Johnson III: The Abelianization of Torelli

Main Goal: $\mathcal{I}(S_g^1)^{ab} \cong \mathbb{Z}^{\binom{2g}{3}} \oplus \mathbb{Z}/2^{\binom{2g}{2}+\binom{2g}{1}+\binom{2g}{0}}$

and: $\mathcal{I}(S_g)^{ab} \cong \mathbb{Z}^{\binom{2g}{3}-\binom{2g}{1}} \oplus \mathbb{Z}/2^{\binom{2g}{2}+\binom{2g}{1}}$

These are isomorphisms as abelian groups, but not as Sp-reps. To understand $\mathcal{I}(S_g^1)^{ab}$ as an Sp-rep, we need some setup.

Let $A_g^1 = \mathcal{I}(S_g^1)^{ab} = H_1(\mathcal{I}(S_g^1);\mathbb{Z})$

U_g^1 = mod 2 abelianization $= H_1(\mathcal{I}(S_g^1);\mathbb{Z}/2)$

T_g^1 = image of $K(S_g^1)$ in A_g^1

B_g^3 = boolean polys of deg 3 in H

Clearly $\mathrm{Mod}(S_g^1) \curvearrowright A_g^1$ & $\mathcal{I}(S_g^1)$ acts trivially

$\implies A_g^1$ is an Sp-rep

The main work of the paper is:

Prop 1. $T_g^1 \cong \bigoplus^n \mathbb{Z}/2$

Prop 2. $U_g^1 \cong B_g^3$

Note: There is automatically a $\sigma: U_g^1 \twoheadrightarrow B_g^3$ by univ. property of U_g^1

Some Setup

Fact 1. The seq.

$$0 \to T_g' \longrightarrow A_g' \longrightarrow \Lambda^3 H \longrightarrow 0$$

is split exact.

Pf. We know

$$1 \longrightarrow K(S_g') \longrightarrow I(S_g') \xrightarrow{\tau} \Lambda^3 H \longrightarrow 0$$

is exact. Divide by $I(S_g')'$ to get the desired seq. It is split since $\Lambda^3 H$ is free. $\square$

Fact 2. $U_g' \cong A_g' / 2A_g' \cong A_g' \otimes \mathbb{Z}/2$

Thus there are induced maps

$$T_g' \to U_g' \quad , \quad U_g' \xrightarrow{\tau \otimes \mathbb{Z}/2} \Lambda^3 H \bmod 2 \qquad \square$$

Fact 3. There is a split exact seq.

$$0 \to T_g' \longrightarrow U_g' \longrightarrow \Lambda^3 H \bmod 2$$

Pf. Tensor Fact 1 with $\mathbb{Z}/2$, use Fact 2, & Prop 1 above $\Rightarrow T_g' \otimes \mathbb{Z}/2 \cong T_g'$ $\square$

Note: Can regard T_g' as subgp of A_g' or U_g'

Statement of the Main Theorem

Theorem. The square

$$\begin{tikzcd}
& A_g^1 \arrow[dl, "\otimes \mathbb{Z}/2"'] \arrow[dr, "\tau"] & \\
B_g^3 \cong U_g^1 \arrow[dr, "\tau"'] & & \Lambda^3 H \arrow[dl, "\otimes \mathbb{Z}/2"] \\
& \Lambda^3 H \bmod 2 &
\end{tikzcd}$$

is a pull back diagram.

Again, the point is that this diagram makes sense as an Sp-rep.

In general, the pullback of

$$\begin{tikzcd}
A \arrow[dr, "p"'] & & B \arrow[dl, "q"] \\
& C &
\end{tikzcd}$$

is $D = \{(a,b) \in A \times B : p(a) = q(b)\} \leq A \times B$

This D is universal in that any other D' factors through D:

$$\begin{tikzcd}
& D' \arrow[d, dashed] \arrow[ddl, bend right] \arrow[ddr, bend left] & \\
& D \arrow[dl] \arrow[dr] & \\
A \arrow[dr, "p"'] & & B \arrow[dl, "q"] \\
& C &
\end{tikzcd}$$

Example: Pullback bundles.

We think of D as combining the fibers of p & q together.

In the theorem, the two maps at the top are labeled. So the theorem means:

The map $\mathcal{A}_g^1 \longrightarrow \mathcal{U}_g^1 \oplus \Lambda^3 H$

$$x \longmapsto (x \otimes \mathbb{Z}/2,\ \tau(x))$$

is injective & image is

$$\{(u,\lambda) : \tau(u) = \lambda \otimes \mathbb{Z}/2\}$$

Pf (assuming Props 1 & 2)

Injectivity. Suppose $(x \otimes \mathbb{Z}/2,\ \tau(x)) = 0$

$\tau(x) = 0 \implies x \in \mathcal{T}_g^1$ by Fact 1.

