Homology 3-Spheres and Torelli

Fix $S_g \subset S^3$ standard $\rightsquigarrow$ Heegaard splitting.
$f \in \mathrm{Mod}(S_g) \rightsquigarrow$ new manifold $M(f)$: change the gluing of the two handlebodies by f.

<u>Fact</u>. $M(f)$ is a homology $S^3 \iff f \in \mathcal{I}(S_g)$.

Since every closed, orientable 3-manifold is an $M(f)$ for some f (in some $\mathrm{Mod}(S_g)$), all homology 3-spheres arise this way.

Let $\mathcal{K}(S_g) = \ker \tau$

<u>Thm</u> (Morita). Every homology 3-sphere is $M(f)$ for some f in some $\mathcal{K}(S_g)$.

In particular, since Dehn twists correspond to Dehn surgeries and $\mathcal{K}(S_g)$ is gen. by Dehn twists (later in the course), the following graph is connected: vertices - hom. 3-spheres
edges - Dehn surgery on a knot

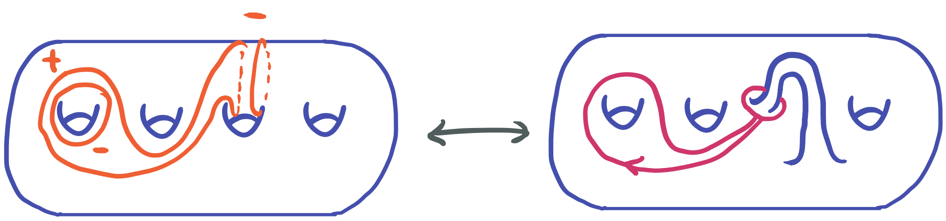
Handlebody Groups

V_g = handlebody, $S_g = \partial V_g$

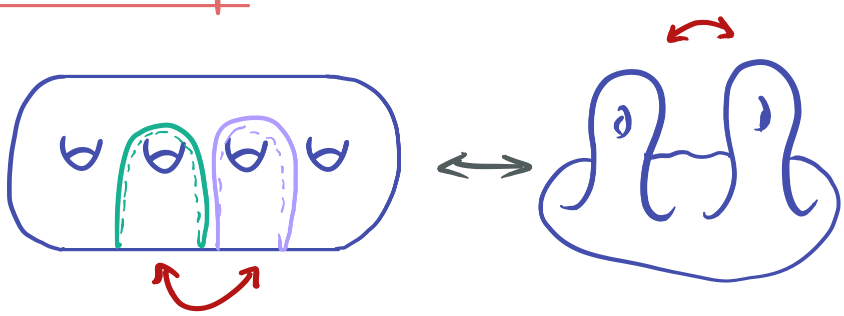
$H(S_g) = \{ f \in \mathrm{Mod}(S_g) : f \text{ extends over } V_g \}$
$\leq \mathrm{Mod}(S_g)$

Fact. If $c \subseteq S_g$ bounds a disk in V_g then $T_c \in H(S_g)$. (converse also true.)

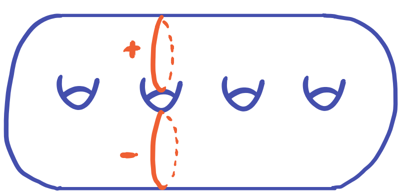
Dragging Feet



Handle Swaps



Bounding Pair Maps



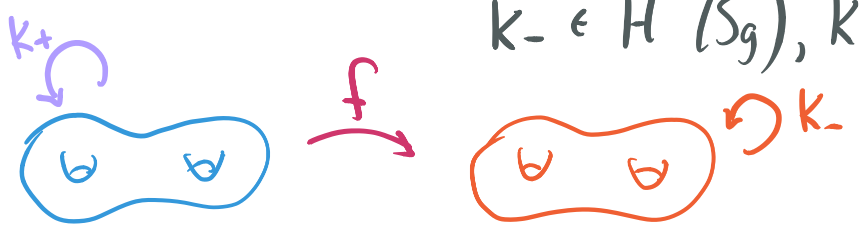
Handlebody Groups & Heegaard Splittings

Say $S_g \subseteq S^3$ Heegaard

$H^+(S_g)$, $H^-(S_g)$ the two handlebody groups.

