Birman-Craggs Homomorphisms

Will exhibit $\binom{2g}{0} + \binom{2g}{1} + \binom{2g}{2} + \binom{2g}{3}$
homomorphisms $\mathcal{I}(S_g^1) \longrightarrow \mathbb{Z}/2$.
The last summand $\longleftrightarrow$ τ mod 2

Johnson: this is all of $\mathcal{I}(S_g^1)^{ab}$.

Fix $h: S_g \longrightarrow S^3$ Heegaard embedding
Let $f \in \text{Mod}(S_g)$
$M(h,f)$ obtained by regluing by f:
Say A, B are the handlebodies
so $\partial A = \partial B = h(S_g)$.
Then $M(h,f) = (A \amalg B) / x \sim hfh^{-1}(x)$

$f \in \mathcal{I}(S_g) \Longrightarrow M(h,f)$ a homology S^3.
$\rightsquigarrow$ Rochlin invariant* $\mu(h,f) \in \mathbb{Z}/2$

$\rho_h = \mu(h,\cdot)$ is a homom. $\mathcal{I}(S_g) \longrightarrow \mathbb{Z}/2$.
Also: the homom. depends on h, but there are only finitely many $\longleftrightarrow$ self-linking forms.

Spin Structures.

A spin structure is a continuous choice of frame on the 1-skeleton that extends to the 2-skeleton

For an n-manifold M this gives a function
$$\pi_1 M \longrightarrow \pi_1 SO(n) \hat{=} \mathbb{Z}/2.$$
$n>2$.

Spin structures on surfaces:

Rokhlin Invariant μ

$W = \mathbb{Z}HS^3$
$\quad = \partial X$ where $X =$ spin (or, parallelizable)

Such X always exists and signature $\sigma(X)$ is multiple of 8. Also $\sigma(X)/8$ mod 2 is indep. of choice of $X \rightsquigarrow \mu$

Note: $\mathbb{Z}HS^3$'s have unique spin structure.

Seifert Linking Forms

Fix $S_g \subseteq S^3$

$\rightsquigarrow$ bilinear Seifert linking form on $H_1(S_g;\mathbb{Z})$

$$L(x,y) = \text{linking }\#\,(x, y^+)$$

pushoff

Use $\mathbb{Z}/2$ coeffs and restrict to $y = x$

$\rightsquigarrow$ mod 2 self-linking form

= quadratic form on $H_1(S_g;\mathbb{Z}/2)$:

$$\omega(x+y) = \omega(x) + \omega(y) + \hat{i}(x,y)$$

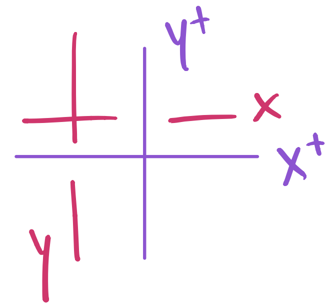
Johnson:

Quad. forms

$\updownarrow$

Spin struct.'s

Can see directly:



each intersection of x & y adds 2 crossings hence $+1$ to linking number.

Arf invariant: Fix a symplectic basis $\{x_i, y_i\}$

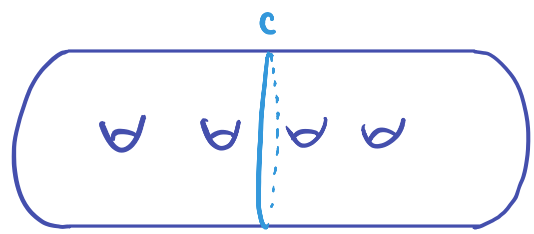
$$\alpha = \sum \omega(x_i)\,\omega(y_i)$$

For a knot K, the Arf invt is the Arf of the self linking # of any Seifert surface.

Computations

Fix $h : S_g \hookrightarrow S^3$
with Seifert linking form ω

Separating Twists



$M(h, T_c)$ obtained by Dehn surgery along c.
Gordon '75 & Gonzalez-Acuna '70:

$$\mu(M(h, T_c)) = \mathrm{Arf}(c)$$

But $h(S_g) \rightsquigarrow$ Seifert surface

$$\Longrightarrow \rho_h(T_c) = \sum_{i=1}^{k} \omega(x_i)\omega(y_i)$$

BP maps.