Fact 3 $\implies x = 0$.

Surjectivity. Let (u,λ) with $\tau(u) = \lambda \bmod 2$.

Fact 1 $\implies \tau$ surj $\rightsquigarrow f_1 \in \mathcal{A}$ s.t. $\tau(f_1) = \lambda$

$\implies \tau(f_1) = \tau(u) \bmod 2 \implies \tau(f_1 \otimes \mathbb{Z}/2 - u) = 0$

Fact 3 $\implies f_1 \otimes \mathbb{Z}/2 - u = t \in \mathcal{T}_g^1$

By Fact 1, may consider t as elt of $\mathcal{A}$.

Let $f = f_1 - t \rightsquigarrow$ image in $\mathcal{U}$ is

$f_1 \otimes \mathbb{Z}/2 - t = u$ & $\tau(f) = \tau(f_1) = \lambda$ $\square$

Conjugacy Relations

Working towards Prop 1. Need to know when Dehn twists & BP maps are conjugate in $\mathcal{I}(S_g')$.

Homology Chains. A chain in H or H mod 2 is a seq $(c_1, \ldots, c_n)$ s.t.

(a) $\hat{\imath}(c_i, c_{i+1}) = 1$ & $\hat{\imath}(c_i, c_j) = 0$ o.w.

(b) If n odd $c_1 + c_3 + \cdots + c_n$ primitive

Twists. For n even, get a well-def elt of A_g', call it $[c_1, \ldots, c_n]$.
Indeed: homology chains $\rightsquigarrow$ symplectic subspaces $\rightsquigarrow$ Dehn twists up to conj in $\mathcal{I}(S_g')$.

BP maps. For n odd, again get a well-def elt of A_g' called $[c_1, \ldots, c_n]$ or $[c_1, \ldots, c_{n-1} | c_1 + c_3 + \cdots + c_n]$
Indeed: homology chains $\rightsquigarrow$ geometric chains $\rightsquigarrow$ BP map, unique up to conj in $\mathcal{I}(S_g')$.

Note: $[c_1, c_2 | -c_3] = -[c_1, c_2 | c_3]$
(get inverse BP map).

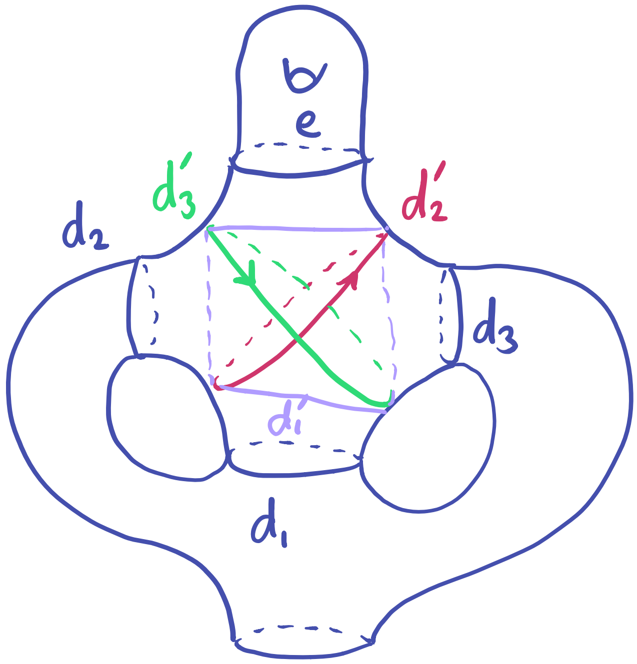
Naturality. For $h \in Sp$: $h * [c_1, \ldots, c_n] = [h(c_1), \ldots, h(c_n)]$

Dehn Twists are 2-Torsion

Lemma 1. If $\hat{i}(a,b)=1$ & $\langle d_1,d_2\rangle$ rationally closed in $\langle a,b\rangle^{\perp}$ & $\hat{i}(d_1,d_2)=0$ then in A_g'

$$[a,b\,|\,d_1+d_2]-[a,b\,|\,d_1]-[a,b\,|\,d_2]=[a,b]$$

Pf. Consider the lantern relation



$$T_e=(T_{d_3'}T_{d_3}^{-1})(T_{d_2'}T_{d_2}^{-1})(T_{d_1'}T_{d_1}^{-1})$$

Let a,b = symp. basis for e-handle

In A:

$$T_{d_1'}T_{d_1}^{-1}=-[a,b\,|\,d_1]$$
$$T_{d_2'}T_{d_2}^{-1}=-[a,b\,|\,d_2]$$
$$T_{d_3'}T_{d_3}^{-1}=-[a,b\,|\,d_3]=[a,b\,|\,d_1+d_2]$$
$$T_e=[a,b]$$

$\square$

Lemma 2. If $\hat{i}(a,b)=1$ then $2[a,b]=0$ in $\mathcal{A}_g'$.

