Generating Torelli

Goal: \( I(S_g) \) is gen. by BP maps (and Dehn twists about sep curves)

Original proof: 1971 Birman gives presentation for \( Sp_{2g}(\mathbb{Z}) \)
1978 Powell interprets relations
1980 Johnson, lantern relation

Want a proof analogous to \( Mod(S_g) \) case.

Complex of homologous curves

Fix (primitive) \( x \in H_1(S_g; \mathbb{Z}) \)
\( C_x(S_g) = \) subgraph of \( C(S_g) \) spanned by (unoriented) reps of \( x \).

Goal: Connected.

Will use auxiliary complex \( B_x(S_g) \), the complex of cycles. Points of \( B_x(S_g) \) are simple, irredundant reps of \( x \).
The Complex of Cycles

$C = \text{oriented multicurve, } n \text{ components}$

$\mapsto [0, \infty)^n \rightarrow H_1(S_g; \mathbb{Z}) \text{ orthant } O(c)$

$A(S_g) = \prod_c O(c) / \sim$

example.

$A_x(S_g) \leq A(S_g) \text{ reps of } x.$

Say $x = [c_1]$
The cells of $A_x(S_g)$ are not necessarily compact:

If $[c_1] = x$ then $[c_1 + bc_2] = x \forall b \in \mathbb{R}$

Or:

An oriented multicurve is **reduced** if

1. the corresponding cell is compact
2. it has no homologically trivial subset
3. the dual directed graph is recurrent

Dual graphs:

The complex of cycles $B_x(S_g)$ is the subcomplex of $A_x(S_g)$ whose cells correspond to reduced oriented multicurves.

We'll show $B_x(S_g)$ is contractible.
Examples of cells

$X = [c_1]$ 

$X = [c_1] + [c_2]$ 

Q. Which polytopes arise?
**Properties of Cells**

**Prop.** The dim. of a cell $= \# \text{compl. comp.'s} - 1$

**Pf.** Defn of homology.

$\Rightarrow$ vertices $\leftrightarrow$ nonsep. multicurves.

**Prop.** Vertices of $B_x(S_g)$ are oriented multicurves with integral weights.

**Pf.** Given a vertex, consider a loop intersecting in one point.

**Prop.** $\text{Dim } B_x(S_g) = 2g-3$.

**Pf.** $x = [c,]$

$\Rightarrow B_x(S_2)$ is a graph.
**Contractibility**

**Theorem.** $B_x(S_g)$ is contractible.

**Surgery on 1-cycles**

Say $c, d \in A_x(S_g)$. Thicken $c, d$ according to weights and then:

![Diagram of surgery on 1-cycles](image)

If $[c] = [d] = x$, this procedure will result in a 1-cycle rep'ing $x$. Why?

$$H_1(S_g; \mathbb{Z}) \cong H'_1(S_g; \mathbb{Z}) \cong \text{Hom}(H_1(S_g; \mathbb{Z}), \mathbb{Z}) \leftrightarrow [S_g, S']$$

The original $c, d$ give maps $S_g \to S'$ by integrating against width of annuli. The surgered picture corresponds to the map $S_g \to S'$ obtained by integrating against both widths.
Prop. \( A_x(S_g) \) is contractible

Pf. Fix some \( c \in A_x(S_g) \). Consider:
\[
F_t(d) = \text{Surger} \left( tc + (1-t)d \right)
\]

Draining 1-cycles

Suppose \( c \in A_x(S_g) \) is not reduced.
\( \sim \) \{\( R_i \)\} subsurfaces with \( \partial R_i \leq c \)
\[
\text{Drain}_t(c) = c - t \Sigma \partial R_i
\]

Prop. \( A_x(S_g) \) def. retracts to \( B_x(S_g) \).
In partic. \( B_x(S_g) \) is contractible.

Pf. Drain

In particular, \( B_x(S_2) \) is a tree.
Connectivity of $C_x(S_g)$

Basic strategy

Define $\text{weight} : B_x(S_g) \rightarrow \mathbb{Z}$

$$\sum w_i c_i \mapsto \sum w_i$$

Note: $C_x(S_g) = \text{weight}^{-1}(1)$.

