Generating Torelli

Goal: $\mathcal{I}(S_g)$ is gen. by BP maps (and Dehn twists about sep curves)

Original proof: 1971 Birman gives presentation for $Sp_{2g}(\mathbb{Z})$
1978 Powell interprets relations
1980 Johnson, lantern relation

Want a proof analogous to $\mathrm{Mod}(S_g)$ case.

Complex of homologous curves

Fix (primitive) $x \in H_1(S_g;\mathbb{Z})$

$\mathcal{C}_x(S_g)$ = subgraph of $\mathcal{C}(S_g)$ spanned by (unoriented) reps of x.

goal: Connected.

"borrowing complex"

Will use auxilliary complex $\mathcal{B}_x(S_g)$, the complex of cycles. Points of $\mathcal{B}_x(S_g)$ are simple, irredundant reps of x.

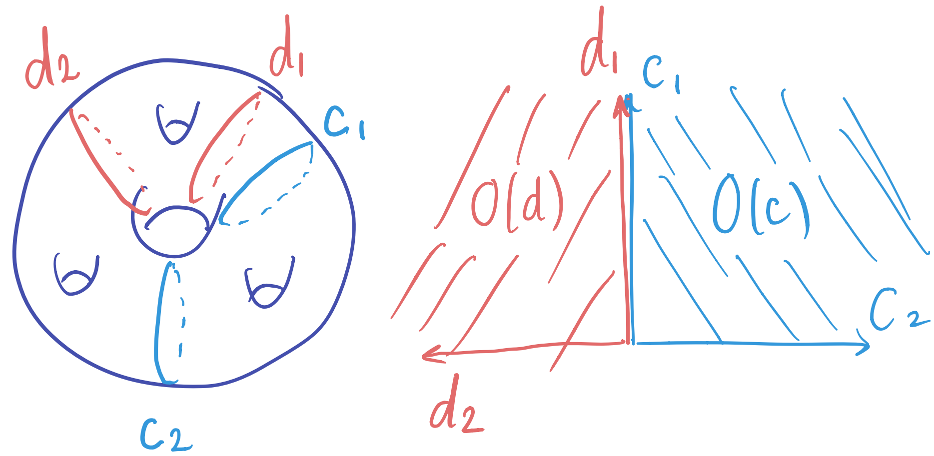
The Complex of Cycles

c = oriented multicurve, n components

$\rightsquigarrow \quad [0,\infty)^n \longrightarrow H_1(S_g;\mathbb{Z})$ orthant $O(c)$

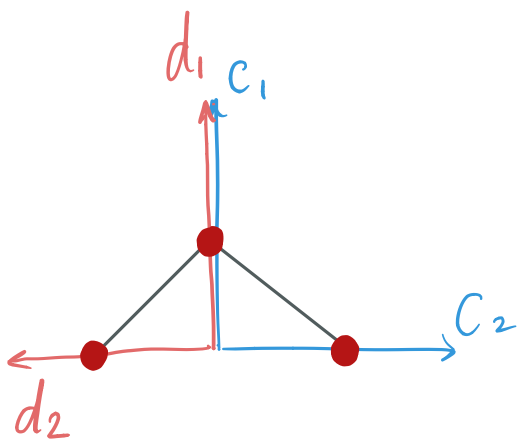
$$A(S_g) = \coprod_c O(c) \;/\sim$$

example.

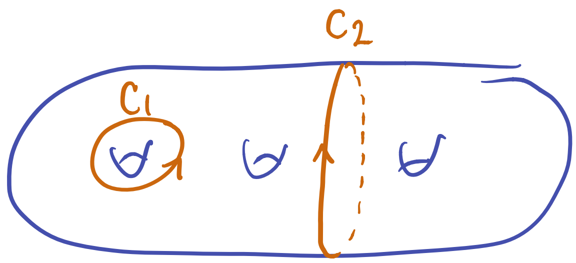


$$A_x(S_g) \subseteq A(S_g) \quad \text{reps of } x.$$

Say $x = [c_1]$

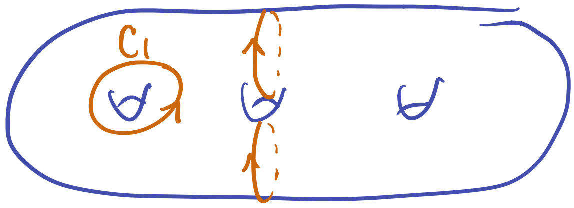


The cells of $A_x(S_g)$ are not necessarily compact:



If $[c_1] = x$ then $[c_1 + bc_2] = x$ $\forall\, b \in \mathbb{R}$

Or:



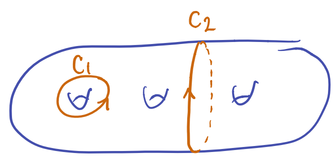
An oriented multicurve is <u>reduced</u> if

