Generating MCG

<u>Alexander Trick</u> <u>Prop</u>. $\mathrm{Mod}(D^2) = 1$.

<u>Pf</u>. For $\varphi \in \mathrm{Homeo}^+(D^2, \partial D^2)$ consider

$$\Phi(x,t) = \begin{cases} (1-t)\,\varphi\left(\frac{x}{1-t}\right) & 0 \le |x| \le 1-t \\ x & \text{o.wise} \end{cases}$$

<u>g=0</u> <u>Lemma</u> $\mathrm{Mod}(\mathbb{R}^2) = 1$

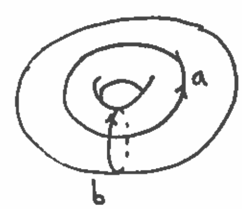
<u>Pf</u>. Straight line homotopy

$\Rightarrow$ <u>Prop</u> $\mathrm{Mod}(S^2) = 1$.

<u>g=1</u> <u>Thm</u>. The map $\psi: \mathrm{Mod}(T^2) \longrightarrow SL_2\mathbb{Z}$ (action on H_1) is an $\cong$

<u>Pf</u>. Injectivity: $K(G,1)$ theory (note $H_1 \cong \pi_1$)

Surjectivity: $T_a \longmapsto \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ $T_b \longmapsto \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$



In particular: $\mathrm{Mod}(T^2)$ is gen. by Dehn twists.

<u>g>1</u> Much more complicated!

Two ingredients: Complex of curves

Birman exact sequence.

Complex of Curves

$C(S_g)$ vertices: homotopy classes of scc in S_g
edges: disjoint reps

<u>Thm</u> (Lickorish '64) For $g \geq 2$ $C(S_g)$ is connected.

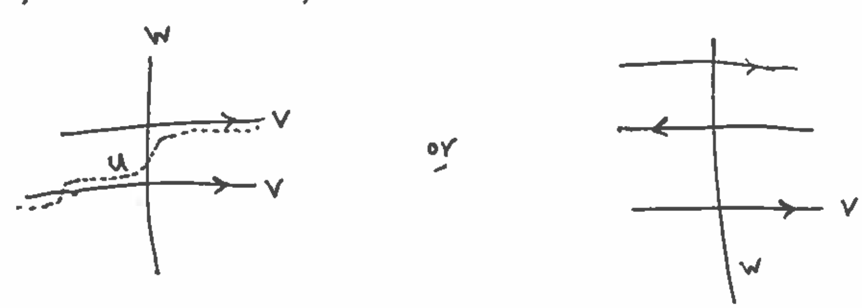
<u>Pf.</u> v, w vertices
To show v, w lie in same component.
Induct on $i(v,w)$

Base cases: $i(v,w) \leq 2$. follow from

<u>Lemma</u>. v, w fill $S_g \Rightarrow i(v,w) \geq 2g-1$
<u>Pf.</u> $v, w \rightsquigarrow$ cell decomp of S_g

$$2-2g = i(v,w) - 2i(v,w) + F \geq -i(v,w) + 1.$$

Now assume $i(v,w) \geq 3$. Must see:



In first case $i(u,v) = 1$ $i(u,w) < i(v,w)$ ▨

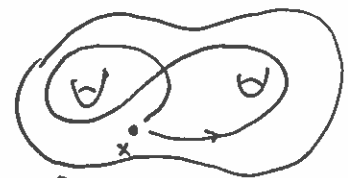
<u>Cor.</u> Complex of nonsep curves $N(S_g)$ is conn. $g \geq 2$

<u>Cor.</u> $\hat{N}(S_g)$ is conn. $g \geq 1$ vertices: nonsep curves
edges: $i = 1$.

Birman Exact Sequence

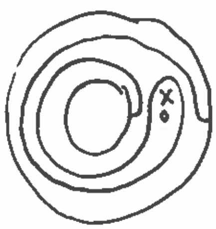
$\mathrm{Mod}(S,x) = \pi_0 \mathrm{Homeo}^+(S,x)$.

<u>Push map</u> $\quad \mathrm{Push}: \pi_1(S,x) \longrightarrow \mathrm{Mod}(S,x)$



example: α = simple loop

$\mathrm{Push}(\alpha) = T_c T_d^{-1}$ $\quad c,d$ = left, right pushoffs



<u>Forgetful map</u> $\quad \mathrm{Forget}: \mathrm{Mod}(S,x) \longrightarrow \mathrm{Mod}(S)$

note $\mathrm{Im}\,\mathrm{Push} \subseteq \ker \mathrm{Forget}$.

<u>Thm</u>. For $\chi(S) < 0$: $\quad 1 \longrightarrow \pi_1(S,x) \longrightarrow \mathrm{Mod}(S,x) \longrightarrow \mathrm{Mod}(S) \longrightarrow 1$

is exact.

(For $\chi(S) \geq 0$, lose injectivity.)

<u>Cor</u>. $\mathrm{Mod}(S_{g,n})$ is fin. gen. by Dehn twists $g = 0, 1$.

<u>Pf of Thm</u>. Long exact seq. for fiber bundle

$$\begin{array}{ccc} \mathrm{Homeo}(S,x) & \longrightarrow & \mathrm{Homeo}(S) \\ & & \downarrow \\ & & S \end{array}$$

plus: $\pi_{0,1}\mathrm{Homeo}(S) = 1$. ∎