Get a surgery description of the desired 4-manifold. End result:

$$\rho_h(T_c T_d^{-1}) = \begin{cases} 0 & \text{if } \omega(c) = 1. \\ \sum_{i=1}^{k} \omega(x_i)\omega(y_i) & \text{o.w.} \end{cases}$$

Upshot: ρ_h only depends on ω! Call it ρ_ω.

BCJ Maps are Distinct

Lemma Fix ω = quad. form on H ← mod 2
Let $0 \neq a \in H$.
Then $\exists\, b \in H$ s.t. $\hat{i}(a,b) = 1$ & $\omega(b) = 1$.

Proof. Choose b_0 s.t. $\hat{i}(a,b) = 1$.
If $\omega(b_0) = 1$, done. Suppose not.
Extend a, b_0 to symplectic basis a, b_0, c, d.
Either $\omega(c)$, $\omega(d)$, or $\omega(c+d)$ must $= 1$.
Say $\omega(c) = 1$. Let $b = b_0 + c$. □

Prop. If $\omega \neq \omega'$ then $\rho_\omega \neq \rho_{\omega'}$

Proof. Choose a s.t. $\omega(a) \neq \omega'(a)$.
Choose b as in Lemma.
Realize both by curves that intersect once.
$\rightsquigarrow$ sep curve c
Note $\rho_\omega(T_c) \neq \rho_{\omega'}(T_c)$. □

Arf Zero

Prop. For a Heegaard embedding $S_g \subset S^3$
The Seifert linking form ω has $\mathrm{Arf} = 0$.
Conversely, all such ω arise.

Pf. Forward direction. Waldhausen's theorem says all Heegaard surfaces are standard. The standard basis $\{x_i, y_i\}$ has $\omega(x_i) = \omega(y_i) = 0 \implies \mathrm{Arf} = 0$.

Reverse direction. Arf showed that all quadratic forms with same Arf invt differ by Sp. $\square$

Counting BCJ Maps

So we now want to count quadratic forms with Arf $=0$. Counting all quadratic forms is easy:

Fix a basis $e_1, \dots, e_{2g}$.

$$\omega\left(\sum \alpha_i e_i\right) = \sum \alpha_i \omega(e_i) + \sum_{i<j} \alpha_i \alpha_j \hat{\imath}(e_i, e_j)$$

So...

A. 2^{2g} quad forms, determined by values on $\{e_i\}$.

Prop. There are $2^{g-1}(2^g+1)$ BCJ maps.

Proof. Induct on g. $\square$

These are not linearly indep!

Will show they form a $\binom{2g}{3} + \binom{2g}{2}$ dim space.

For open surfaces: $\binom{2g}{3} + \binom{2g}{2} + \underbrace{\binom{2g}{1} + \binom{2g}{0}}_{\text{disk pushing}}$.

The Space of BCJ maps

Let $H = H_1(S_g; \mathbb{Z}/2)$

Will define a $\mathbb{Z}/2$-algebra B based on H.

B is commutative w/ unit 1

has a generator $\bar{x}$ for each $x \in H$,

and relations:

① $\bar{x}^2 = \bar{x}$ $\forall\ x \in H$

② $\overline{x+y} = \bar{x} + \bar{y} + \hat{\imath}(x,y)$ $\forall\ x,y \in H$

An element of B is a polynomial in the gens

$\rightsquigarrow$ degree

$B_k = \{ b \in B \mid \deg b \le k \}$ vector space.

Goal: BCJ maps can be assembled to a surjective map $\sigma : \mathcal{I}(S_g^1) \longrightarrow B_3$ with naturality.

For $\mathcal{I}(S_g)$ we get a subspace of B_3, as some elts are 0

$\rightsquigarrow$ subspace of dim $\binom{2g}{3} + \binom{2g}{2}$

The Affine Space of Quadratic Forms

$\Omega = \{\text{quad. forms on } H\}$

not a vector space.

Still want to define linear functions on it.

