Johnson II

will write this for $\mathcal{I}(S_g')$

Let $K_g' = \ker \tau : \mathcal{I}_g' \to \Lambda^3 H$

$T_g' = \langle T_c : c \text{ sep} \rangle \le \mathcal{I}(S_g')$

Theorem. $K_g' = T_g'$ & $K_g = T_g$

Strategy. We know $T_g' \le K_g'$.
So τ factors:

$$\mathcal{I}_g' \to \mathcal{I}_g' / T_g' \twoheadrightarrow \Lambda^3 H \cong \mathcal{I}_g' / \ker \tau$$

Want: 2^{nd} map is $\cong$.
Will show: $\mathcal{I}_g' / T_g' \cong \mathbb{Z}^{\binom{2g}{3}}$

i.e. it ① has $\binom{2g}{3}$ gens & ② is abelian.

Consider the $\binom{2g}{3}$ straight chain maps

$$[i \;\; i{+}1 \;\; j \;\; k]$$

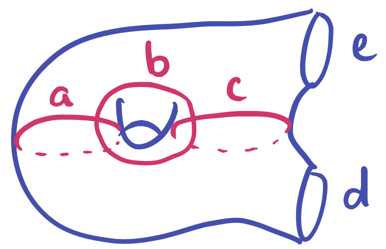
We will first observe that the τ-images span $\Lambda^3 H$, so they at least have a chance of generating $\mathcal{I}_g' / T_g'$.

Purported Generators for $\mathcal{I}_g' / \mathcal{T}_g'$

Prop. The τ-images of $[i \; i{+}1 \; j \; k]$ span $\Lambda^3 H$.

Note: these are all BP maps of genus 1.

Observation. The τ-image of the BP map



$T_d T_e^{-1}$

is $a \wedge b \wedge c$ (ignoring signs)

Since: $a + c = d$ in H $\rightsquigarrow$

$$a \wedge b \wedge d = a \wedge b \wedge (a+c) = a \wedge b \wedge c$$

By the definition of the chain maps:

$$\tau([i \; i{+}1 \; j \; k]) = c_i \wedge (c_{i+1} + \cdots + c_{j-1}) \wedge (c_j + \cdots + c_{k-1})$$

and these form a basis. Indeed, the $c_i \wedge c_j \wedge c_k$ do.

Here we get $c_i \wedge c_{i+1} \wedge c_{i+2}$ & $c_i \wedge c_{i+1} \wedge (c_{i+2} + c_{i+3})$

$= \cancel{c_i \wedge c_{i+1} \wedge c_{i+2}} + c_i \wedge c_{i+1} \wedge c_{i+3} \rightsquigarrow c_i \wedge c_{i+1} \wedge c_{i+3}$

& $c_i \wedge (c_{i+1} + c_{i+2}) \wedge c_{i+3} = \cancel{c_i \wedge c_{i+1} \wedge c_{i+3}} + c_i \wedge c_{i+2} \wedge c_{i+3}$

A Birman Exact Sequence for K_2^1

Will use induction on g, with base case $g=2$.
We already know $\mathcal{I}_2 = \mathcal{T}_2$ and so $K_2 = \mathcal{T}_2$.
Next: $\mathcal{I}(S_2^1)$. Recall:

$$1 \longrightarrow \pi_1 UT(S_2) \longrightarrow \mathcal{I}_2^1 \longrightarrow \mathcal{I}_2 \longrightarrow 1$$

Need to determine $\pi_1 UT(S_2) \cap K_2^1$.

First, we have:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \langle T_\partial \rangle & \longrightarrow & \mathrm{Mod}(S_g^n) & \longrightarrow & \mathrm{Mod}(S_{g,1}^{n-1}) & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi_1 UT(S_g^{n-1}) & \longrightarrow & \pi_1(S_g^{n-1}) & \longrightarrow & 1
\end{array}$$

We know that the $\mathbb{Z}$ is in K_2^1, so we can ignore it and deal with π_1.

Let $\alpha \in \pi_1(S_2^1)$.

did this when defining τ for closed S_g.