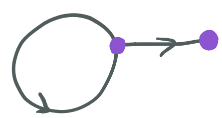
(1) the corresponding cell is compact

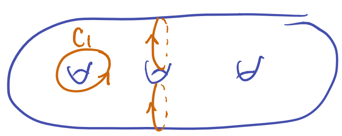
$\iff$ (2) it has no homologically trivial subset

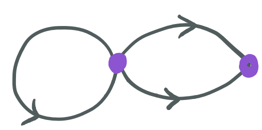
$\iff$ (3) the dual directed graph is recurrent

Dual graphs:







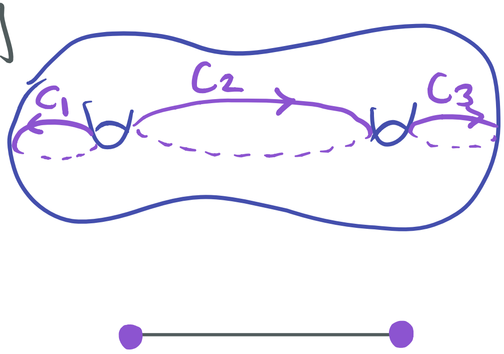


The complex of cycles $B_x(S_g)$ is the subcomplex of $A_x(S_g)$ whose cells correspond to reduced oriented multicurves.

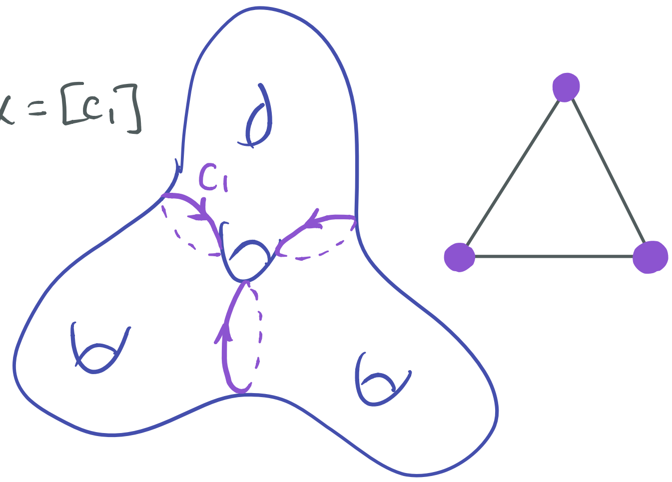
We'll show $B_x(S_g)$ is contractible.