<u>Fact</u>. $M(f) = M(h) \iff h = k_- f k_+$

$k_- \in H^-(S_g)$, $k_+ \in H^+(S_g)$



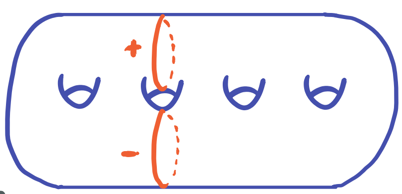
Handlebody Torelli Groups

V_g = handlebody

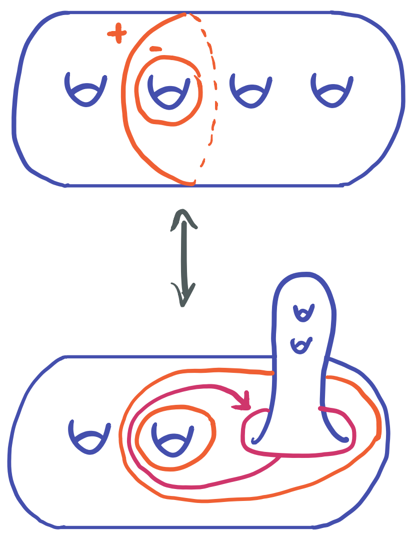
$S_g = \partial V_g$

$HI(S_g) = H(S_g) \cap I(S_g)$

Sample Elements



clearly in $H(S_g)$



Handlebody Torelli Under Johnson

Let $W_y \subseteq \Lambda^3 H$ subspace spanned by basis elts with a y.

Prop. $\tau(HI(S_g)) = W_y$. (only need $\supseteq$).

Pf. $\boxed{\subseteq}$ $HI(S_g)$ preserves $\langle\langle \beta_i \rangle\rangle$...

$\boxed{\supseteq}$ By naturality & existence of handle swaps in $H(S_g)$, enough to exhibit:

$x_1 \wedge y_1 \wedge x_2$, $x_1 \wedge y_1 \wedge y_2$, $x_1 \wedge x_2 \wedge y_3$, $x_1 \wedge y_2 \wedge y_3$, and $y_1 \wedge y_2 \wedge y_3$

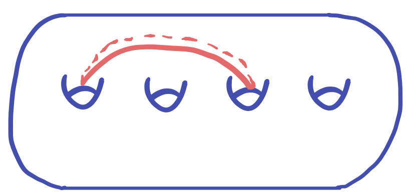
The first 2 are given by the sample elts of $HI(S_g)$.

The foot drag above acts on H by:

$$y_1 \longmapsto y_1 + y_3, \quad x_3 \longmapsto x_3 - x_1$$

Apply to the first 2 targets gives next 2.

Finally apply the twist:

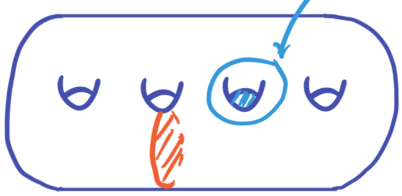


The action on H takes the second target to the sum of the second & fifth. $\square$

The Proof of Morita's Theorem

$S_g \subseteq S^3$ std/Heegaard.

$H^+(S_g) = H(S_g)$
$H^-(S_g) = \iota H^+(S_g) \iota^{-1}$ where ι_* (action on H) swaps x's & y's.



But then $\tau(\langle H^+(S_g), H^-(S_g)\rangle) = W_x \cup W_y$
$= \Lambda^3 H.$

So can modify a given $M(f)$ so that $\tau(f') = 0$.