Johnson I

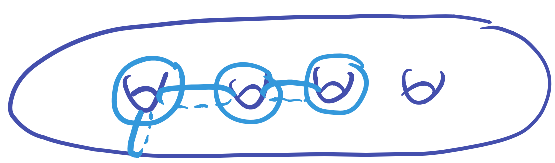
Thm $\mathcal{I}(S_g)$ is fin. gen. by Dehn twists for $g \geq 3$.

Basic strategy 1. List prospective generators $\{g_i\}$.
s.t. g_1 is a BP map of genus 1.
2. Show $\langle g_i \rangle \trianglelefteq \mathrm{Mod}(S_g)$.

This suffices since $\langle\langle g_i \rangle\rangle_{\mathrm{Mod}(S_g)} = \mathcal{I}(S_g)$.

Chains and BP maps.

A chain:



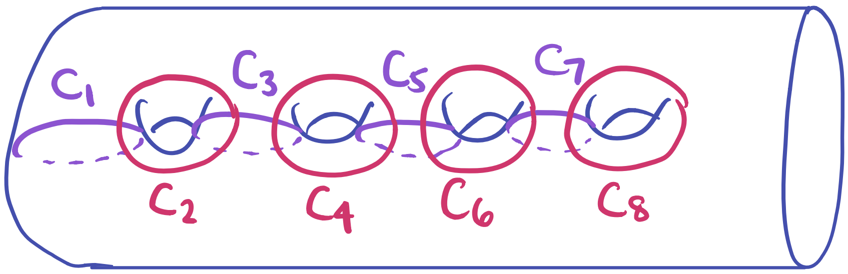
$\rightsquigarrow$ BP map.

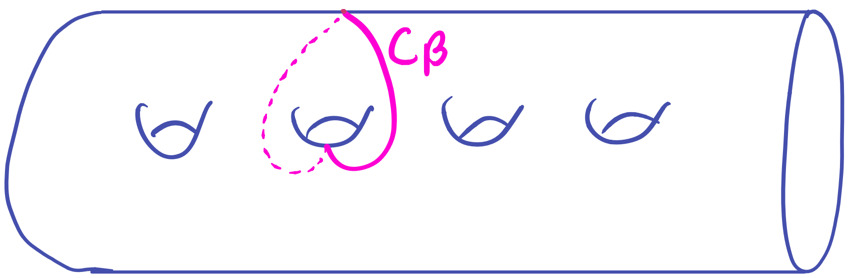
Given a chain, can resolve intersections to get another chain. Can also take subchains.

Given a chain $ch(c_1, \ldots, c_n)$
$ch(i_1 i_2 \cdots i_{k+1})$
denotes the chain you get by combining $c_{i_1}, \ldots, c_{i_2 - 1}$, $c_{i_2}, \ldots, c_{i_3 - 1}$ etc. dropping $c_{k+1}, \ldots, c_n$. "subchain"

Denote the BP-map $[i_1 i_2 \cdots i_{k+1}]$

Listing the Generators





Consider the chains:

$(c_1, \dots, c_{2g})$ straight chain

$(c_\beta, c_5, \dots, c_{2g})$ β-chain

Use same notation for subchains of β-chain:

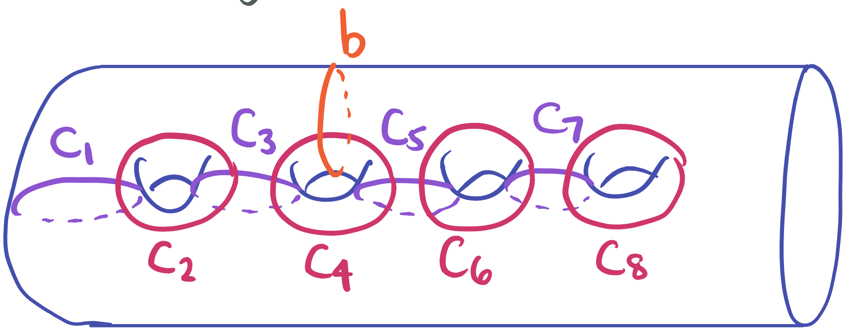
$(\beta i_1) =$ surger $c_\beta, c_5, \dots, c_{i_1-1}$

Theorem. For $g \geq 3$ the odd subchain maps of straight chain & β-chain generate $\mathcal{I}(S_g^1)$

Since $\mathcal{I}(S_g^1) \twoheadrightarrow \mathcal{I}(S_g)$ this gives closed case as well.

