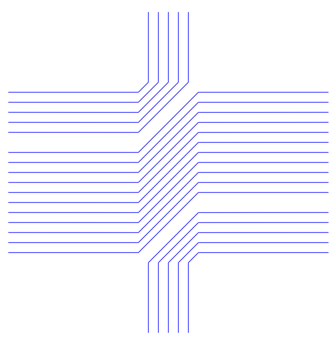

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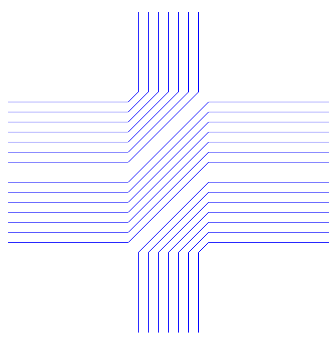

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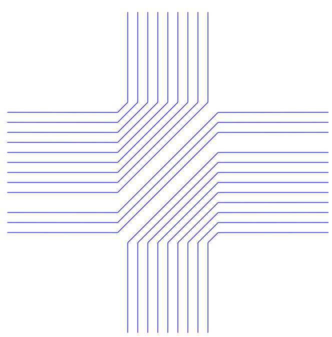

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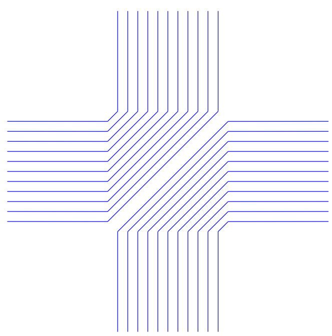

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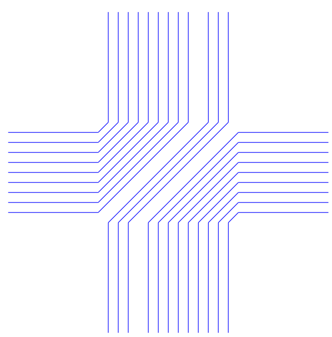

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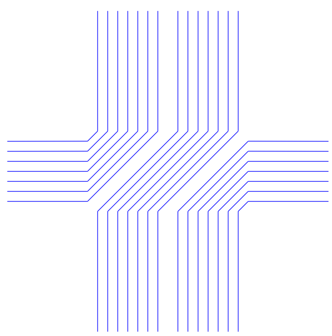

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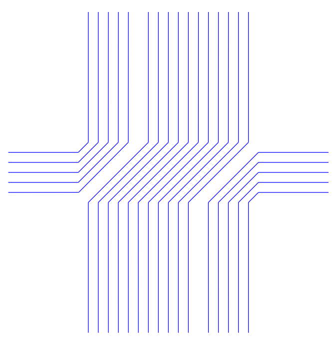

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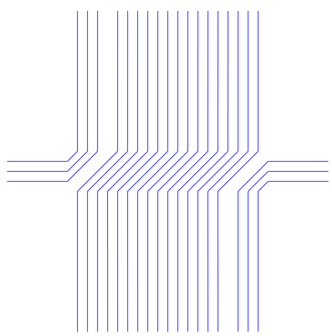

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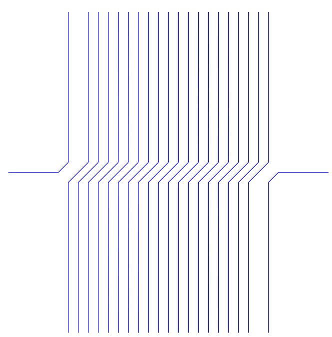

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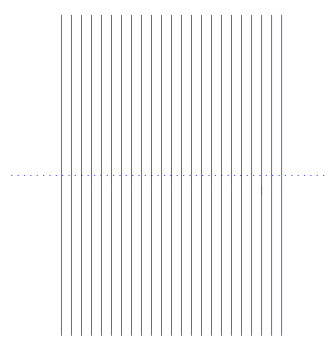