Examples of cells

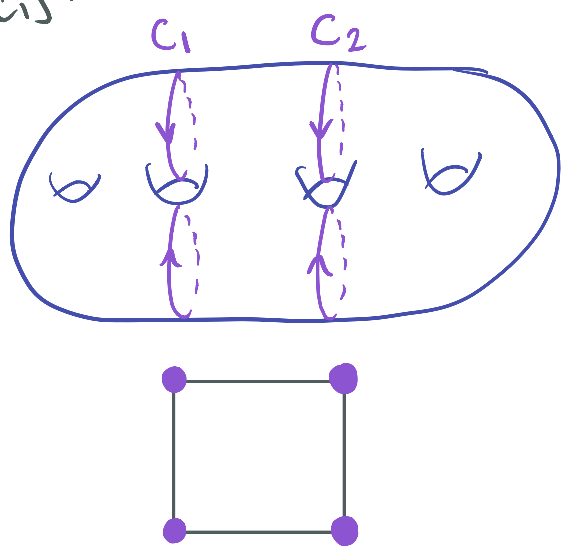
$X=[c_1]$

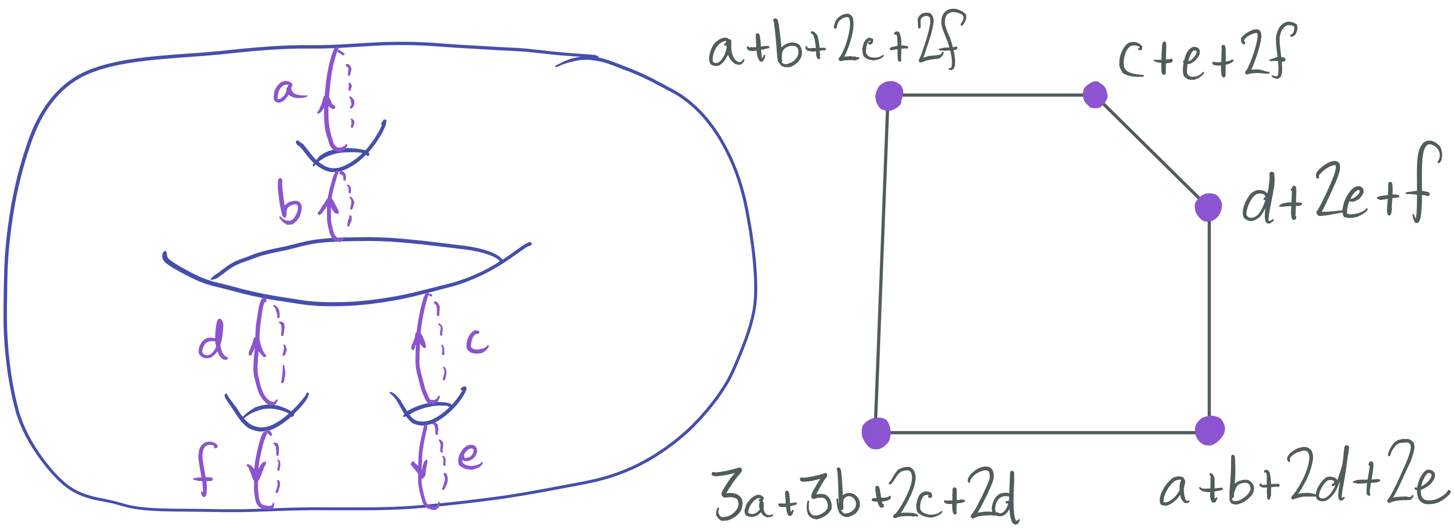


$X=[c_1]$



$X=[c_1]+[c_2]$





Q. Which polytopes arise?

Properties of Cells

Prop. The dim. of a cell = # compl. comp.'s − 1.

Pf. Defn of homology.

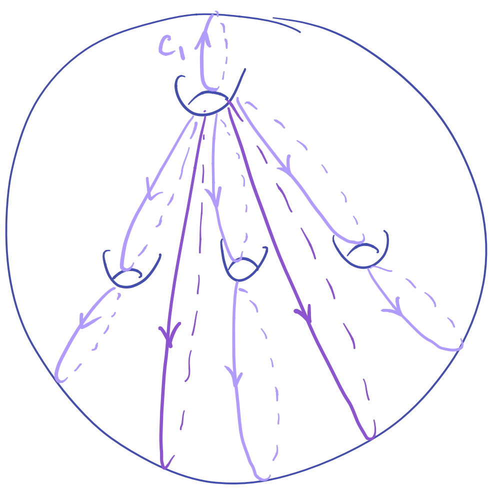
$\Longrightarrow$ vertices $\longleftrightarrow$ nonsep. multicurves.

Prop. Vertices of $B_x(S_g)$ are oriented multicurves with integral weights.

Pf. Given a vertex, consider a loop intersecting in one point.

Prop. $\operatorname{Dim} B_x(S_g) = 2g-3$.

Pf.



$x = [c_1]$.

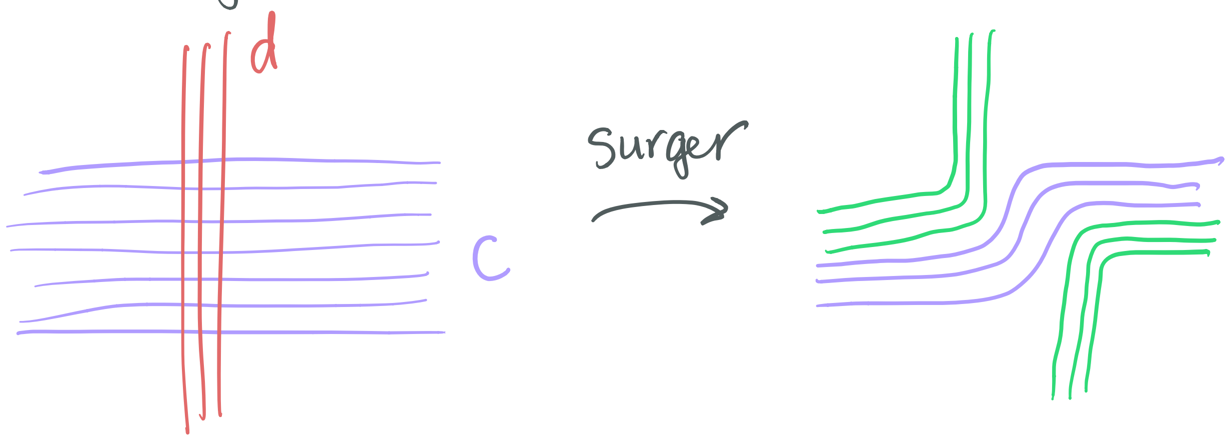
$\rightsquigarrow B_x(S_2)$ is a graph.

Contractibility

Theorem. $B_x(S_g)$ is contractible.

Surgery on 1-cycles

Say $c, d \in A_x(S_g)$. Thicken c, d according to weights and then:



If $[c] = [d] = x$, this procedure will result in a 1-cycle rep'ing x. Why?

$$H_1(S_g;\mathbb{Z}) \cong H^1(S_g;\mathbb{Z}) \cong \mathrm{Hom}(H_1(S_g;\mathbb{Z}),\mathbb{Z}) \longleftrightarrow [S_g, S^1]$$

The original c, d give maps $S_g \to S^1$ by integrating against width of annuli. The surgered picture corresponds to the map $S_g \to S^1$ obtained by integrating against both widths.