$\tau(\mathrm{Push}(\alpha)) = [\alpha] \wedge \Theta \qquad \Theta = x_1 \wedge y_1 + \cdots + x_g \wedge y_g$

$\Rightarrow \mathrm{Push}(\alpha) \in K_g \iff [\alpha] = 0 \iff \alpha \in \pi'$.

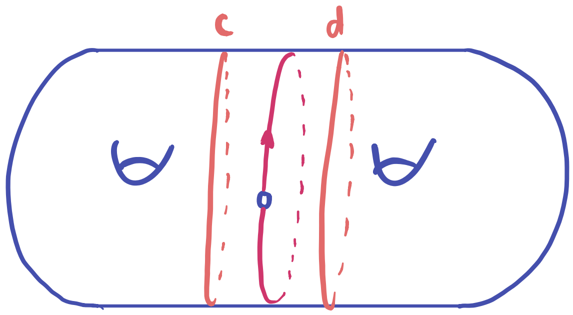
Intuitively: Push $\tilde{\alpha}$ conjugates by α. To be in K_g we must conjugate by a commutator.

Generating the Birman Kernel for K_2'

We just established

$$1 \to \pi_1 S_2' \to K_{2,1} \to K_2 \to 1.$$

We already saw $\pi_1(S_2)'$ is generated by simple separating loops. Now:



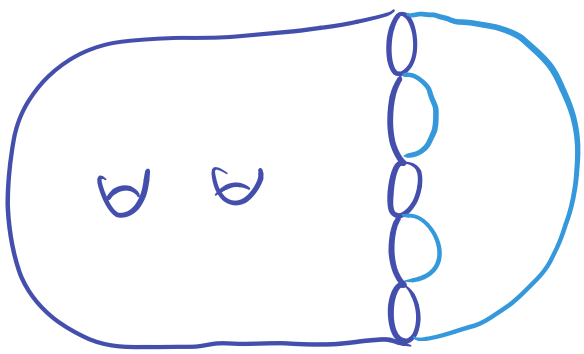
$T_c T_d^{-1}$

So $\pi_1(S_2)' \subseteq T_{2,1}$. It follows that $K_{2,1} \subseteq T_{2,1}$ hence $K_2' \subseteq T_2'$

Using the fact that each Dehn twist about a sep. curve in S_2 can be lifted to a Dehn twist about a sep curve in S_2'.

Two Boundary Components

In this paper, we define $\mathcal{I}_g^n$ as follows.
Embed S_g^n in S_{g+k} s.t. complement is connected.
Then $\mathcal{I}_g^n = \{ f \in \mathcal{I}_{g+k} : \text{supp } f \subseteq S_g^n \}$



This is not the subgroup acting trivially on homology
Eg a Dehn twist about a boundary curve.

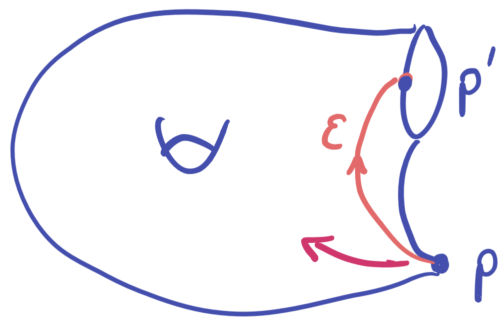
$\mathcal{H}$ is the subgroup acting trivially on $H_1(S_g^n, P)$
where P consists of one point in each component
of the boundary

Since the boundary twists are not in $\mathcal{I}_g^n$
we don't lose any info if we do point
pushing in $\mathcal{I}_{g,1}^{n-1}$ instead of $\mathcal{I}_g^n$

A Birman Exact Sequence for K_2^2

As above, suffices to consider

$K_{2,1}^1 \longrightarrow K_2^1$



$P = \{p, p'\}$

An elt of $\mathrm{Mod}(S_{2,1})$ lies in $K_{2,1}$
$\Longleftrightarrow$ it acts trivially on

$$\pi / [\pi, [\pi, \pi]]$$

More precisely if $f(x)x^{-1} \in [\pi, [\pi, \pi]]$ $\forall$ $x \in \pi_1(S_2^1, p)$ or $x =$ arc from p to p'.