Now, given $v, w \in C_x^{(0)}(S_g)$, we connect them in $B_x^{(1)}(S_g)$:

We then push the highest vertex down inductively until the path lies in $C_x(S_g)$. 
Key idea: If we cut along a vertex of $B^0_x(Sg)$, we get "cobordism"

What does an edge in $B_x(Sg)$ look like?

The region between transfers weight from one side to the other $\Rightarrow$ the new vertex will have smaller weight iff there are fewer interior curves than boundary curves.

Call the edge on the right a pants edge. This is the simplest way to reduce weight.
Proof that \( C_x(S_g) \) is connected

**Step 1.** Make maxima isolated, by making pants edges/triangles.

**Step 2.** Make highest edges into pants edges in same way

**Step 3.** Given

Connect \( uv \) to \( v \) by a seq of pants triangles

Can then push \( v \) down. Apply this process inductively.
To this end, consider the graph with vertices: pants edges emanating from \( v \)
\[ \leftrightarrow \] certain arcs in \( S \setminus v \)
edges: disjoint arcs.

To show: connected

Notes. • Every vertex is adjacent to one connecting first two components of \( \partial (S \setminus v) \).
  • Push maps (corresponding to \( 1^\text{st} \) \( \partial \) - comp act transitively on these.
  • \( \Pi_1 \) (punctured sphere) has a simple genset \( \{ x_i \} \)

So: suffices to show that each \( \text{Push}(x_i) \cdot \alpha \) lies in same component as \( \alpha \).

Sample case: \( x_i \) lies on LHS of \( S \setminus v \). Then if \( \beta \) lies on RHS we have

\[ \alpha \quad \beta \quad \text{Push}(x_i) \cdot \alpha \]
Proving Torelli is gen. by BP maps

Ingredient #1. $C_x(S_g)$ is connected $\checkmark$

Ingredient #2. Fact. Say $G \subseteq X = \text{graph}$

$A \subseteq G$ s.t. $\forall$ edges $vw$

$\exists \ g \in A$ with $g \cdot v = w$.

Then $G = \langle A, \text{vertex stabs} \rangle$

Pf. Same as before

Ingredient #3. If $vw$ is an edge of $C_x(S_g)$

$\exists$ BP map taking $v$ to $w$.

Thus it suffices to show $\text{Stab}_{I(S_g)}(v)$ is gen. by BP maps and Dehn twists about sep curves.
Two Birman Exact Sequences for Torelli

One Marked Point

\[ 1 \rightarrow \pi_1(S_g, p) \rightarrow \Gamma(S_g, p) \rightarrow \Gamma(S_g) \rightarrow 1 \]

(restriction of usual BES)

Two Marked Points

\[ 1 \rightarrow K \rightarrow \Gamma(S_g, \{p, q\}) \rightarrow \Gamma(S_g, p) \rightarrow 1 \]

\[ K = \{ \pi_1(S_g \setminus p, q) \} \]

What is defn of \( \Gamma(S_g, \{p, q\}) \)? It is the image of \( \Gamma(S_g, c) \) under the cutting map:

\[ \rightarrow \Gamma(S_g, \{p, q\}) = \ker \left( \text{Mod}(S_g, \{p, q\}) \rightarrow \text{Aut} \: \Gamma_1(S_g, \{p, q\}) \right) \]

Can define \( \Gamma(S_g, p) \) in same way, as further image under forgetting \( q \). Get usual defn.
Stabilizers are gen. by BP maps...

First, $\text{Stab}_{I(S_g)}(c) \cong I(S_{g-1}, \{p, q\})$ since:

$$1 \rightarrow \langle T_c \rangle \rightarrow \text{Mod}(S_g, c) \rightarrow \text{Mod}(S_g, \{p, q\}) \rightarrow 1$$

and $T_c \notin I(S_g)$.