Prop. $A_x(S_g)$ is contractible

Pf. Fix some $c \in A_x(S_g)$. Consider:

$$F_t(d) = \mathrm{Surger}\big(tc + (1-t)d\big) \qquad \square$$

Draining 1-cycles

Suppose $c \in A_x(S_g)$ is not reduced.
$\rightsquigarrow \{R_i\}$ subsurfaces with $\partial R_i \subseteq c$
$\mathrm{Drain}_t(c) = c - t \sum \partial R_i$

Prop. $A_x(S_g)$ def. retracts to $B_x(S_g)$.
In partic. $B_x(S_g)$ is contractible.

Pf. Drain $\square$

In particular, $B_x(S_2)$ is a tree.

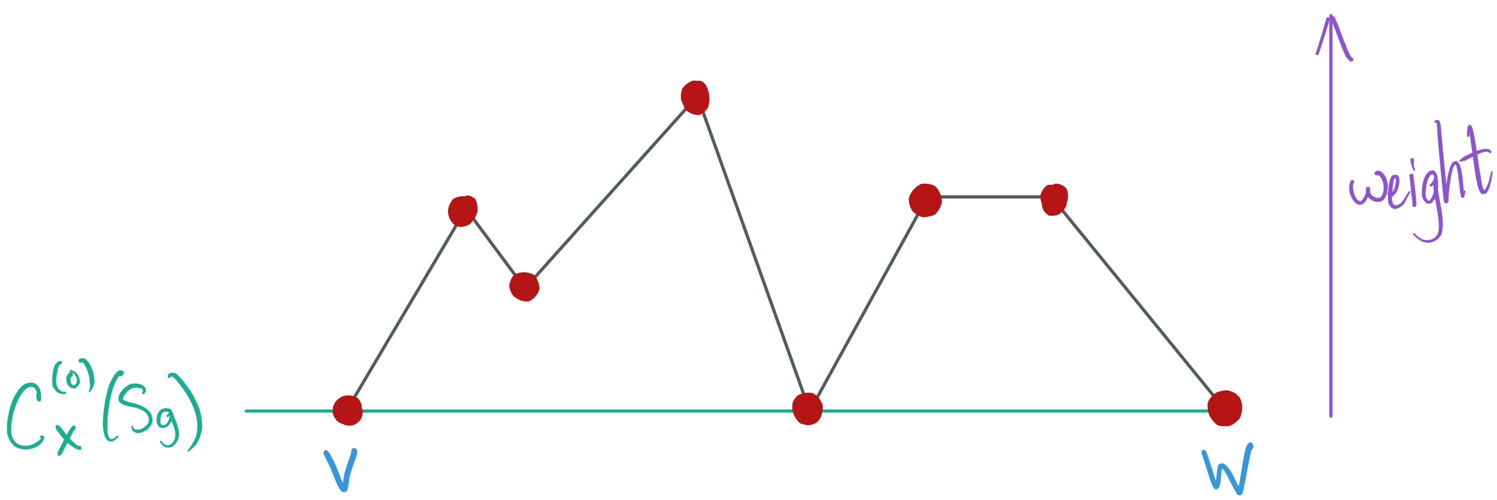
Connectivity of $C_x(S_g)$

Basic strategy

Define weight : $B_x(S_g) \longrightarrow \mathbb{Z}$
$$\sum w_i c_i \longmapsto \sum w_i$$

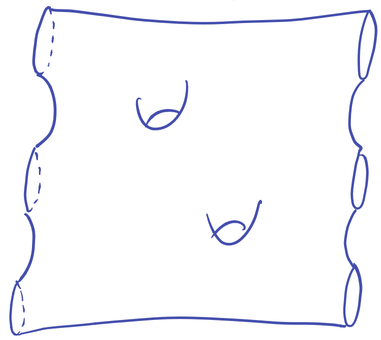
Note: $C_x(S_g) = \text{weight}^{-1}(1)$.

Now, given $v, w \in C_x^{(0)}(S_g)$, we connect them in $B_x^{(1)}(S_g)$:



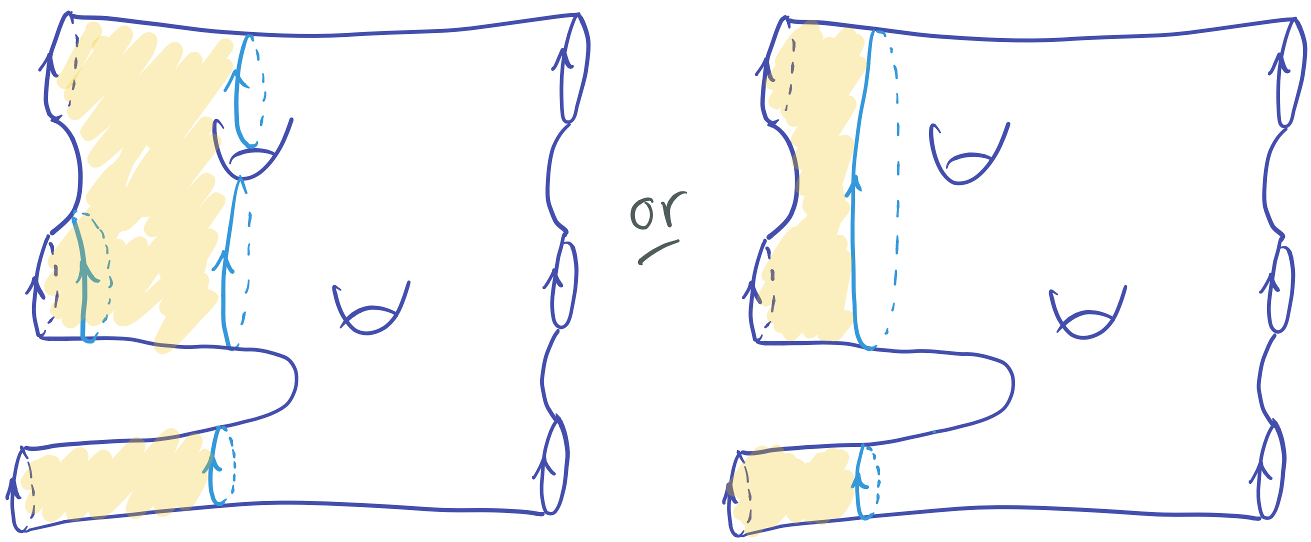
We then push the highest vertex down inductively until the path lies in $C_x(S_g)$.

Key idea: If we cut along a vertex of $B_x^{(0)}(S_g)$ we get



"cobordism"

What does an edge in $B_x(S_g)$ look like?

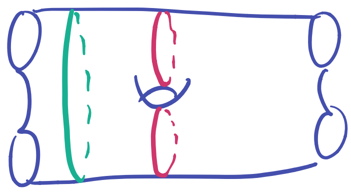


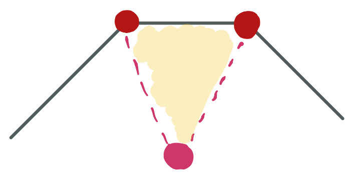
The region between transfers weight from one side to the other $\Longrightarrow$ the new vertex will have smaller weight iff there are fewer interior curves than boundary curves.

Call the edge on the right a <u>pants edge</u>.
This is the simplest way to reduce weight.

Proof that $Cx(S_g)$ is connected

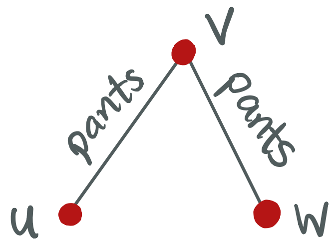
Step 1. Make maxima isolated, by making pants edges/triangles.



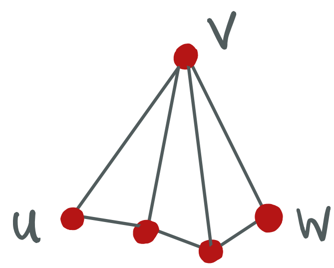


Step 2. Make highest edges into pants edges in same way

Step 3. Given



Connect uv to v by a seq of pants triangles



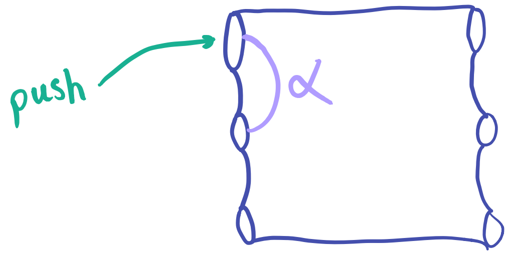
Can then push v down. Apply this process inductively.

To this end, consider the graph with

Vertices: pants edges emanating from v

$\longleftrightarrow$ certain arcs in $S \setminus v$

edges: disjoint arcs.



To show: connected

Notes.
- Every vertex is adjacent to one connecting first two components of $\partial(S \setminus v)$.
- Push maps (corresponding to 1^{st} ∂-comp act transitively on these.
- π_1(punctured sphere) has a simple gen set $\{x_i\}$

So: suffices to show that each $\mathrm{Push}(x_i)\cdot\alpha$ lies in same component as α.