For $\alpha \in \pi_1(S_2^1, p)$, action on π is conjugation for loops based at p & multiplication for arcs from p to p'.

So to act trivially on $\pi / [\pi, [\pi, \pi]]$ need $\alpha \in [\pi, [\pi, \pi]]$.

(So far this is necessary but not suff because of ∂ twisting)

Generating the Birman Kernel for K_2^2

Need to show we can realize each elt of $[\pi,[\pi,\pi]]$ as elt of K_2^2 and at the same time (a forteriori) as elt of T_2^2.

Want generators for $[\pi,[\pi,\pi]]$.
We already said the Mod-orbit of γ_1 generates $[\pi,\pi]$. So $[\pi,[\pi,\pi]]$ is gen. by conjugates of $[x, f(\gamma_1)]$ $\quad x \in \pi$
hence Mod-orbit of $[x, f(\gamma_1)]$

or $[f^{-1}(x), \gamma_1]$

or $[x, \gamma_1]$.

But γ_1 is in T_2^2. As the latter is normal, the commutator $[x,\gamma_1]$ is as well.

↑ This sentence seems to eliminate much of Johnson's argument.

Summary:

$$1 \longrightarrow \pi_1(S_2^1, p) \longrightarrow K_2^2 \longrightarrow K_2^1 \longrightarrow 1$$

$$\pi_1(S_2^1, p) \subseteq T_2^2$$

Thus $K_2^2 = T_2^2$.

The Induction / Main Argument

As above we consider $\mathcal{I}_g' / \mathcal{T}_g'$

We want ① gen. by 2g choose 3 elts
② abelian

Begin with ①. We already listed our elts:

$$[i \;\; i{+}1 \;\; j \;\; k] \qquad 1 \le i < j < k \le 2g+1.$$

Let $\mathcal{J}_g'$ = subgp of $\text{Mod}(S_g')/\mathcal{T}_g'$ gen. by these

As in Johnson I, suffices to show:

Prop. $\mathcal{J}_g'$ is normal in $\text{Mod}(S_g')/\mathcal{T}_g'$ $\quad g \ge 2$.

Pf. Base case $g = 2$. The $\binom{4}{3} = 4$ elts are (images of)

$$W_5 = [1\,2\,3\,4]$$
$$W_4 = [1\,2\,3\,5]$$
$$W_3 = [1\,2\,4\,5]$$
$$W_1 = [2\,3\,4\,5]$$

From Johnson I, these & $W_2 = [1\,3\,4\,5]$ generate kernel of $\text{Mod}(S_2') \longrightarrow \text{Mod}(S_2)$.

Also from Johnson I:

$$W_5 W_4^{-1} W_3 W_2^{-1} W_1 = T_\partial \quad ✓$$

Now assume $g \geq 3$. Have a map:

$$\mathcal{J}^1_{g-1} \longrightarrow \mathcal{J}^1_g$$

This is injective since

Where is the injectivity used?

$$\begin{array}{ccc} \mathcal{J}^1_{g-1} & \longrightarrow & \mathcal{J}^1_g \\ \cong \text{ by induction} \downarrow & \circlearrowleft & \downarrow \\ \Lambda^3 H_{g-1} & \hookrightarrow & \Lambda^3 H_g \end{array}$$

So $\mathcal{J}^1_k$ & $\mathcal{J}^2_k \hookrightarrow \mathcal{J}^1_g$ $\quad 2 \leq k < g$

The proof has 5 parts:

1. All straight 3-chain maps in $\mathcal{J}^1_g$
2. $\mathcal{J}^1_g$ is normalized by T_{c_i}
3. $W_7 = [1\,2\,3\,4\,5\,6] \in \mathcal{J}^2_2 \subseteq \mathcal{J}^1_g$
4. $W_1 = [2\,3\,4\,5\,6\,7]$, $W_2 = [1\,3\,4\,5\,6\,7]$, $\dots$, $W_6 = [1\,2\,3\,4\,5\,7]$ & $T_b * W_1$, hence all 5-chain maps (= push maps) in $\mathcal{J}^1_g$
5. T_b normalizes $\mathcal{J}^1_g$

(2) & (5) imply the theorem.