Step 1. K gen by BP maps & Dehn twists about seps.

$$\pi_1(S_g \setminus p, q) \cong H_1(S_g, \{p, q\}) \cong H_1(S_g) \oplus \mathbb{Z}$$

- trivial on first factor,
- action on 2$^{nd}$ factor is $\alpha \cdot v = [\alpha] + v$.

Check:

$$\Rightarrow K = [\pi_1, \pi_1].$$

Now need: K gen. by simple (sep) loops.
Realize $S_g$ as 4g-gon with opp sides id
d and use the fact that commutator subgps
are normally gen. by commutators of gens.
Step 2. \( \mathcal{H}_i(\Sigma_g,p) \leq I(\Sigma_g,p) \) gen. by BP maps.

Even better:

**Thm.** \( I(\Sigma_g) \) is gen. by BP maps of genus one.

**Pf.**

\[ Ta T_b^{-1} = (Ta T_c^{-1})(T_c T_d^{-1})(T_d T_b^{-1}) \]

Still need to address base case \( g=2 \)!
Genus 2

$C_x(S_2)$ is not connected 😐
Will instead use $I(S_2) \cong B_x(S_2)$.
Already showed $B_x(S_2)$ is a tree.
Will show $B_x(S_2)/I(S_2)$ is a tree.

**Fact.** If $G \triangleleft T = \text{tree}$ and $T/G = \text{tree}$
then $G$ is (freely) gen. by vertex stabilizers.

**Pf.** Key point: $T/G \hookrightarrow T$
So $T$ covered by translates of $T/G$

![Diagram]

Fix $X$, a copy of $T/G$ in $T$. (“tile”)
Let $g \in G$. Suppose first that $g \cdot X \cap X \neq \emptyset$
$\Rightarrow g \cdot X \cap X = \{v\} \Rightarrow g \in \text{Stab}(v)$
(otherwise $T/G$ would have a loop). Induct on tile distance. Free b/c no loops
Remains: ① $\mathbf{Bx}(S_2) / \mathbf{I}(S_2) = \text{tree}$
   ② Stab's gen by Dehn twists (and free)

Proof of ① Weight fn. descends to quotient.
   And $\exists!$ vertex of weight 1 (in quotient).

To show: if $\text{weight}(v) > 1 \exists! w$ with
   \[ \text{weight}(w) < \text{weight}(v) \text{ and } \vDash_w \vDash_v \]

$v$ looks like:

\[ \vDash_a a b \vDash_p p, q \in \mathbb{Z} \]

$w$ given by nonsep curve $C$ in middle.

\[ \leftrightarrow \text{curve in cut surface.} \]

$C$ nonsep $\implies$ $C$ does not sep. left from right

$\text{weight}(w) < \text{weight}(v) \implies C$ separates top from bottom.

Can think of $C$ as an arc connecting top two circles.

Make a graph: vertices: arcs as above
   edges: disjointness.

This connected (use usual trick with $\mathbf{PMod}(S_{0,4})$ action).
   Adjacent vertices differ by sep twist.
Proof of \( \circ \) Two kinds of vertices

**Connected multicurves:** Use Birman exact seq. as before (no change).

**Disconnected multicurves:** By above argument can assume (up to Dehn twists about sep curves) that a stabilizer of \( a \) & \( b \) also fixes \( c \).

\[ \Rightarrow \text{it is trivial} . \]

**Note:** \( \pi_1 (S^1, 1) \) is free.

**Note:** \( PMod (S_0, 4) \) free.

We have shown \( \mathbb{I}(S_2) = \langle T_c \mid c \text{ sep} \rangle \)

With some more bookkeeping, we show it is freely gen. by Dehn twists, one for each symplectic splitting. \( \square \)
The Farey Graph and $B_x(S_2)/I(S_2)$.

Vertices of $B_x(S_2)/I(S_2)$ are minimal bases for Lagrangian subspaces of $H_1(S_2; \mathbb{Z})$ containing $x$.

(Minimal means that if the basis contains two elts, neither is $x$.)

Edges are for $\{a, b\} \rightarrow \{a, a+b\}$
(If one of these is $x$, just drop it.)