Sample case: x_i lies on LHS of $S \setminus v$.
Then if β lies on RHS we have

$\square$

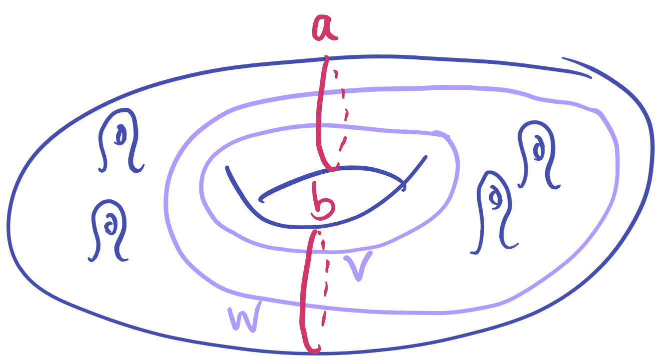
Proving Torelli is gen. by BP maps

Ingredient #1. $C_x(S_g)$ is connected ✓

Ingredient #2. Fact. Say $G \circlearrowright X$ = graph
$A \subseteq G$ s.t. $\forall$ edges $\overline{vw}$
$\exists\ g \in A$ with $g \cdot v = w$.
Then $G = \langle A, \text{vertex stabs} \rangle$

Pf. Same as before

Ingredient #3. If vw is an edge of $C_x(S_g)$
$\exists$ BP map taking v to w.



$T_a T_b^{-1}(v) = w$

Thus it suffices to show $\text{Stab}_{I(S_g)}(v)$ is gen. by BP maps and Dehn twists about sep curves.

Two Birman Exact Sequences for Torelli

One Marked Point

$$1 \to \pi_1(S_g, p) \to \mathcal{I}(S_g, p) \to \mathcal{I}(S_g) \to 1$$

(restriction of usual BES)

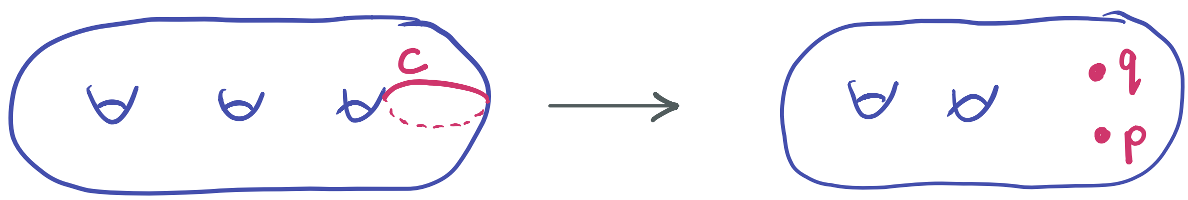
Two Marked Points

$$1 \to K \to \mathcal{I}(S_g, \{p,q\}) \to \mathcal{I}(S_g, p) \to 1$$

$K = [\pi, \pi] \qquad \pi = \pi_1(S_g \setminus p, q)$

$\boxed{\text{will verify below!}}$

What is defn of $\mathcal{I}(S_g, \{p,q\})$? It is the image of $\mathcal{I}(S_g, c)$ under the cutting map:



$\rightsquigarrow \mathcal{I}(S_g, \{p,q\}) = \ker\left(\mathrm{Mod}(S_g, \{p,q\}) \to \mathrm{Aut}\, H_1(S_g, \{p,q\})\right)$

Can define $\mathcal{I}(S_g, p)$ in same way, as further image under forgetting q. Get usual defn.

Stabilizers are gen. by BP maps...

First, $\mathrm{Stab}_{\mathcal{I}(S_g)}(c) \cong \mathcal{I}(S_{g-1},\{p,q\})$ since:

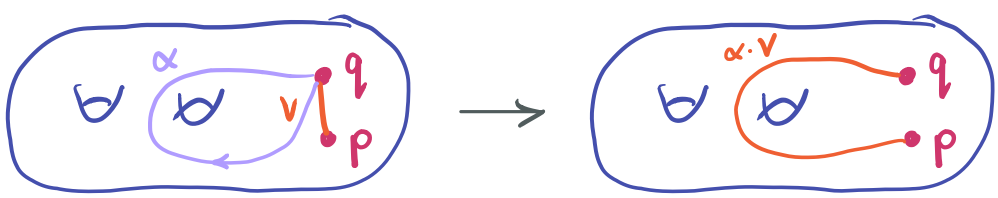
$$1 \to \langle T_c \rangle \to \mathrm{Mod}(S_g, c) \to \mathrm{Mod}(S_g,\{p,q\}) \to 1$$

and $T_c \notin \mathcal{I}(S_g)$.

<u>Step 1.</u> K gen by BP maps & Dehn twists about seps.

$$\pi_1(S_g \setminus p,q) \curvearrowright H_1(S_g,\{p,q\}) \cong H_1(S_g) \oplus \mathbb{Z}$$

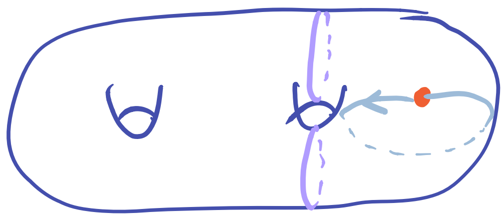
- trivial on first factor.
- action on 2nd factor is $\alpha \cdot v = [\alpha] + v$.