(1) used for (2)

(3) used in (4) used in (5)

Proof of ① Want all $[i\ j\ k\ l]$.
Induct on $j-i$. Base case $j-i=1$ vacuous.
The 5-chain $(i\ i{+}1\ j\ k\ l)$ gives an S_2^1.
$[i\ j\ k\ l]$ is a push map. But push maps are gen. by $[i\ i{+}1\ j\ k]$, $[i\ i{+}1\ j\ l]$, $[i\ i{+}1\ k\ l]$ & $[i{+}1\ j\ k\ l]$ & T_∂.
All these BPs have "shorter" first curve.

Proof of ② In Johnson I we showed the T_{c_i} normalize the group gen. by straight 3-chains.

Proof of ③ $W_7 = [1\ 2\ 3\ 4\ 5\ 6]$ lives on S_2^2
So it suffices to find a product of 3-chains in there whose product has same τ-image as W_7:
$$[1\,2\,3\,4][3\,4\,5\,6][1\,2\,3\,6][1\,2\,3\,5]^{-1}[1\,4\,5\,6][2\,4\,5\,6]^{-1}.$$

Here is where we use $K_2^2 = T_2^2$

Proof of ④ Can hit W_7 with the T_{c_i} to get the other W_i (use part ②).
Another relation from Johnson I:
$$T_6 * W_1 = W_4 W_3^{-1} W_2 T_{\partial_3}$$
← ∂S_3^1

Proof of ⑤ As in Johnson I, reduce to $T_b * f$ where f is a consecutive 3-chain (controlled change of coords).

$\rightsquigarrow$ 3 nontrivial cases: [2 3 4 5], [3 4 5 6], & [4 5 6 7]

The first lies in S_2^1 ✓

The other required relations are in Johnson I.

Prop. I_g'/T_g' is abelian

Proof. Want: all straight 3 chains centralize I_g'/T_g'.

But: $Z(I_g'/T_g')$ is characteristic in I_g'/T_g', hence normal in $Mod(S_g')/T_g'$.

Thus: suffices to show [1 2 3 4] normalizes I_g'/T_g'.

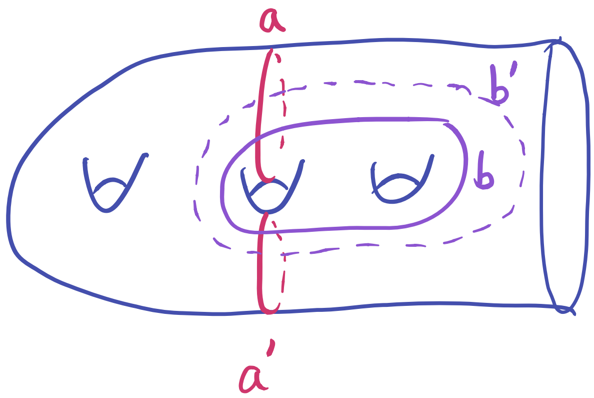
i.e. [[1 2 3 4], f] $\forall f \in$ gen set for I_g'/T_g'

Controlled change of coords $\rightsquigarrow$ 3 nontrivial cases:

[2 3 4 5], [3 4 5 6], [4 5 6 7]

First two are in S_2^1 & S_2^2, & image of a commutator is trivial under τ ✓

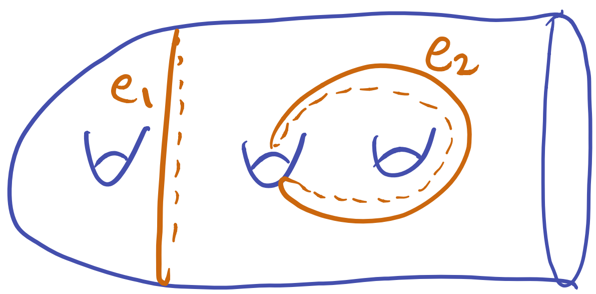
Last one is the following commutator:



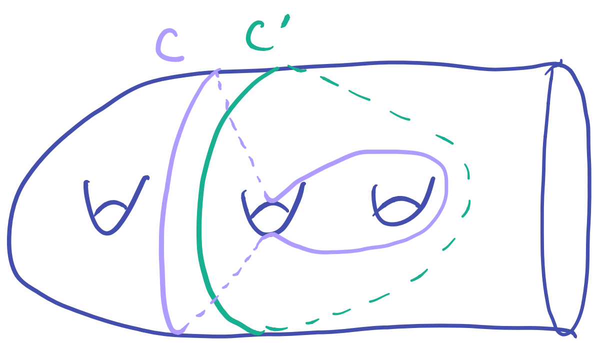
$[T_a T_{a'}^{-1}, T_b T_{b'}^{-1}]$

We use a lantern relation:

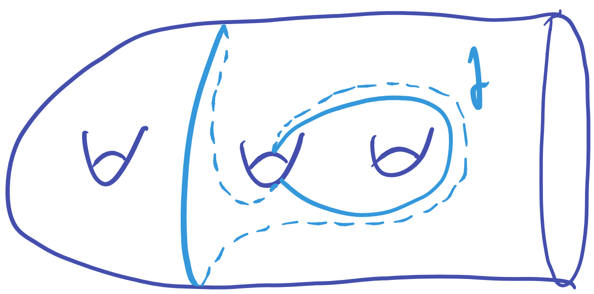
$$T_c T_{b'} T_d = T_b T_{c'} T_{e_1} T_{e_2}$$



b' is a band surgery of c' & e_1



$T_{a'} T_a^{-1}(\{b, b'\})$



Band surgery of e_1 & e_2

Rewrite as: $T_{c'}^{-1} T_b^{-1} T_c T_{b'} = T_d^{-1} T_{e_1} T_{e_2}$

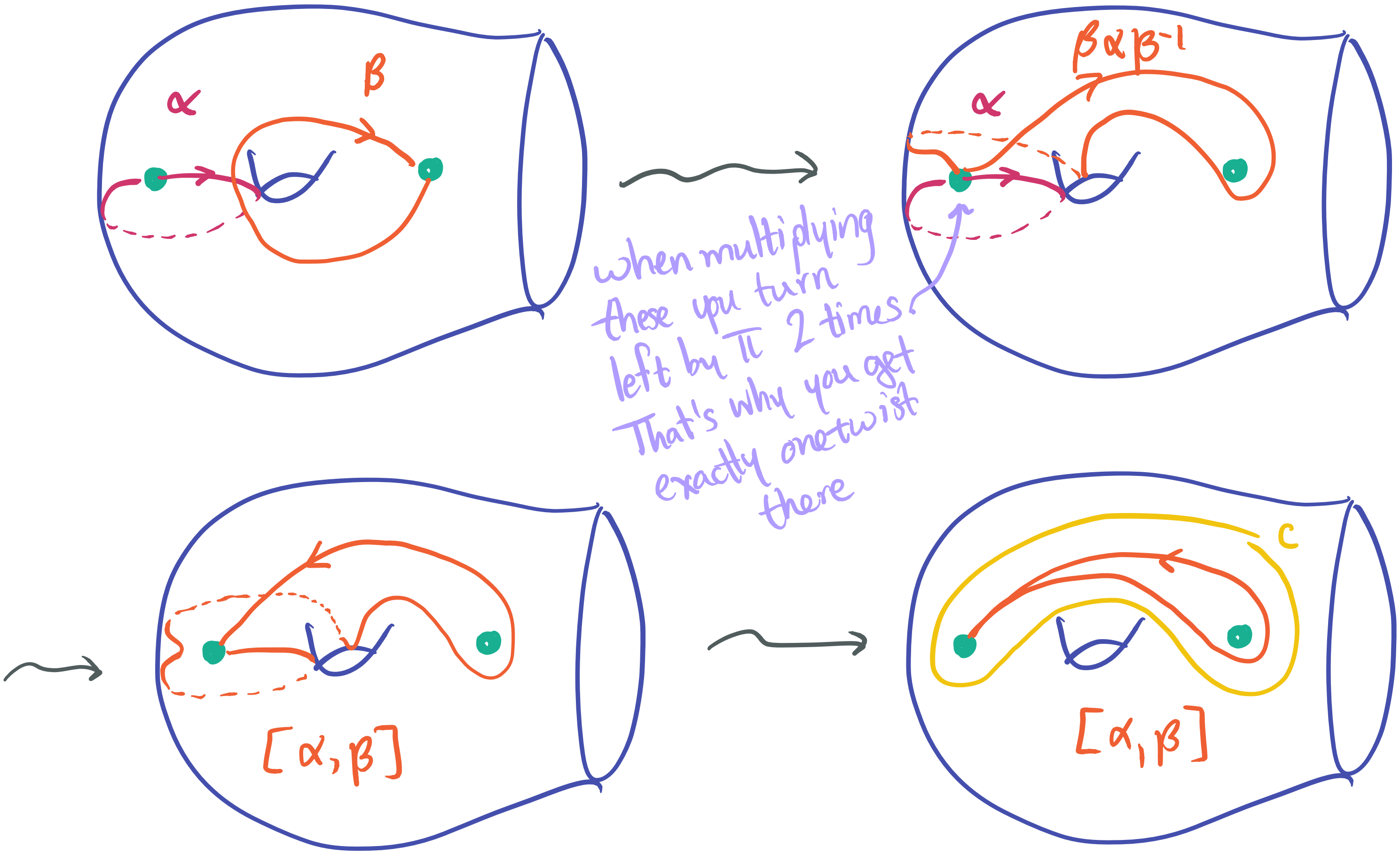
$$\begin{aligned} &T_{c'}^{-1} T_b^{-1} T_c T_{b'} \\ &= T_c T_{c'}^{-1} T_{b'} T_b^{-1} \\ &= [T_a^{-1} T_{a'}, T_b T_{b'}^{-1}] \end{aligned}$$