Pf. By change of coords, can replace d_i with $-d_i$ in Lemma 1. Thus plus $-[a,b|x]=[a,b|-x]$ gives
$$[a,b] = -[a,b|d_1+d_2]+[a,b|d_1]+[a,b|d_2]$$
$$= -[a,b] \qquad \square$$

Lemma 3. If a_1,b_1,a_2,b_2 is a symplectic subspace of H then $[a_1,b_1,a_2,b_2]=[a_1,b_1]+[a_2,b_2]$ in $\mathcal{A}_g'$.

Pf. See Johnson II, Cor to Thm 4B.

We have thus shown:

$\mathcal{T}_g'$ is a $\mathbb{Z}/2$ vector space, and it is generated by the 2-chain maps $[a,b]$

The Action of Level 2 on $\mathcal{U}$ is Trivial

$$\mathcal{M}_g^1[2] = \mathrm{Mod}(S_g^1)[2]$$
$$\mathrm{Sp}_g[2] = \mathrm{Sp}_{2g}(\mathbb{Z})[2]$$

Prop. $\mathcal{M}_g^1[2]$ acts trivially on $\mathcal{U}_g^1$.

If we believe that $\mathcal{U}_g^1 \cong \mathcal{B}_g^3$ then this must be true!

Since $\mathcal{I}_g^1$ acts trivially on $\mathcal{U}$, the prop could be:

$\mathrm{Sp}_{2g}[2]$ acts trivially on $\mathcal{U}_g^1$

Prop implies $\mathcal{U}_g^1$ is an $\mathrm{Sp}_g(\mathbb{Z}/2)$-module.

$\rightsquigarrow$ projection $\tau : \mathcal{U}_g^1 \to \mathcal{B}_g^3$ is $\mathrm{Sp}_g(\mathbb{Z}/2)$-module homomorphism.

Lemma. $\mathrm{Sp}_{2g}[2]$ is generated by squares of transvections

Cor. $\mathrm{Sp}_{2g}[2]$ is the normal closure in $\mathrm{Sp}_{2g}(\mathbb{Z})$ of any square transvection.

Pf of Prop. The kernel of $Sp_{2g}(\mathbb{Z}) \longrightarrow \mathrm{Aut}\ \mathcal{U}$ is normal in Sp. $\rightsquigarrow$ suffices to show a single square transv. is in kernel.

Suffices: $T_{c_1}^2$ acts trivially on images of straight & β-chain gens

β-chains: trivial since c_1 disjoint.

Straight chains: by controlled change of coords, reduce to consecutive chains starting w/ 2, namely $[2\ 3 \cdots 2k{+}1]$. Will write here for $k=1$, but proof works in general.

Johnson I: In $\mathcal{I}_g'$

$$T_{c_1} * [2\ 3][2\ 3]^{-1} = \left(T_{c_1}^{-1} * [2\ 3] \cdot [2\ 3]^{-1}\right)^{-1}$$

$\mathcal{U}_g'$ is abelian, so in $\mathcal{U}_g'$:

$$T_{c_1} * [2\ 3] = T_{c_1}^{-1} * [2\ 3] \quad \cancel{[2\ 3]^2}$$

$$T_{c_1}^2 * [2\ 3] = [2\ 3] \qquad \square$$

Cor. $[\mathcal{M}_g'[2], \mathcal{I}_g'] = (\mathcal{I}_g')^2$

Pf. $\boxed{\subseteq}$ Immediate from prop

$\boxed{\supseteq}$ Enough to show $\mathcal{I}_g' / [\mathcal{M}_g'[2], \mathcal{I}_g']$ is a $\mathbb{Z}/2$ vector space (since $\mathcal{U}_g'$ is universal).
H is abelian since $(\mathcal{I}_g')' \subseteq [\mathcal{M}_g'[2], \mathcal{I}_g']$.
H is gen. by images of BP maps.
So: suffices to show the square of any BP map is in $[\mathcal{M}_g'[2], \mathcal{I}_g']$.

