Generating the Johnson Kernel

Thm (Ershov-He '17, Church-Ershov-Putman '17)
$K(S_g)$ is finitely generated for $g \geq 4$.

Notes. ① Not known for $g = 3$.
② No explicit finite gen set is known.

History ① Biss-Farb '05 $K(S_g)$ is not f.g.
② Erratum '09
③ Dimça-Papadima '13: $K(S_g)^{ab}$ is f.g.
④ Morita-Sakasai-Suzuki '17: $K(S_g)^{ab}$ has $O(g^5)$ generators (after EH & CEP).

The CEP results are more general:

Main Theorem. For $k \geq 3$, $g \geq 2k+1$, $b \in \{0,1\}$ every subgp of I_g^b containing k^{th} term of LCS of I_g^b is fin. gen.

$k=2$ case: subgps containing $[I_g^b, I_g^b]$ also proved for $g=4$.

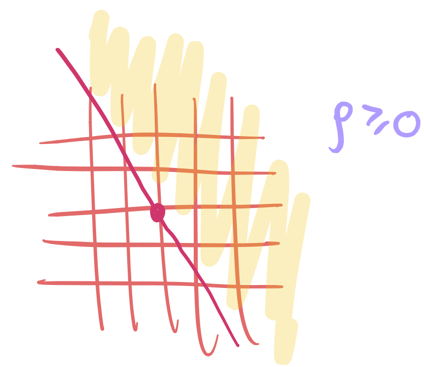
Special case: terms of Johnson filtration.

Bieri-Neumann-Strebel Invariants

G = group $\rightsquigarrow$ $G^* = \mathrm{Hom}(G, \mathbb{R})$

$\Sigma(G) = \{ \rho \in G^* :$ the sugraph of the Cayley graph of G spanned by $g \in G$ with $\rho(G) \geq 0$ is connected $\}$

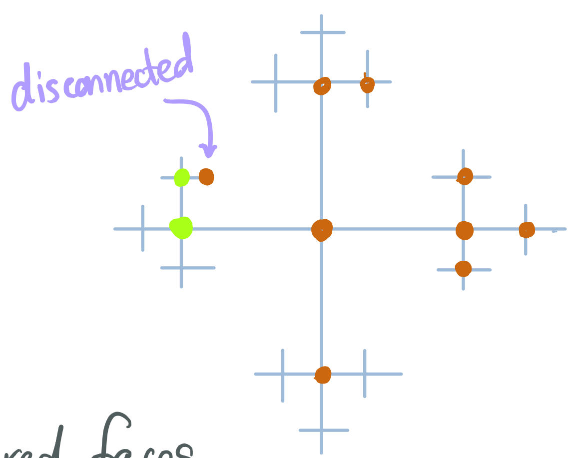
Examples. ① $\Sigma(\mathbb{Z}^n) = (\mathbb{Z}^n)^*$



② $\Sigma(F_2) = \{0\}$

e.g. consider $\rho : F_2 \longrightarrow \mathbb{R}$

$\rho(a) = 1, \ \rho(b) = 0$



③ M = hyperbolic 3-manifold.

$\Sigma(\pi_1(M))$ = cone on open fibered faces.

Theorem (BNS). Say G = fin gen group & $[G,G] \leq H \leq G$

Then: H f.g. $\iff \{ \rho \in G^* : \rho|_H = 0 \} \subseteq \Sigma(G)$

Prop. $\Sigma(G)$ is invariant under $\mathrm{Aut}(G)$.

Proof Outline

Will show $\Sigma(\mathcal{I}_g) = \mathcal{I}_g^*$ $g \geq 4$.
Then apply above theorem.

Claim 1. $\exists$ finite gen set S for $\mathcal{I}_g$ consisting of genus 1 BPs s.t.

$$\{\rho \in \mathcal{I}_g^* : \rho(s) \neq 0 \ \forall s \in S\} \subseteq \Sigma(\mathcal{I}_g)$$

For next step note $\mathrm{Mod}(S_g) \circlearrowright \mathcal{I}_g$

$\rightsquigarrow \mathrm{Mod}(S_g) \circlearrowright \mathcal{I}_g^*$

$\rightsquigarrow \mathrm{Mod}(S_g) \longrightarrow GL(\mathcal{I}_g^*)$

Pull back the Zariski topology to $\mathrm{Mod}(S_g)$.