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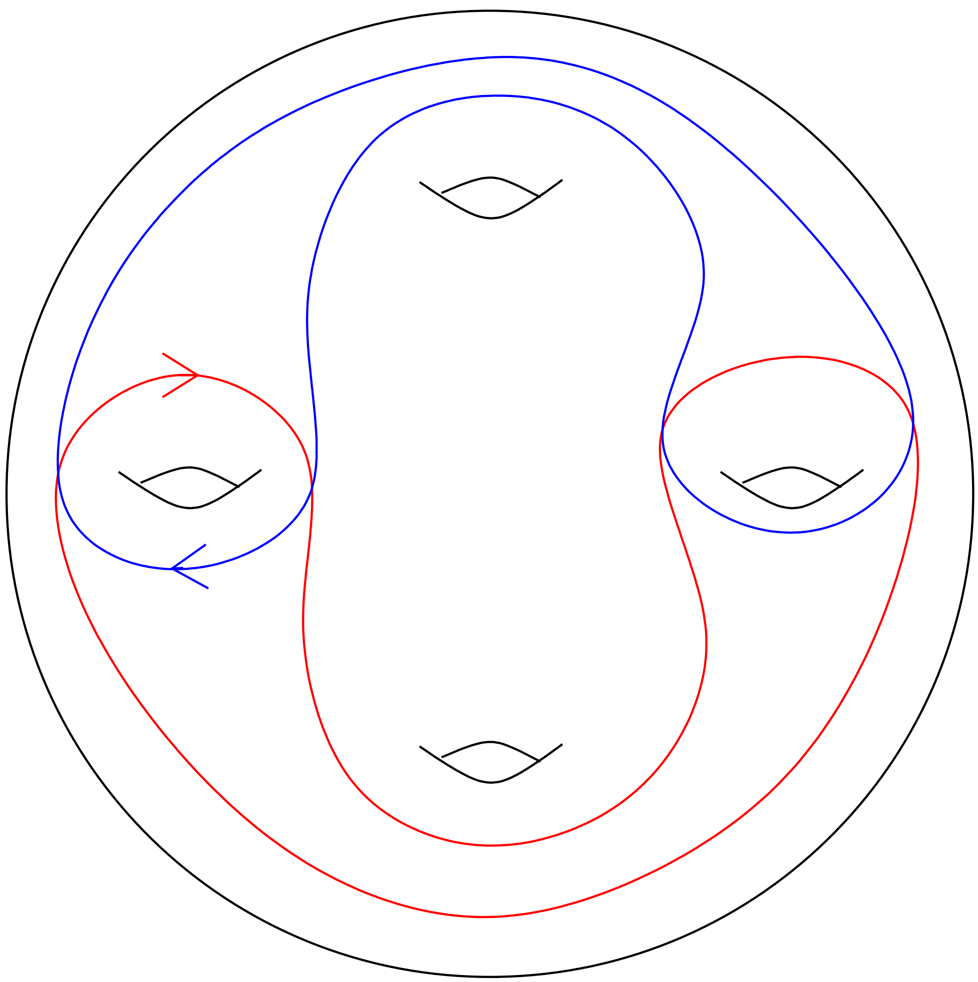

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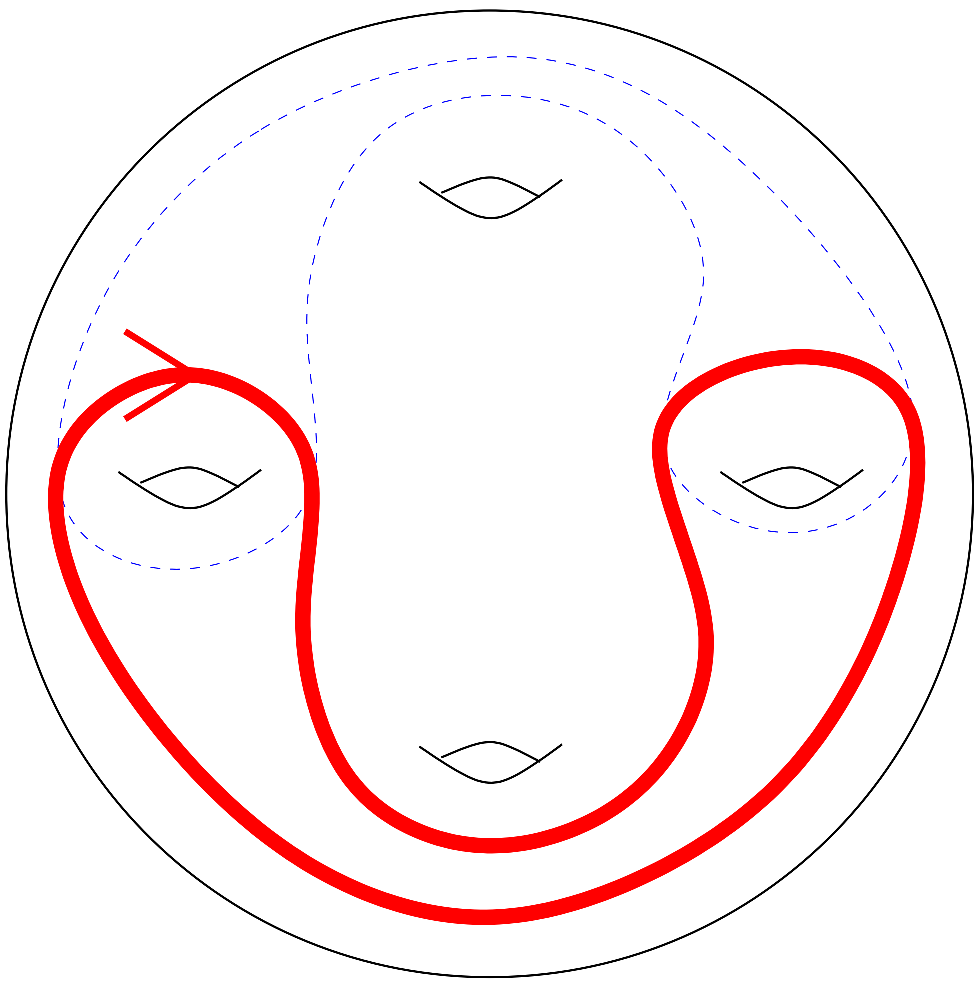