Check: 

$\Rightarrow K = [\pi_1, \pi_1]$.

Now need: K gen. by simple (sep) loops.
Realize S_g as $4g$-gon with opp sides id'd.
and use the fact that commutator subgps are normally gen. by commutators of gens.

Step 2. $\pi_1(S_g,p) \subseteq \mathcal{I}(S_g,p)$ gen by BP maps.

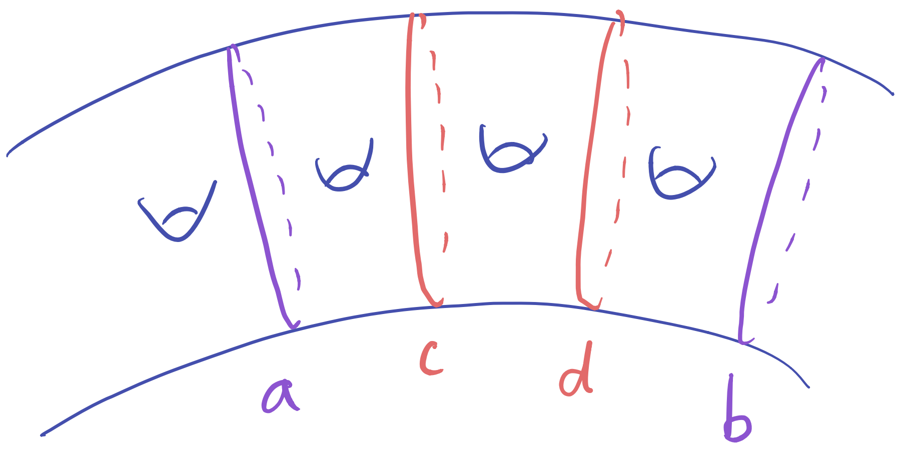
easy: 

$\square$

Even better:

Thm. $\mathcal{I}(S_g)$ is gen. by BP maps of genus one.

Pf.



$$T_a T_b^{-1} = (T_a T_c^{-1})(T_c T_d^{-1})(T_d T_b^{-1}).$$

Still need to address base case $g=2$!

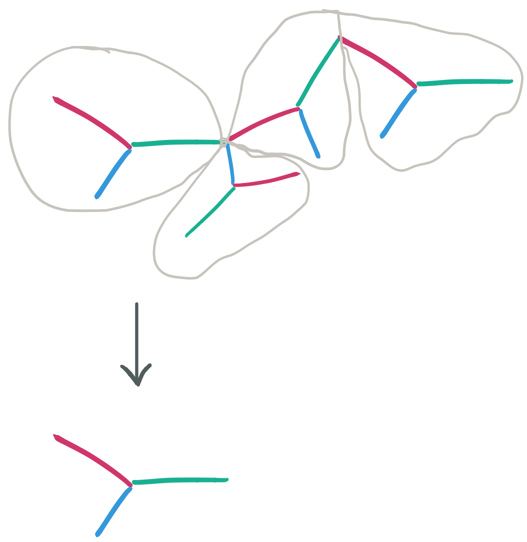
Genus 2

$C_x(S_2)$ is not connected ☹
Will instead use $\mathcal{I}(S_2) \circlearrowright B_x(S_2)$.
Already showed $B_x(S_2)$ is a tree.
Will show $B_x(S_2)/\mathcal{I}(S_2)$ is a tree.

Fact. If $G \circlearrowright T$ = tree and T/G = tree
then G is (freely) gen. by vertex stabilizers.

Pf. Key point: $T/G \hookrightarrow T$
So T covered by translates of T/G

T



T/G

Fix X, a copy of T/G in T. ("tile")
Let $g \in G$. Suppose first that $g \cdot X \cap X \neq \emptyset$
$\Rightarrow g \cdot X \cap X = \{v\} \Rightarrow g \in \mathrm{Stab}(v)$
(otherwise T/G would have a loop). Induct
on tile distance. Free b/c no loops $\square$

Remains: ① $Bx(S_2)/\mathcal{I}(S_2)$ = tree

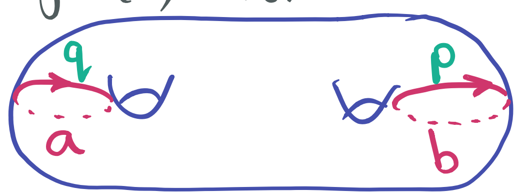
② Stab's gen by Dehn twists (and free)

Proof of ① Weight fn. descends to quotient.

And $\exists!$ vertex of weight 1 (in quotient).

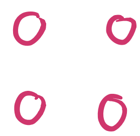
To show: if weight(v) > 1 $\exists!$ w with

weight(w) < weight(v) and 

v looks like: 

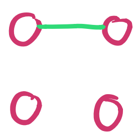
$p, q \in \mathbb{Z}$

w given by nonsep curve c in middle.