Claim 2. In this topology, $\mathrm{Mod}(S_g)$ is an irreducible space (i.e. it is not the union of proper closed subspaces.)

Fix $\rho \neq 0$ in $\mathcal{I}_g^*$. For each $s \in S$ let

$$Z_s = \{f \in \mathrm{Mod}(S_g) : f \cdot \rho(s) = 0\}$$

Claim 3. Z_s is a proper, closed subset of $\mathrm{Mod}(S_g)$.

Claims 2 & 3 imply:

$$\bigcup_{s \in S} Z_s \subsetneq \mathrm{Mod}(S_g)$$

Choose $f \in \mathrm{Mod}(S_g)$ in the complement.
That is: $f \cdot \rho(s) \neq 0 \ \forall\, s \in S$.
Claim 1 $\Rightarrow f \cdot \rho \in \Sigma(\mathcal{I}_g^*)$.

But $\Sigma(\mathcal{I}_g^*)$ is invariant under automorphisms of $\mathcal{I}_g$
$\Rightarrow \rho \in \Sigma(\mathcal{I}_g^*)$. But ρ was arbitrary. $\square$

We know $\mathcal{I}_g$ is fin. gen.
So by BNS theorem, all subgroups of $\mathcal{I}_g$ containing the commutator subgroup, including $K(S_g)$ are finitely generated.

More About BNS

Prop. For G a group, $\Sigma(G)$ is indep. of gen set.

Pf. Say T, S are gen sets.
Fix $\rho \in G^*$. Say $\rho \in \Sigma(G,T)$. Want $\rho \in \Sigma(G,S)$.

Idea. Say $\rho(g), \rho(g') \geq 0$. Choose $s \in S$ s.t. $\rho(s) > 0$.
For big N, $\rho(s^N g), \rho(s^N g') \gg 0$. Connect using s^N-translate of T-path from g to g' with $\rho \geq 0$.

Claim 1. $\forall\, n \in \mathbb{Z}$ $\{g \in G : \rho(g) \geq n\}$ is T-connected.

Pf. Follows by translating $\{g : \rho(g) \geq 0\}$.

Claim 2. $\exists\, n$ s.t. for any $g \in G$ & $t \in T$ $\exists$ path in S-Cayley graph from g to gt s.t. if v is a vertex on this path then $\rho(v) > \max\{\rho(g) - n, \rho(gt) - n\}$

Pf. For each $t \in T$ choose word w_t in S rep'ing t. Choose n larger than $|\rho(w)|$ whenever w is a prefix or suffix of any w_t. Connect g to gt by $g \cdot w_t$.

Claim 3. $\exists\ s \in S \cup S^{-1}$ s.t. $\rho(s) > 0$.

Pf. $\rho \neq 0$.

Finish the proof as in the idea above.

As a cor, we obtain the Prop which says $\Sigma(G)$ is invariant under automorphisms. Indeed, say $\rho \in \Sigma(G)$, $\alpha \in \operatorname{Aut} G$. Then $\{g : \rho(g) \geq 0\}$ is connected wrt some gen set X. So $\{g : \rho(\alpha^{-1}(g)) \geq 0\}$ is connected wrt to $\alpha^{-1}(X)$. Want it to be connected wrt X. But this is just a change of gen set.

A Lemma about BNS

Lemma (Ershov-He) G = fin. gen. group, $\rho \in G^*$ nonzero.
Say $\exists\ x_1, \dots, x_n \in G$ s.t.
① G is gen. by the x_i.
② $\rho(x_1) \neq 0$.
③ For $2 \le i \le n$ $\exists\ j < i$ s.t. $\rho(x_j) \neq 0$ and
$[x_j, x_i] \in \langle x_1, \dots, x_{i-1} \rangle$
Then $\rho \in \Sigma(G)$.