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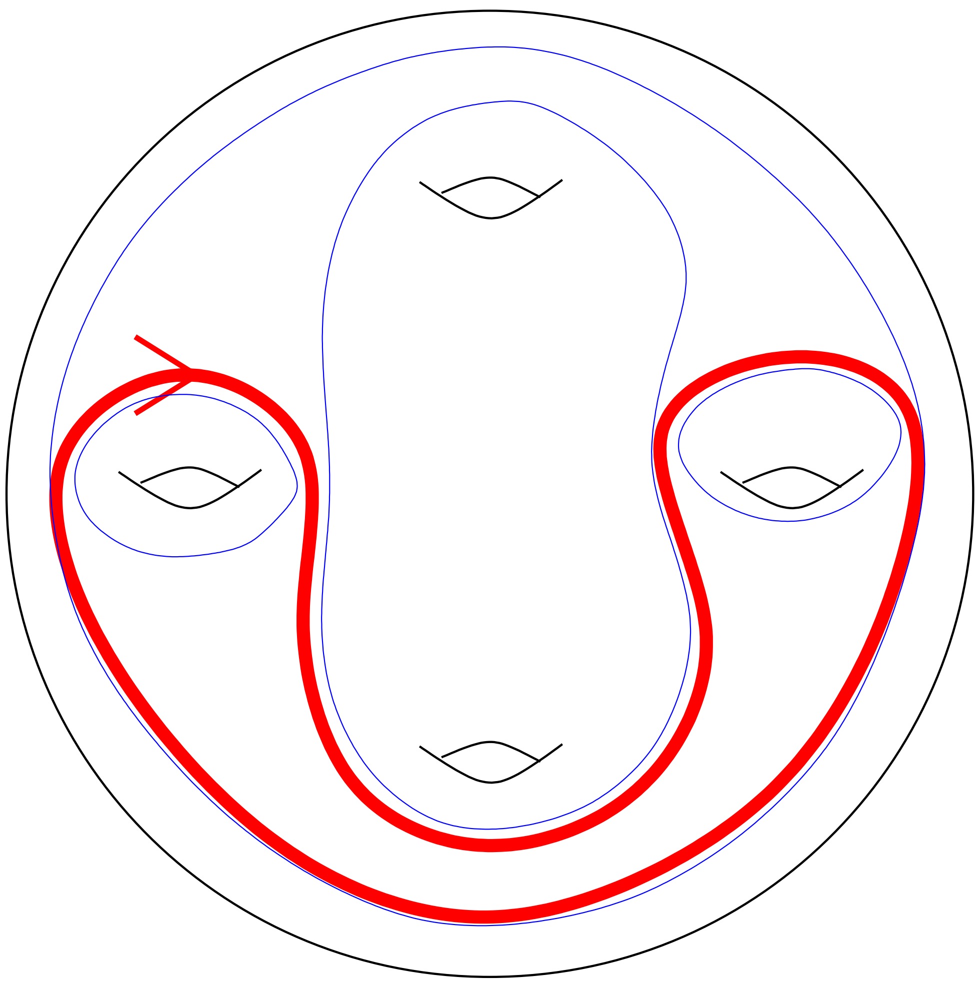