Setup.

Humpries generators:



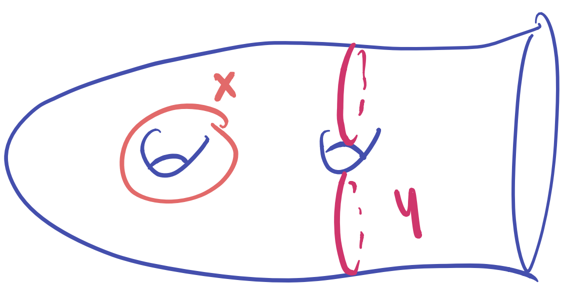
Let $J(S_g^1)$ & $J(S_g)$ denote groups gen by Johnson's generators.

As above, need to show

$$T_x * y = T_x y T_x^{-1} \in J(S_g)$$

$\forall$ $x \in$ Humphries set
$y \in$ Johnson set

In many cases $T_x * y$ equals y or is another Johnson gen:



Conjugating by a Positive Twist in the Chain

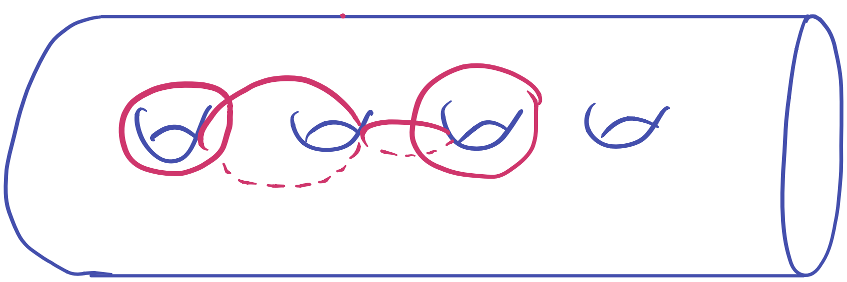
Prop

j in $\{i_1,\dots\}$	$j+1$ in $\{i_1,\dots\}$	$T_{c_j} * [i_1\ i_2 \cdots]$
✓	✓	$[i_1\ i_2 \cdots]$
✗	✗	$[i_1\ i_2 \cdots]$
i_m	✗	$[i_1 \cdots i_{m-1}\ \ i_m+1 \cdots]$
✗	i_m	$[i_1 \cdots i_{m-1}\ \ i_m-1 \cdots]$

The lemma completely characterizes commuting among straight Johnson & Humphries gens.

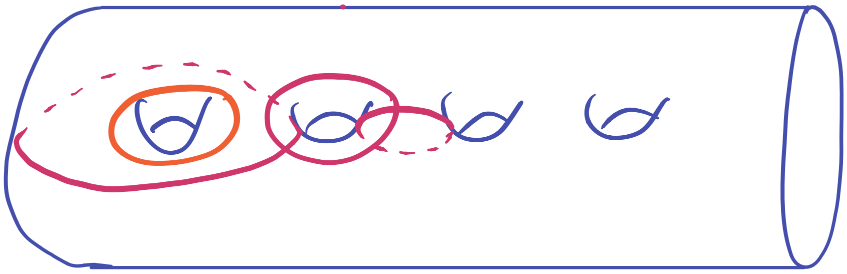
Pf (by examples)