Special case of ③: $x_i \leftrightarrow x_j$.
This goes back to Koban-McCammond-Meier (essentially).

Actions on $\mathbb{R}$-trees

T = $\mathbb{R}$-tree (a space w/ unique paths b/w pts)
$G \curvearrowright T$ by isometries.
$\rightsquigarrow \ell : G \to \mathbb{R}$ (translation) length fn.
The action is...
nontrivial if no global fixed pts
exceptional if no invariant lines
abelian if $\exists\ \rho \in G^*$ s.t. $\ell = |\rho|$
(say action is associated to ρ).

Lemma. (Brown '87) Say $\rho \in G^*$. $\exists$ exceptional, nontrivial, abelian action of G on an $\mathbb{R}$-tree assoc. to ρ $\Longleftrightarrow$ $\rho \notin \Sigma(G)$.

Say $G \circlearrowright T = \mathbb{R}$-tree, $g \in G$.

$\rightsquigarrow$ characteristic subtree T_g

g elliptic $\rightsquigarrow$ fixed pts

g hyperbolic $\rightsquigarrow$ axis

Facts ① $g \leftrightarrow h$ hyps $\Longrightarrow T_g = T_h$

② $g \leftrightarrow h$, h hyp $\Longrightarrow T_g \supseteq T_h$

Commuting graph $X \subseteq G \rightsquigarrow C(X) =$ graph with vertex set X and edges for commuting.

Domination $X, Y \subseteq G$. Say Y dominates X if every elt of X commutes with some elt of Y

Lemma (KMM) $G =$ group, $\rho \in G^*$. If $\exists$ $X, Y \subseteq G$ s.t.

① $\rho(y) \neq 0$ $\forall y \in Y$

② $C(Y)$ connected

③ Y dominates X

④ X generates G

Then $\rho \in \Sigma(G)$.

Pf. Suppose $\exists$ abelian action of G on $\mathbb{R}$-tree T assoc. to ρ.

By ① each $y \in Y$ acts as hyperbolic.

By ② there is a common characteristic subtree T_Y.

By ③ $T_x \supseteq T_Y \quad \forall\, x \in X$.

By ④ T_Y invariant under G

$\Longrightarrow$ action is not exceptional. $\square$

Proof of Ershov-He is essentially same.

CLAIM 1

Claim 1. $\exists$ finite gen set S for $\mathcal{I}_g$ consisting of genus 1 BPs s.t.

$$\{\rho \in \mathcal{I}_g^* : \rho(s) \neq 0 \ \forall s \in S\} \subseteq \Sigma(\mathcal{I}_g)$$

Pf. By Johnson I there is a finite set X of genus 1 BP maps that generates $\mathcal{I}_g$.

Make a graph Γ w/ vertices the genus 1 BPs in S_g and edges for disjointness. Putman trick $\Rightarrow$ Γ connected. Let S be a set of BP maps that contains X and corresponds to a connected subset of Γ.

Enumerate the elts of S as $s_1, \ldots, s_n$ s.t. $\forall i \ \exists$ $j<i$ with $s_i \leftrightarrow s_j$ (enumerate by increasing distance from a basepoint in Γ).

Choose ρ with $\rho(s_i) \neq 0$ $\forall i$. Apply the Ershov-He lemma.

Claim 2

Claim 2. In this topology, $\mathrm{Mod}(S_g)$ is an irreducible space (i.e. it is not the union of proper closed subspaces.)

Facts about irred. spaces

① Y irred. top. space, $X \to Y$ set map $\Rightarrow$ pullback topology on X is irred.

② $Y \to Z$ cont., Y irred $\Rightarrow$ $\mathrm{im}(Y)$ irred.

③ $Z \subseteq W$ subsp. irred $\Leftrightarrow$ $\overline{Z}$ irred.

Pf. By ①, enough to show image of $\mathrm{Mod}(S_g^1)$ in $GL(\mathcal{I}_g^{1\,*})$ is irred.