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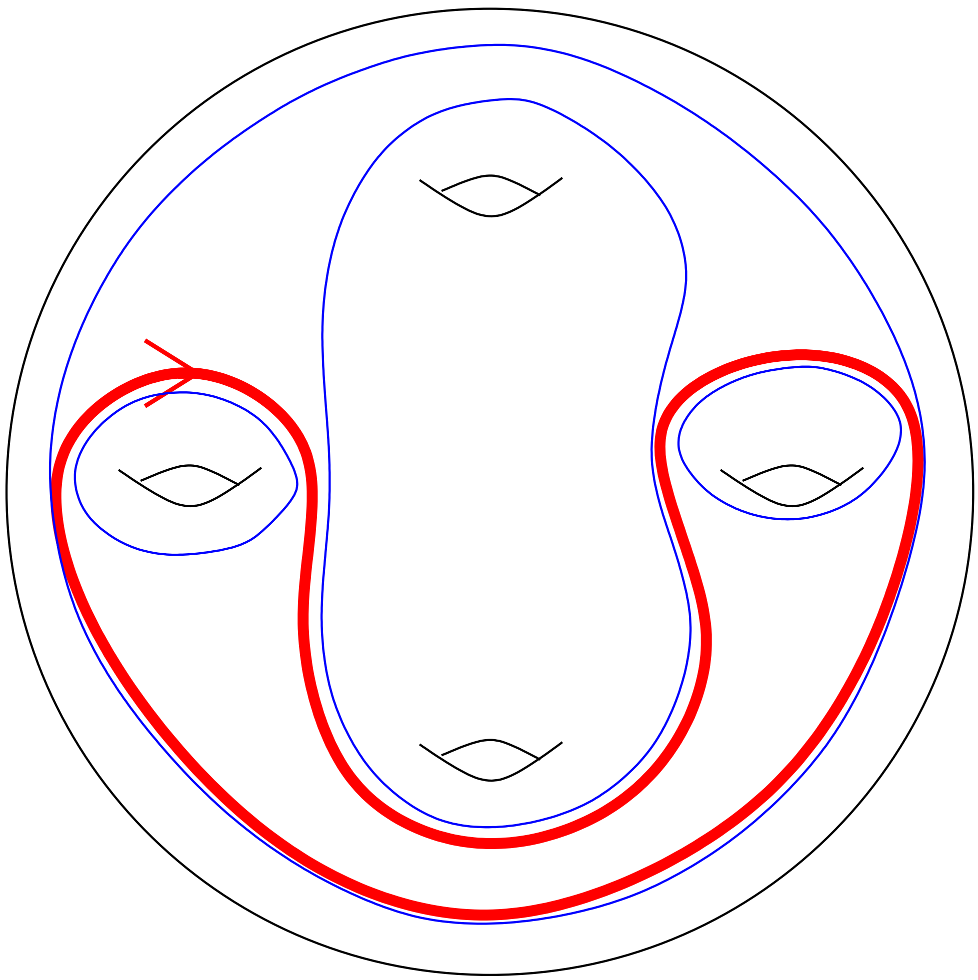

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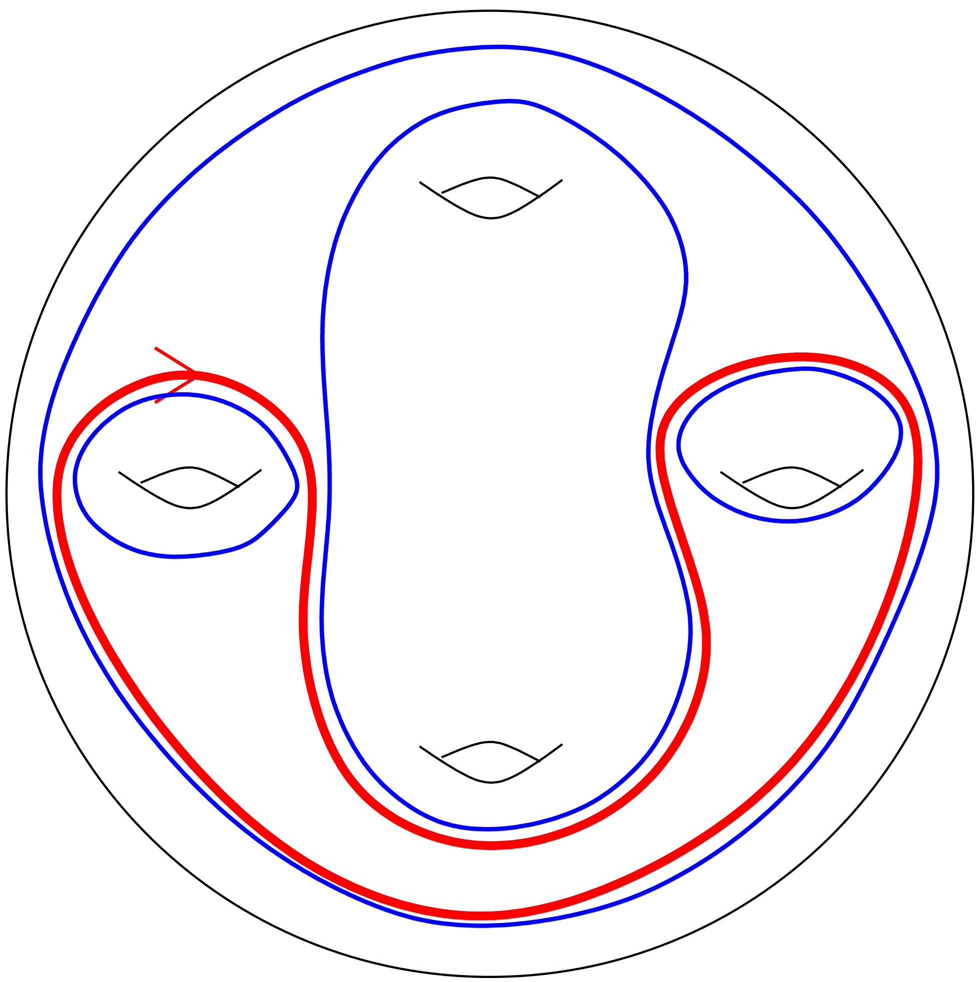