In particular, we have: $\mathcal{U}_g' = \mathcal{I}_g' / [\mathcal{M}_g'[2], \mathcal{I}_g']$

follows from Prop $\longrightarrow$ and: $\mathcal{U}_g'$ is an $Sp_g(\mathbb{Z}/2)$-module

Any chain $(c_1, \ldots, c_n)$ in H mod 2 determines a unique element $[c_1 \cdots c_n]$ in $\mathcal{U}_g'$. This is in $\mathcal{T}_g'$ if n is even.

As usual, have naturality. So

$$\sigma : \mathcal{U}_g' \longrightarrow B_g^3 \text{ is an } Sp_g(\mathbb{Z}/2)\text{-module hom.}$$

σ IS AN ISOMORPHISM FOR $g=3$.

It remains to show σ is an isomorphism.

Lemma. $\sigma : U_3' \longrightarrow B_3^3$ is an isomorphism.

Proof. Johnson I: I_3' is gen. by 42 elts
$\Rightarrow \dim U_3' \leq 42$
But σ is surjective &

$$\dim B_3^3 = \sum_{i=0}^{3} \binom{6}{i} = 20+15+6+1 = 42 \qquad \square$$

Strategy for general case: Use Sp-module structure to show $\ker \sigma$ is generated by elements of U_3', then apply the lemma.

Subalgebras of B_g^3 from Subsurfaces.

Let X = symplectic subsp. of H mod 2.

$\rightsquigarrow B_X^k$ = k-nomials in X.

$\rightsquigarrow B_X^k \longrightarrow B_g^k$ injective.

Let $S_k^1 \subseteq S_g^1$ any subsurf of genus k

$\rightsquigarrow \mathcal{I}_k^1 \longrightarrow \mathcal{I}_g^1$

$\rightsquigarrow \mathcal{U}_k^1 \longrightarrow \mathcal{U}_g^1$

Lemma. (1) Image only depends on $H_1(S_k^1; \mathbb{Z}/2)$.

(2) For $k=3$, the map $\mathcal{U}_3^1 \to \mathcal{U}_g^1$ is injective.

Pf. (1) $\mathcal{M}_g^1[2]$ acts trivially on $\mathcal{U}_g^1$

(2) Commutativity of

$$\begin{tikzcd}
\mathcal{U}_3^1 \arrow[r, "\cong"] \arrow[d] & B_3^3 \arrow[d, hook] \\
\mathcal{U}_g^1 \arrow[r] & B_g^3
\end{tikzcd}$$

$\square$

So for X as above, makes sense to define

$\mathcal{U}_X$ = image of $\mathcal{U}_k^1 \longrightarrow \mathcal{U}_g^1$.

Maybe this page is not so essential for understanding

CARRYING

X = 2k-dim symp. subsp. of H mod 2
Say X carries $f \in \mathcal{U}_k'$ if $f \in \mathcal{U}_X$
i.e. $\exists\ S_k' \subseteq S_g'$ s.t. $H_1(S_k'; \mathbb{Z}/2) = X$
& $\tilde{f}$ with $\mathrm{supp}(\tilde{f}) = S_k'$ s.t.
$\tilde{f} \longmapsto f$

Cor. Say $\dim X \le 6$. If $f, h \in \mathcal{U}_g'$ carried by X,
then $f = g \iff \sigma(f) = \sigma(g)$ in $\mathcal{B}_g^3$.

Pf. Extend X so $\dim X = 6$.
$\Rightarrow \sigma : \mathcal{U}_X \longrightarrow \mathcal{B}_X^3$ is an isomorphism. $\square$

Note. $h \in Sp_g[2]$, $f \in \mathcal{U}_g'$, f carried by X
$\Rightarrow h * f$ carried by $h(X)$.

Maybe this page is not so essential
for understanding

σ IS AN ISOMORPHISM

Basic outline. ① Show $T = T_g^1$ is ker of

$$\sigma : U_g^1 \longrightarrow B_g^3 / B_g^2$$

Have gens for T_g^1: 2-chain maps

② Find gens for S = Kernel of

$$\sigma : U_g^1 \longrightarrow B_g^3 / B_g^1$$

$$= \ker \quad \sigma : T_g^1 \longrightarrow B_g^3 / B_g^1$$

As a module over $Sp_g^1[2]$ it is gen. by

$$\partial(a,b,c) = [b,c] - [a+b,c] + [a,b+c] - [a,b]$$

for any 3-chain (a,b,c).

(Check this is really in the kernel!)