Example $j=2$ $\quad [2\ 3\ 5\ 6]$

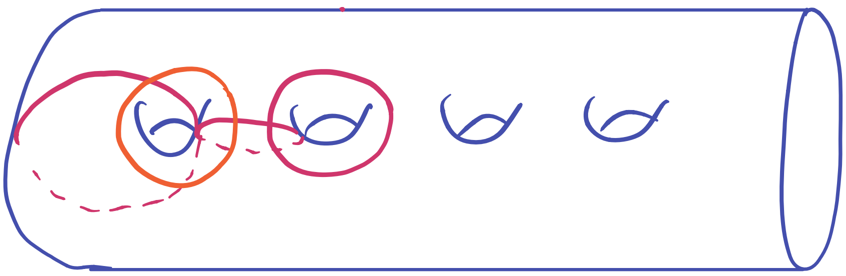


✓

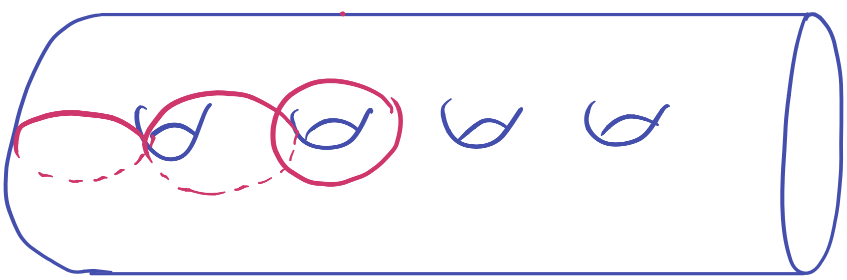
Example $j=2$ $[1\ 4\ 5\ 6]$



Example $j=2$ $[1\ 3\ 4\ 5]$



$\downarrow$ $[1\ 2\ 4\ 5]$



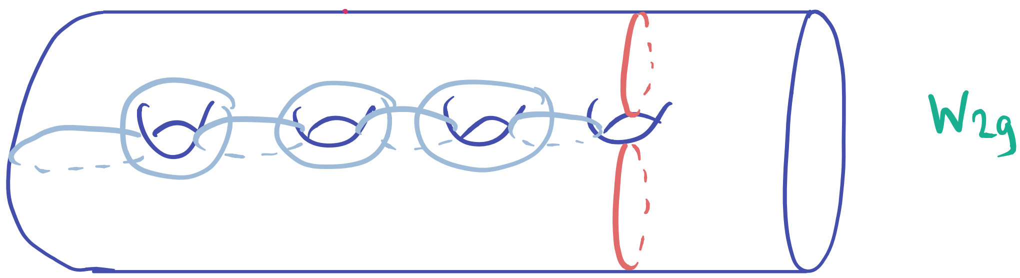
Third case similar.

Upshot of Lemma: If we conj a chain map by a positive twist about a curve in the chain we get a subchain map. What about negative twists?

Generating the Kernel of $\mathcal{I}(S_g^1) \longrightarrow \mathcal{I}(S_g)$

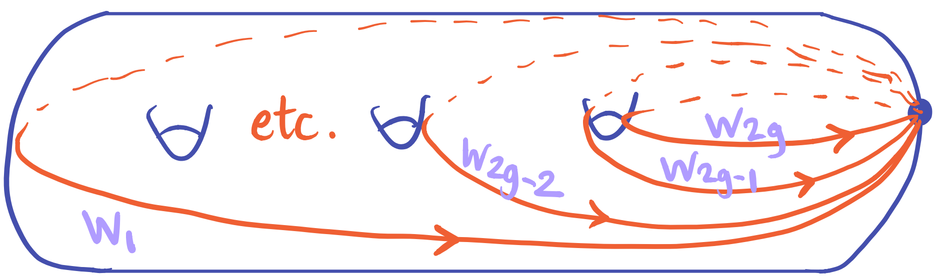
Define $W_i = [1\ \ 2\ \ 3 \cdots \hat{i} \cdots 2g+1]$
for $i \in \{1, \dots, 2g+1\}$. "maximal odd subchain maps"

Each lies in $\pi_1 UT(S_g)$



Lemma. $\pi_1 UT(S_g)$ is gen by $W_1, \dots, W_{2g}, T_b * W_1$

Pf. The corresponding push maps are



tangentially framed.

These clearly generate π_1. Need to get the fiber of $\pi_1 UT \longrightarrow \pi_1$.