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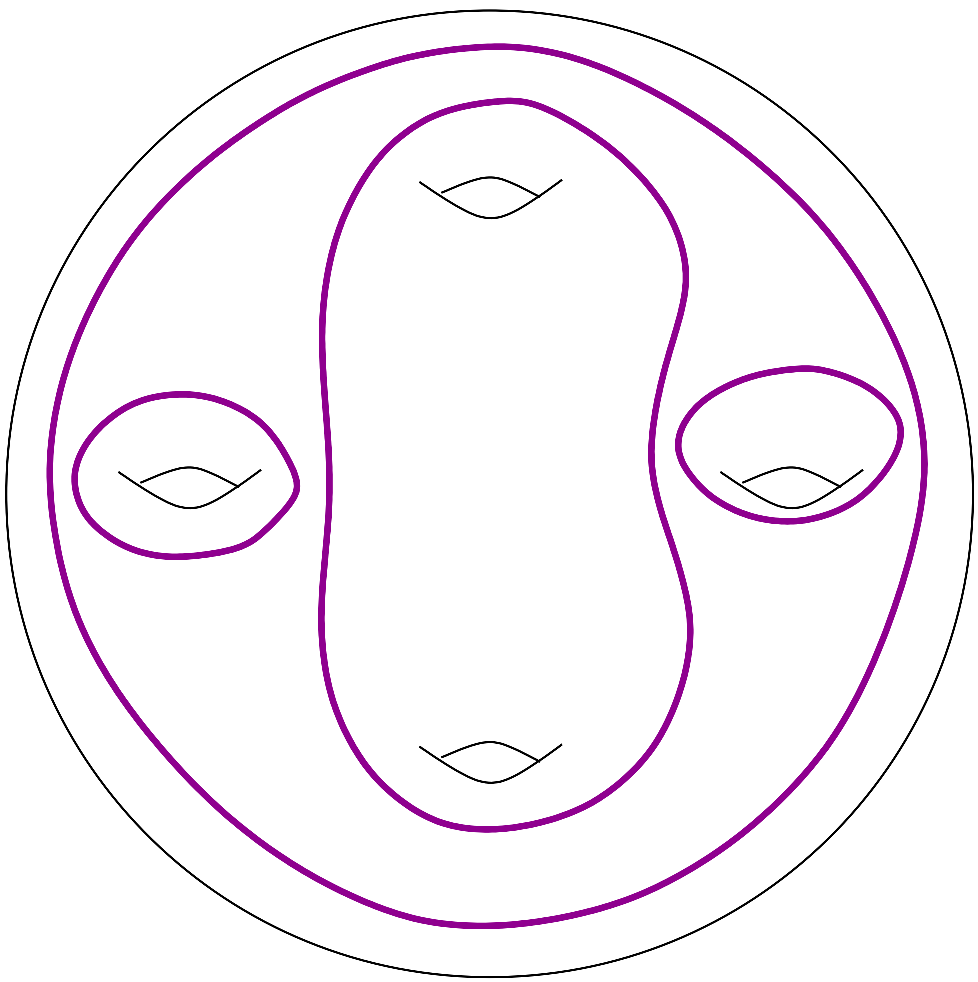