So the desired commutator is trivial in $\mathcal{I}(S_g^1)/K(S_g^1)$.

This relation will be used in Johnson III to show that the image of $K(S_g^1)$ is $\mathcal{I}(S_g^1)^{ab}$ is generated by images of genus 1 maps. In $\mathcal{I}(S_g^1)^{ab}$ the LHS is trivial and RHS gives $T_{e_1} T_{e_2} = T_d$ is $\mathcal{I}(S_g^1)^{ab}$.

An Intuitive Version of the Last Relation

After crushing two handles:

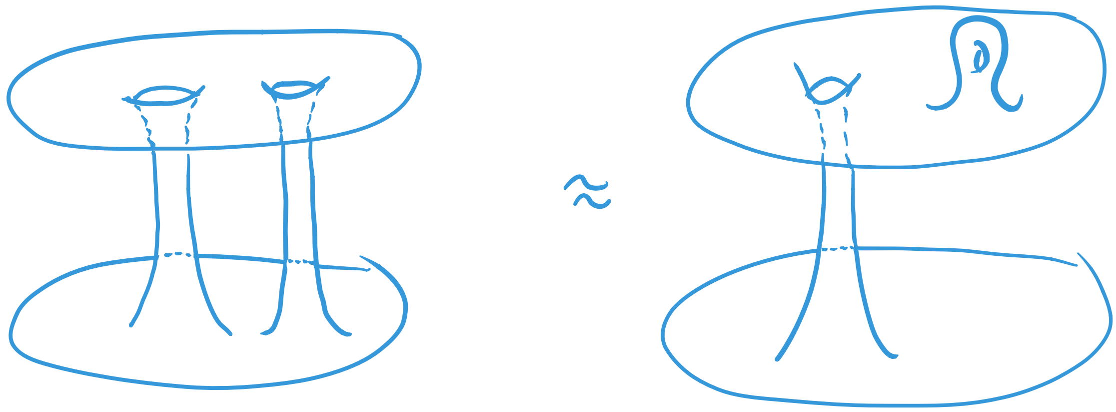


So back in S_3^1 the original commutator of BP maps is a multitwist about two genus one twists and a single genus 2 twist T_c & so the commutator is in $K(S_g^1)$.