Claim. $T_b * W_1 = W_4 W_3^{-1} W_2 T_\partial$ $\square$

Cor. $\pi_1 UT(S_g^1) \le \mathcal{J}(S_g^1)$

Pf. Use lantern to get $T_b * W_1$ (Cor 1 in paper).

Conjugating by a Negative Twist in the Chain

Prop. Fix a chain (c_i). The subgroup of $\mathcal{I}(S_g^1)$
$\langle$odd subchain maps of length $2k-1\rangle$
is normalized by $\langle T_{c_i}\rangle$.

Lemma. $\langle w_1, \dots, w_{2g+1}\rangle$ is normalized by
$\langle T_{c_1}, \dots, T_{c_{2g}}\rangle$

Pf. Need to check $T_{c_i}^{\pm 1}(w_j) \in \langle w_1, \dots, w_{2g+1}\rangle$
in $\pi_1 UT(S_g^1)$.

Pf of Prop. Let $f = [i_1 \; i_2 \cdots i_{2k}]$
Fix a c_j in original chain.
As above $T_{c_j} * f = f$ unless
exactly one of $j, j+1$ is an $i_\square$.
Enlarge $(i_1, i_2, \dots, i_{2k})$ to include both j & $j+1$.
$\rightsquigarrow$ subchain of length $2k$ (c_i') w/ c_j basic

Lemma $\Rightarrow$ $T_{c_j}^{\pm 1} * f$ is a product of maximal
subchain maps of (c_i'). But these
are all subchains of original chain $\square$

Two More Tools.

① Induction

If our Johnson & Humphries gens both lie in S'_{g-1}, can apply induction. Works because both sets of gens agree with $S'_{g-1} \hookrightarrow S'_g$ (this is one specific inclusion).

N.B. Base case is subgp of $\mathcal{I}(S'_2)$ gen. by BP maps i.e. $\pi_1 UT(S_2)$ i.e. $\langle W_1, \dots, W_5 \rangle$.

② Controlled Change of Coords

Suppose $T_{c_i} \in$ Humphries
$f \in \mathrm{Mod}(S'_g)$ already known to normalize $\mathcal{J}(S'_g)$.
$h \in$ Johnson

Then checking $T_{c_i} * h \in \mathcal{J}(S'_g)$ is equiv. to checking $f * (T_{c_i} * h)$
$= T_{c_i} * (f * h) \in \mathcal{J}(S'_g)$

Proof of the Theorem

Need to conjugate all Johnson gens by $T_{c_i}^{\pm 1}, T_b^{\pm 1} \in$ Humphries.

Step 1. $T_{c_1}, T_{c_2}, T_{c_5}, \ldots, T_{c_{2g}}$ normalize $\mathcal{J}(S_g^1)$.

T_{c_1}, T_{c_2} in the straight chain, disjoint from β-chain.
$T_{c_5}, \ldots, T_{c_{2g}}$ in both chains.
Apply Prop.

Step 2. T_{c_3} normalizes $\mathcal{J}(S_g^1)$.

T_{c_3} normalizes straight chains by Prop.
By controlled change of coords, can restrict to consecutive β-chain maps.

$$[\beta \ 5 \ 6 \ \cdots].$$

If chain not maximal, use induction.
If it is, need a relation (Cor 2 in paper).
But this is one check.

Step 3. T_{c_4} normalizes $\mathcal{J}(S_g^1)$ similar to T_{c_3}

Step 4. T_b normalizes $\mathcal{J}(S_g^1)$.

β-chains: controlled change of coords reduces consecutive β-chains & induction reduces to
$T_b^{\pm 1} * [\beta\ 5\ \cdots\ 2g+1]$ single check

Straight chains $[i_1 \cdots i_{2k}]$
Can assume $i_1 \leq 4$ $i_{2k} \geq 5$
(otherwise T_b commutes)

Change of coords reduces to: consecutive straight chain starting with 1, 2, 3, or 4. and going up to $2g$ or $2g+1$ (induction).

If it starts with 1 or 2 it is maximal, i.e. in $\Pi_1 \cup T$.

Only two remaining cases:
$[4\ 5 \ldots 2g+1]$ & $[3\ 4 \cdots 2g]$.

$\square$