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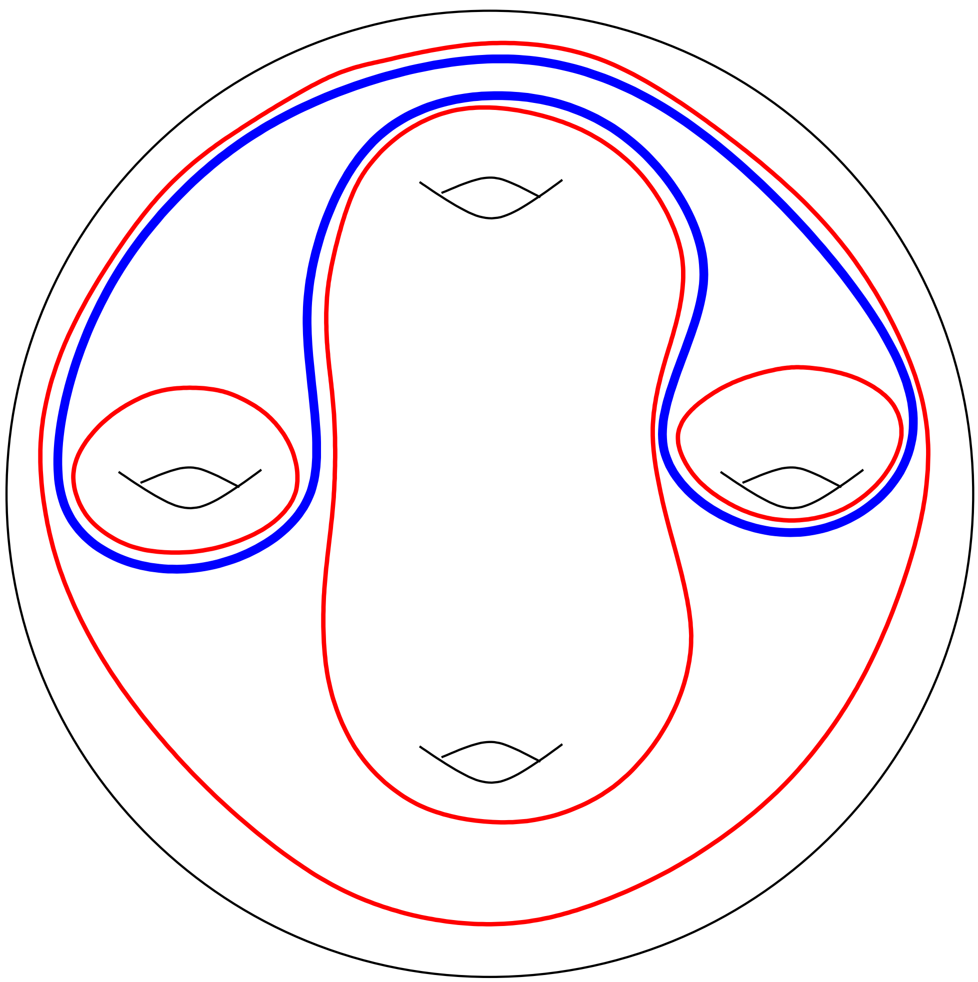

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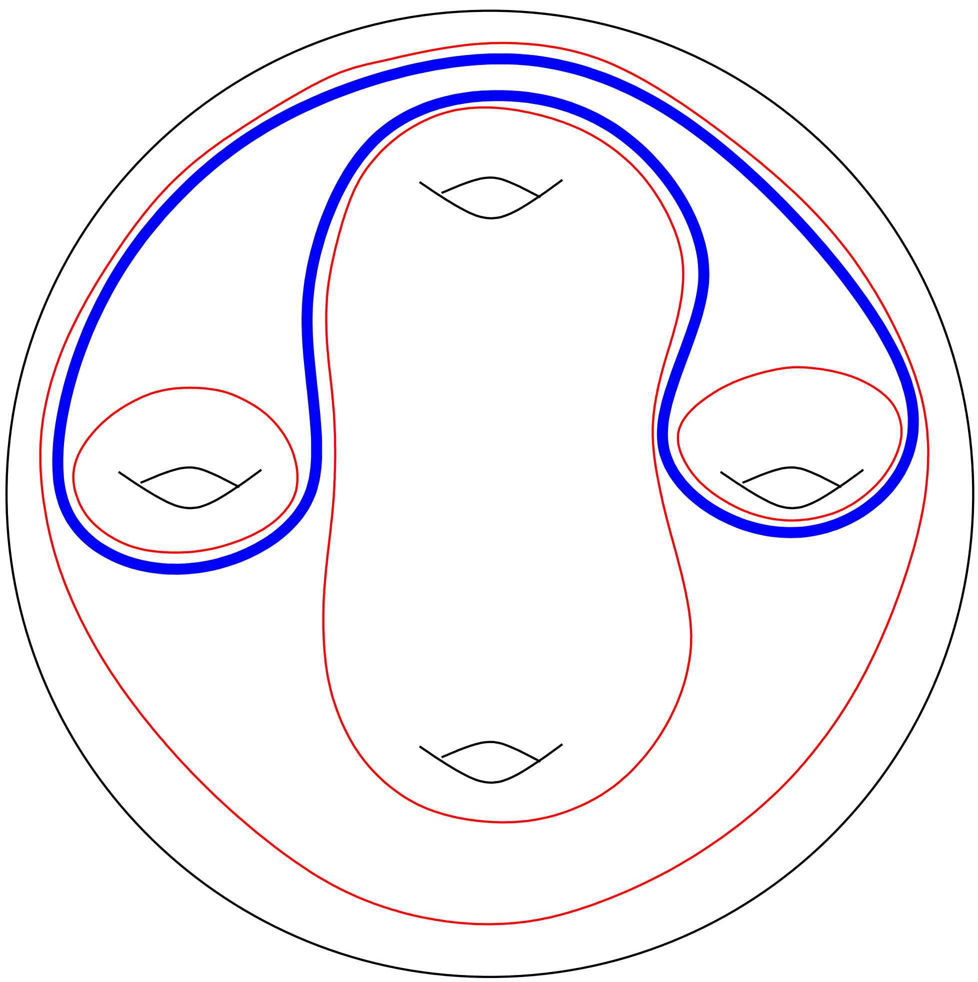