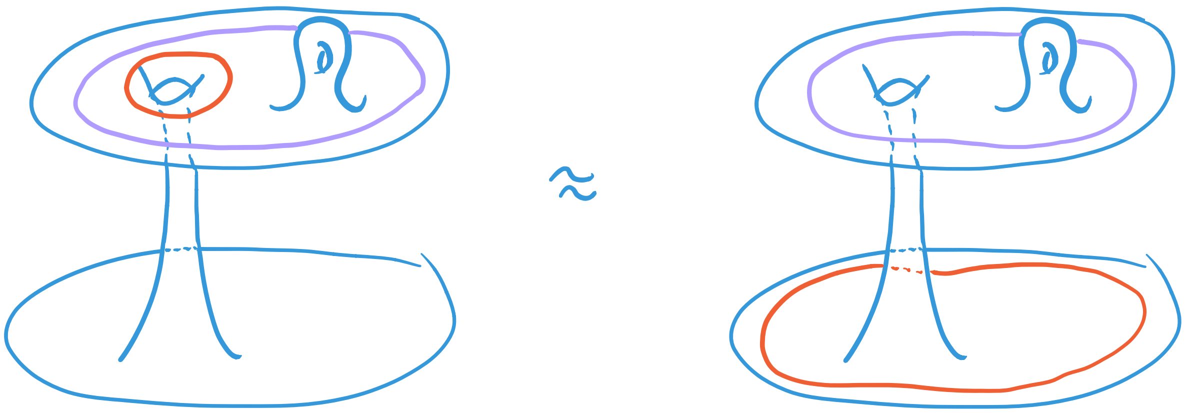
For Johnson III all we need is that in $\mathcal{I}(S_g^1)^{ab}$ T_c is a product of genus 1 twists.

A Finer Point from Previous Page

To get the first picture from Johnson's picture, need to believe:

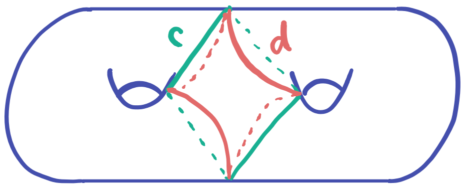


<u>and</u>



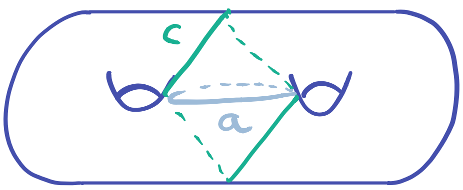
Aside: A Question of Johnson

Q. What is the normal closure in $Mod(S_2)$?



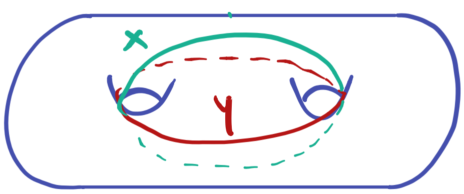
$T_c T_d^{-1}$

or:



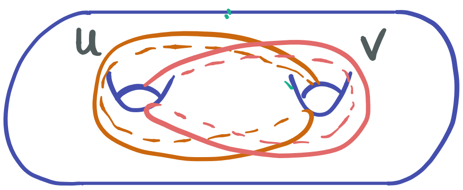
$[T_c, T_a]$

or:



$[T_x, T_y]$

Two different curves complete the lantern:



Lantern relation $\Rightarrow T_c T_d^{-1} = T_u T_v^{-1}$

$\Rightarrow T_c T_d^{-1} \in \ker \mathcal{I}(S_2) \to \mathbb{Z}$

In fact it equals the kernel: By connectedness of the complex with vertices = sep curves & edges for $i=4$, all gens for $\mathcal{I}(S_2)$ have same image in $\mathcal{I}(S_2)/\langle\langle T_c T_d^{-1} \rangle\rangle$.