Recall $(\mathcal{I}_g^1)^{ab} \otimes \mathbb{R} \cong (\mathcal{I}_g^1)^* \cong \Lambda^3 H$ natural.

Image of $\mathrm{Mod}(S_g^1)$ is image of $Sp_{2g}(\mathbb{Z})$ under $\iota : GL_{2g}(\mathbb{R}) \to GL(\Lambda^3 H)$

Classical: Zariski closure of $Sp_{2g}(\mathbb{Z})$ in $GL_{2g}(\mathbb{R})$ is $Sp_{2g}(\mathbb{R})$, which is a connected alg. gp, hence irred.

So ③ $\Rightarrow$ $Sp_{2g}(\mathbb{Z})$ is irred.

The map ι is Zariski continuous.

So ② $\Rightarrow$ $\iota(Sp_{2g}(\mathbb{Z}))$ is irred. $\square$

CLAIM 3

Fix $\rho \neq 0$ in I_g^*, $s \in S \rightsquigarrow Z_s = \{ f \in \mathrm{Mod}(S_g) : f \cdot \rho(s) = 0 \}$

Claim 3. Z_s is a proper, closed subset of $\mathrm{Mod}(S_g)$.

Pf. For fixed s, the condition $\rho(s) = 0$ is Zariski closed

$\Rightarrow Z_s$ closed.

Suppose $Z_s = \mathrm{Mod}(S_g)$.

$\Longrightarrow (f \cdot \rho)(s) = \rho(f s f^{-1}) = 0 \quad \forall f \in \mathrm{Mod}(S_g)$

$\Rightarrow \rho$ vanishes on all BP maps of genus 1

$\Rightarrow \rho = 0$. $\square$

BNS – THE EASY CASE
following Putman

Prop. G = f.g. gp
$\rho : G \to \mathbb{Z}$ surjective
$H = \ker \rho$
$\rho, -\rho \in \Sigma(G)$.
Then H f.g.

Pf. Choose $t \in G$ s.t. $\rho(t) = 1$.
G f.g. $\Rightarrow \exists$ finite $S \subseteq H$ s.t. $S \cup \{t\}$ gens. G.
$\rightsquigarrow$ S normally generates H.
$\rightsquigarrow$ H gen. by

$$\bigcup_{k=-\infty}^{\infty} t^k S t^{-k}$$

Claim. H gen. by $S_+ = \bigcup_{k=0}^{\infty} t^k S t^{-k}$

Pf. Since $\rho \in \Sigma(G)$, any $h \in H$ can be written as $t^{i_1} s_1 t^{i_2} s_2 \cdots t^{i_n} s_n$
where running totals $i_1 + \cdots + i_k$ are ≥ 0
& $i_1 + \cdots + i_n = 0$. The claim follows.

Example: $h = t^2 s_1 t^{-1} s_2 t^3 s_3 t^{-4}$
$= (t^2 s_1 t^{-2})(t s_2 t^{-1})(t^4 s_3 t^{-4})$ //

Similarly, $-\rho \in \Sigma(G) \Rightarrow H$ gen. by
$$S_- = \bigcup_{k=-\infty}^{0} t^k S t^{-k}$$

So $\forall\ s \in S$, can write tst^{-1} as product of elts of S_-. $\Rightarrow \exists\ N \geq 0$ s.t. the gp. gen by
$$S_{-N,0} = \bigcup_{k=-N}^{0} t^k S t^{-k}$$
contains tSt^{-1}.

Let H' = gp. gen by $S_{-N,0}$.
WTS $H' = H$.

Claim. $tH't^{-1} \subseteq H'$

Pf. $tH't^{-1}$ gen by $\bigcup_{k=-N+1}^{1} t^k S t^{-k}$

All in H' by defn except the tSt^{-1}. But we actually chose N, hence H', so this is true.

Of course $H' \subseteq H$. Remains to show $H \subseteq H'$.
Applying claim iteratively $\Rightarrow t^k S t^{-k} \subseteq H'\ \forall\ k \geq 0$.
But as above these generate H. $\square$