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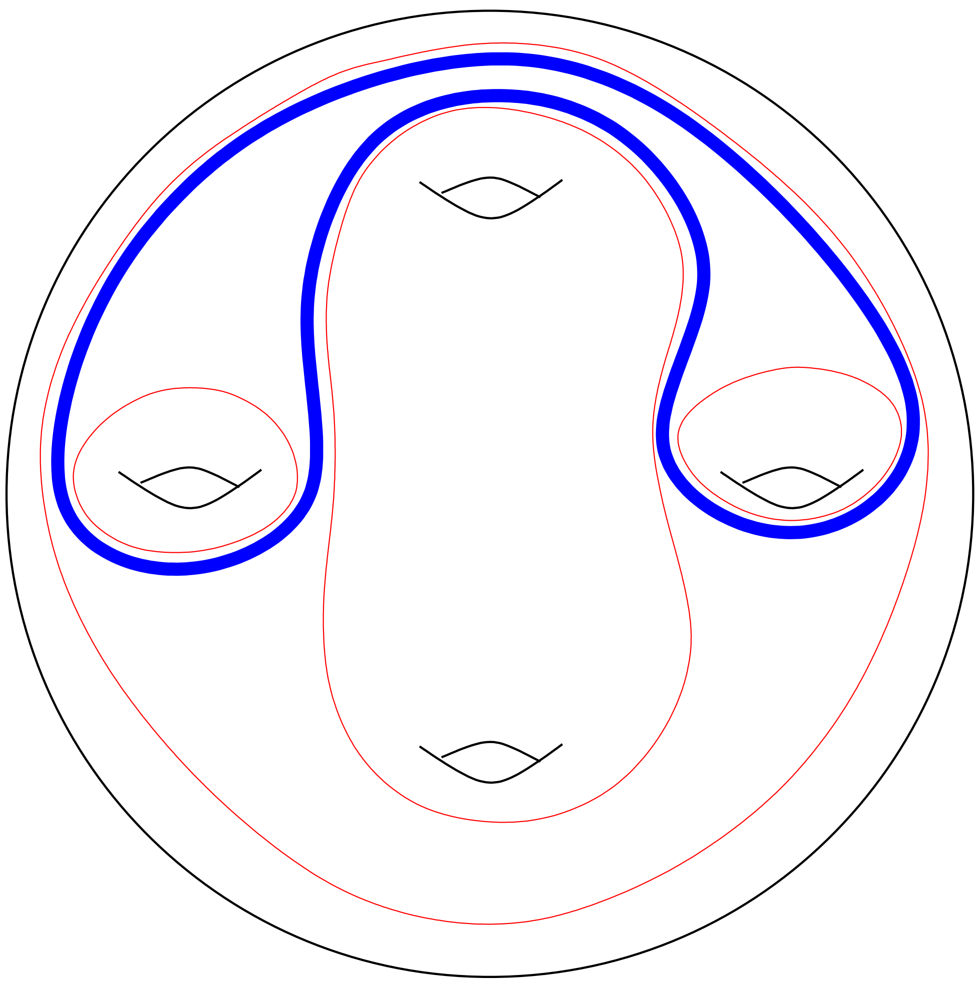