$\longleftrightarrow$ curve in cut surface. 

c nonsep $\Longrightarrow$ c does not sep. left from right

weight(w) < weight(v) $\Longrightarrow$ c separates top from bottom.

Can think of c as an arc connecting top two circles. 

Make a graph: vertices: arcs as above

edges: disjointness.

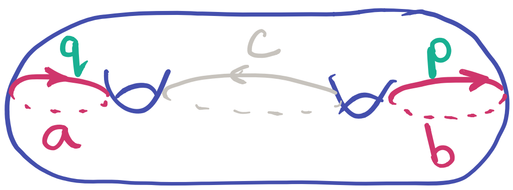
This connected (use usual trick with $PMod(S_{0,4})$ action). Adjacent vertices differ by sep twist.

Proof of ② Two kinds of vertices

Connected multicurves: Use Birman exact seq. as before (no change).

Note: $\pi_1(S_{1,1})$ is free.

Disconnected multicurves: By above argument can assume (up to Dehn twists about sep curves) that a stabilizer of a & b also fixes c.



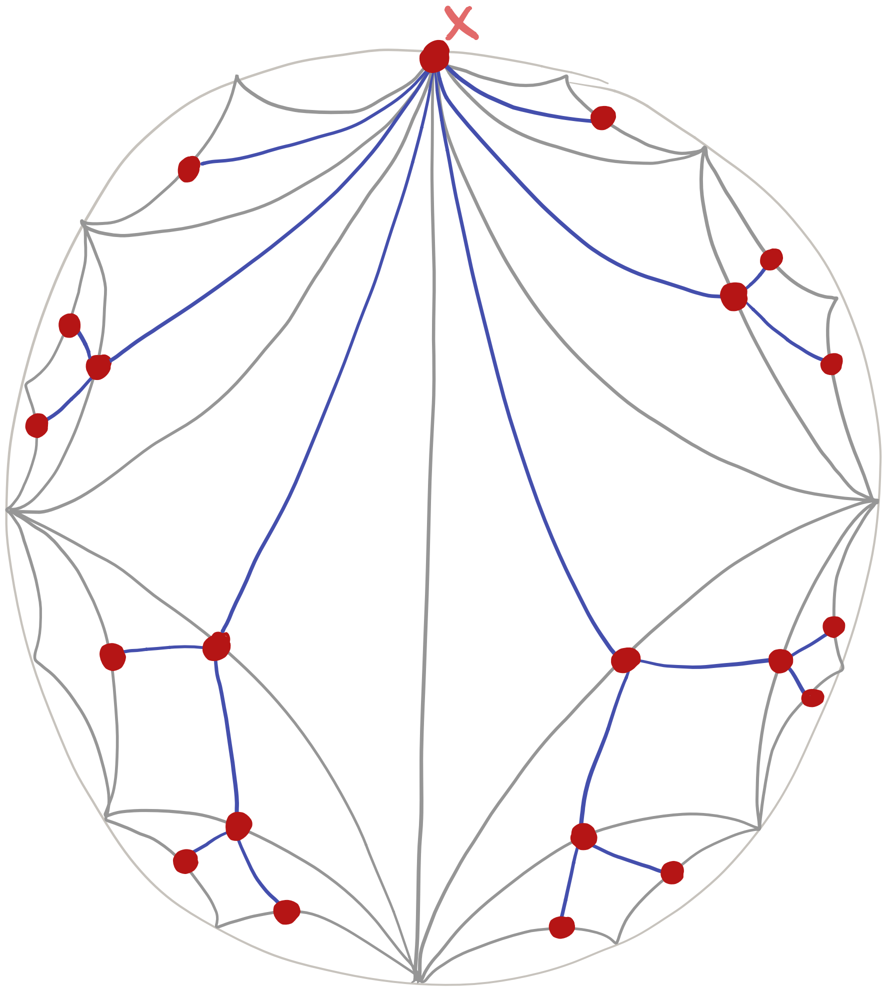
$\Longrightarrow$ it is trival.

Note: $\mathrm{PMod}(S_{0,4})$ free.

We have shown $\mathcal{I}(S_2) = \langle T_c \mid c \text{ sep} \rangle$

With some more bookkeeping, we show it is freely gen. by Dehn twists, one for each symplectic splitting. $\square$

The Farey Graph and $\mathcal{B}_x(S_2)/\mathcal{I}(S_2)$.



one of these for each Lagrangian subspace containing x. All glued at x.

Vertices of $\mathcal{B}_x(S_2)/\mathcal{I}(S_2)$ are minimal bases for Lagrangian subspaces of $H_1(S_2;\mathbb{Z})$ containing x.

(Minimal means that if the basis contains two elts, neither is x.)

Edges are for $\{a,b\} \longrightarrow \{a, a+b\}$
(If one of these is x, just drop it.)