For an abelian gp / vector space L,

a torsor / affine space over L

is a transitive free L-space, that is,

a set K with an action

$$+ : L \times K \longrightarrow K$$

with: $\ell_1 + (\ell_2 + k) = (\ell_1 + \ell_2) + k$

$\forall\, k_1, k_2 \ \exists!\, \ell$ s.t. $\ell + k_1 = k_2$

So K is like L, but with no base pt.

e.g. $L = \mathbb{Z}/n$, $K =$ vertices of n-gon.

Fact. Ω is an affine space over H^1

In other words:

$$\omega_1, \omega_2 \in \Omega \implies \omega_1 - \omega_2 \in H^1$$

$$\omega \in \Omega,\ \theta \in H^1 \implies \omega + \theta \in \Omega$$

(exercise).

Polynomial Functions on Affine Spaces

V = vect. sp. over F
U = affine sp. over V
$f : U \to F$ is linear if $\exists$ linear $g : V \to F$
s.t. $f(v+u) = g(v) + f(u)$

Facts
- f linear, c const $\Rightarrow$ $f + c$ linear
- The linear fns on U form a vect sp. $L(U)$ over F.

Fact. $f \in L(U)$ determined by $g \in V^*$ & $f(u_0)$ some fixed $u_0 \in U$.
$\Rightarrow \dim L(U) = \dim V + 1$.

A polynomial fn on U is a sum of products of linear ones.

Polynomial Functions on Ω

Consider $U = \Omega$, $V \in H'$

$x \in H \rightsquigarrow \bar{x} : \Omega \to \mathbb{Z}/2$

$$\bar{x}(\omega) = \omega(x)$$

Fact. $\bar{x}$ is linear.

$\{e_i\}$ = basis for H

$\Rightarrow \{\bar{e}_i\} \cup \{1\}$ = basis for $L(\Omega)$

Fact. $\overline{x+y} = \bar{x} + \bar{y} + \hat{\imath}(x,y)$

In $\mathbb{Z}/2$, $a^2 = a \rightsquigarrow$ above relation in B.

A Boolean polynomial is one made of square free monomials

The Arf invt is a quadratic Boolean poly:

$$\alpha = \sum_{i=1}^{g} \bar{x}_i \bar{y}_i$$

BCJ maps as Boolean Polys

Recall $\rho_\omega : \mathcal{I}(S_g) \longrightarrow \mathbb{Z}/2$

Dualize as $\sigma_f : \Omega \to \mathbb{Z}/2$

$$\sigma_f(\omega) = \rho_\omega(f).$$

Let $\Psi = \{\omega \in \Omega : \alpha(\omega) = 0\}$

<u>Fact</u>. σ is a homom. from $\mathcal{I}(S_g)$ to vect. space of fns $\Psi \to \mathbb{Z}/2$

We can rephrase our calculations:

$$\rho_\omega(T_c) = \sum \omega(x_i)\omega(y_i) \rightsquigarrow \sigma_{T_c}(\omega) = \sum \bar{x}_i(\omega)\bar{y}_i(\omega)$$

$$\rho_\omega(T_a T_b^{-1}) = \begin{cases} 0 & \omega(a) = 1 \\ \sum \omega(x_i)\omega(y_i) & \end{cases} = \left(\sum \omega(x_i)\omega(y_i)\right)\left(\omega(a)+1\right)$$

$$\rightsquigarrow \sigma_{T_a T_b^{-1}}(\omega) = \left(\sum \bar{x}_i(\omega)\bar{y}_i(\omega)\right)\left(\bar{a}+1\right) \quad \text{cubic!}$$

Use naturality to get all of B_3.

Remains to determine which elts are 0 on Ψ.

Answer: (linear functions) $\cdot\, \alpha$

Arf

BCJ for Open Surfaces

We can include S_g^1 into S_{g+1} and play the same game.

Everything works the same except now there are no cubic polys that are 0 on all of ψ.

$\rightsquigarrow$ image of BCJ is $\sum_{i=0}^{3} \binom{2g}{i}$ dim.

Johnson all gives a complete description of the relations among the ρ_w.