③ Find gens for R = kernel of

$$\sigma : U_g^1 \longrightarrow B_g^3 / B_g^0$$

$$= \ker \quad \sigma : S \longrightarrow B_g^3 / B_g^0$$

If we rename $\partial(a,b,c)$ as $[a+c]$ (since, it turns out, the element only depends on this sum) then R is gen by $\partial(x,y) = [x+y] - [x] - [y]$ for any 2-chain (x,y).

④ $R = \mathbb{Z}/2$ & $\sigma : R \to B_g^0$ is $\cong$. That's it!

STEP ①

Lemma. kernel of $\mathcal{U}_g^1 \longrightarrow B_g^3/B_g^2 = \Lambda^3 H \bmod 2$ is $\mathcal{T}_g^1$

Proof. In his paper defining τ, Johnson shows:

$$\begin{tikzcd}
A_g^1 \arrow[r, "\tau"] \arrow[d, "\sigma"'] \arrow[dr, phantom, "\circlearrowleft"] & \Lambda^3 H \arrow[d, "\otimes \mathbb{Z}/2"] \\
B_g^3 \arrow[r] & \Lambda^3 H \bmod 2
\end{tikzcd}$$

Just check on generators

Now tensor with $\mathbb{Z}/2$:

$$\begin{tikzcd}
\mathcal{U}_g^1 \arrow[d, "\sigma"'] \arrow[dr, "\tau \otimes \mathbb{Z}/2"] \arrow[dr, phantom, "\circlearrowleft"', near start] & \\
B_g^3 \arrow[r] & \Lambda^3 H \bmod 2 \cong B_g^3/B_g^2
\end{tikzcd}$$

So $\tau \otimes \mathbb{Z}/2$ is same as $\mathcal{U}_g^1 \longrightarrow B_g^3 \longrightarrow B_g^3/B_g^2$

By Fact 3, kernel is $\mathcal{T}_g^1$ $\square$

STEP 2

Let's first check $\partial(a,b,c)$ is in $\ker\left(U_g^1 \to B_g^3/B_g^1\right)$

$$\partial(a,b,c) = [b,c] - [a+b,c] + [a,b+c] - [a,b]$$

Why the signs if T_g^1 is a $\mathbb{Z}/2$ vect sp?

$$\rightsquigarrow \bar b \bar c + \overline{a+b}\,\bar c + \bar a\,\overline{b+c} + \bar a \bar b$$
$$= \bar b \bar c + (\bar a + \bar b + 1)\bar c + \bar a(\bar b + \bar c + 1) + \bar a \bar b$$
$$= \cancel{\bar b \bar c} + \cancel{\bar a \bar c} + \cancel{\bar b \bar c} + \bar c + \cancel{\bar a \bar b} + \cancel{\bar a \bar c} + \bar a + \cancel{\bar a \bar b}$$
$$= \bar a + \bar c \qquad \underline{\text{linear}} \text{ in } B_g^1.$$

Since σ is $\cong$ for $g=3$: $\partial(a,b,c)$ only dep. on $a+c$ for $g=3$ (also true in general).

Lemma. S is gen (as a module) by any one $\partial(a,b,c)$.

Pf for $g=3$. In this case $\sigma : T_g^1 \xrightarrow{\cong} B_g^2$
and so $\sigma : S \xrightarrow{\cong} B_g^1$
So enough to show $\sigma(\partial(a,b,c))$ is a module gen for B_g^1. But a module gen. for B_g^1 is any non-0 vector. By above calculation $\sigma(\partial(a,b,c)) = \bar a + \bar c$. This is non-0 by defn of a chain. $\square$

Note: the $g=3$ case is a waste of time since we already know σ is $\cong$ here. But it gives the idea.

For $g>3$. Use naturality & move stuff around.

STEP 3

As above $\partial(a,b,c) = [e]$ where $e = a+c$.

Have naturality: $h \in Sp_g[2] \Longrightarrow h * [e] = [h(e)]$.
(this actually requires e to live in Sp space of dim ≥ 4 for change of coords).

Check $\partial(x,y)$ in $\ker U_g' \longrightarrow B_g^3 / B_g^0$

$\partial(x,y) = [x+y] - [x] - [y] = \hat{\imath}(x,y) \in B_g^0$. ✓

Lemma. $\partial(x,y) = 0$ in U_g' if $\hat{\imath}(x,y) = 0$
$\partial(x,y)$ indep of x,y if $\hat{\imath}(x,y) = 0$
In particular, $R \cong \mathbb{Z}/2$ & τ is $\cong$.

Pf. Make x,y sit in genus 3 subsurf. by change of coords.

This (more or less) proves the theorem. $\square$