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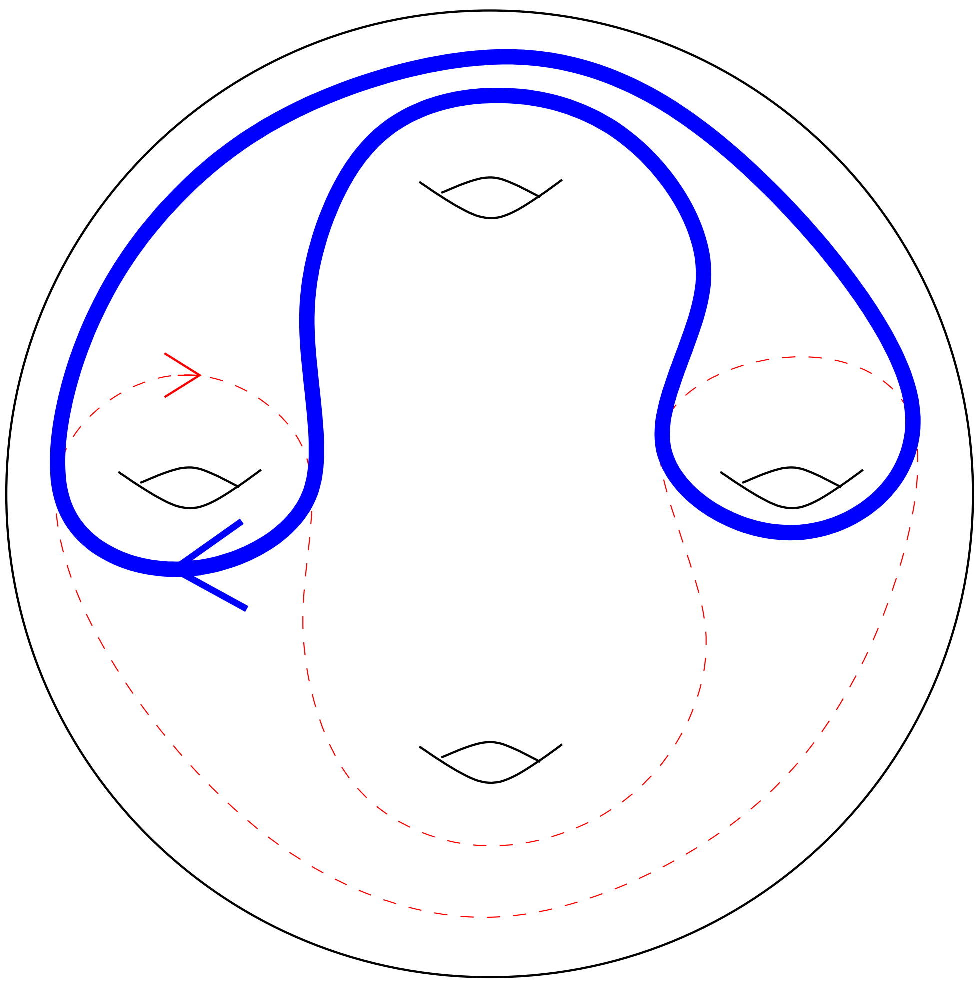

The complex of minimizing cycles

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Quillen:

$$\mathrm{cd}(\mathcal{I}(S_g)) \leq \sup\{\mathrm{cd}(\mathrm{Stab}(\sigma)) + \dim(\sigma)\}$$

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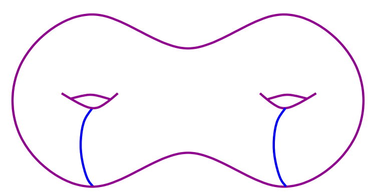

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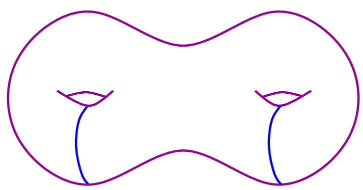

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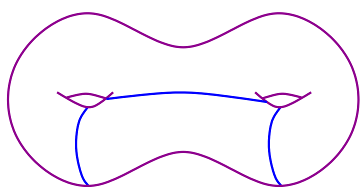

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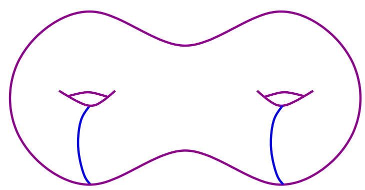

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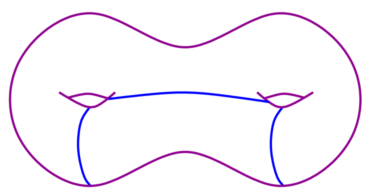

Higher genus: induction, Birman exact sequence.

Infinite generation of top homology

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Infinite generation of top homology

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$$\sup\{\text{cd}(\text{Stab}(\sigma) + \dim(\sigma)\} \leq 1.$$

Let $\{v\}$ be a set of representatives for vertices of G/X.

Cartan–Leray Spectral Sequence $\Rightarrow$

$$\oplus H_1(\text{Stab}(v)) \hookrightarrow H_1(G)$$

Infinite generation of top homology

Say $G \circlearrowleft X$, where X is contractible, and

$$\sup\{\mathrm{cd}(\mathrm{Stab}(\sigma) + \dim(\sigma)\} \leq 1.$$

Let $\{v\}$ be a set of representatives for vertices of G/X.

Cartan–Leray Spectral Sequence $\Rightarrow$

$$\oplus H_1(\mathrm{Stab}(v)) \hookrightarrow H_1(G)$$

Therefore, to prove that (H_1 of) $\mathcal{I}(S_2)$ is infinitely generated, we just need to show that H_1 of some vertex stabilizer is infinitely generated.