THE SYMPLECTIC REPRESENTATION OF MCG

The symplectic group

Consider $\mathbb{R}^{2g}$ with basis $(x_1, \ldots, x_g, y_g, \ldots, y_1)$ and standard symplectic form

$$\omega = \sum_{i=1}^{g} dx_i \wedge dy_i$$

Think of ω as a pairing on $\mathbb{R}^{2g}$ e.g.

$$\omega(x_1 + 2y_2,\ x_1 + y_1 + x_2) = 1 - 2 = -1$$

This is the unique nondegenerate, alternating bilinear form on $\mathbb{R}^{2g}$ up to change of basis.

Connection to surfaces:

$$(\mathbb{R}^{2g}, \omega) \cong (H_1(S_g; \mathbb{R}), \hat{i})$$

$Sp_{2g}(\mathbb{R})$ = subgp of $GL_{2g}\mathbb{R}$ preserving ω:

$$\omega(u, v) = \omega(Mu, Mv)$$

Similar with $\mathbb{Z}$.

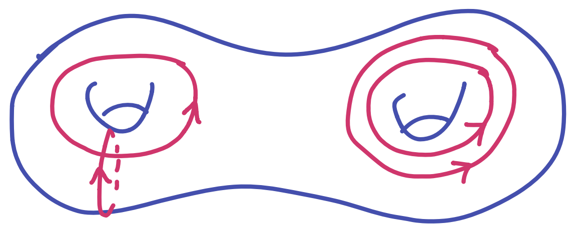
Realizing H_1-classes by curves.

<u>Prop</u>. If $v \in H_1(S_g;\mathbb{Z})$ is primitive then $v = [c]$ where c is an oriented simple closed curve.

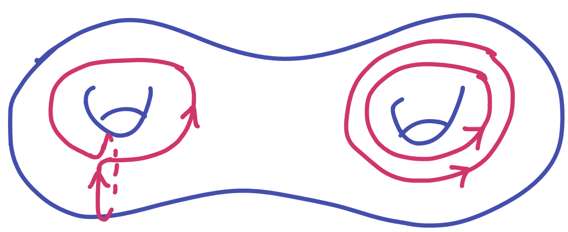
<u>Pf</u> (Meeks–Patrusky). Euclidean algorithm for scc's.

Step 1. Draw v naively:

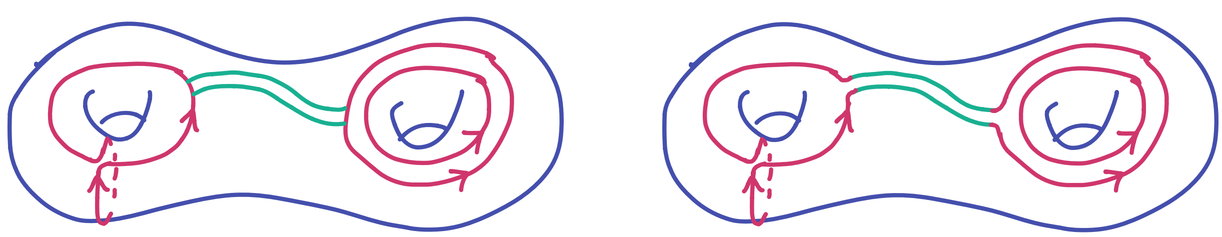
$v = x_1 + y_1 + 2x_2$



Step 2. Surger to remove crossings.



Step 3. Band surgeries to reduce the number of components



By Euclidean algorithm, this terminates in a connected curve! $\square$

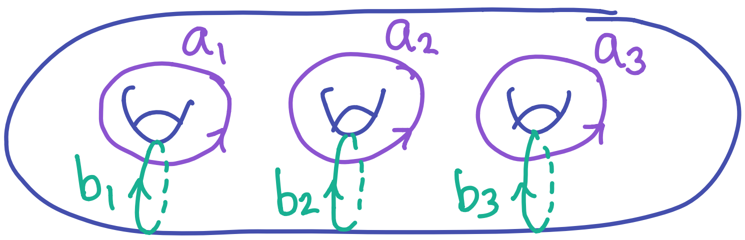
Action of a Dehn Twist

Prop. Say a, b = oriented curves

Then $T_b^k([a]) = [a] + k\,\hat{i}(a,b)[b]$

N.B. Indep of orientation of b!

A geometric symplectic basis:



Proof. Case 1. b separating

Choose geometric symplectic basis for $H_1(S_g)$ disjoint from b.

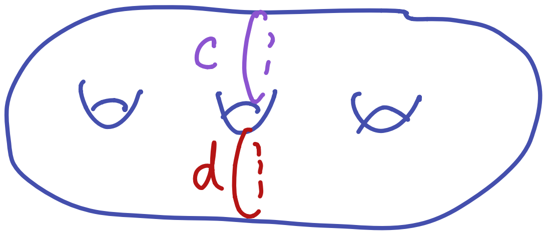
Case 2. b nonseparating.

Choose a geometric symplectic basis so b is one curve. Check for a = basis elt. Apply linearity of $\psi(T_b^k)$.

Some Elements of Torelli.

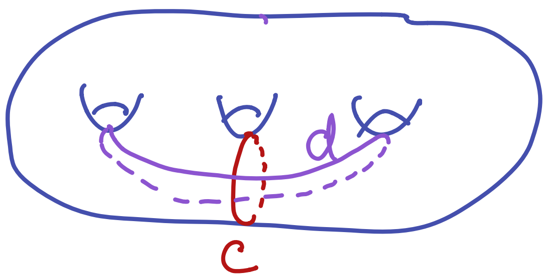
1. Dehn twists about sep curves

2. Bounding pair maps



$T_c T_d^{-1}$

3. Commutators of simply intersecting pairs



$[T_c, T_d]$

2 & 3 are special cases of:

$T_c T_d^{-1}$ where $[c] = [d]$

That all of these lie in Torelli follows immediately from the Prop.

4. $\pi_1(S_g) \trianglelefteq \mathrm{Mod}(S_{g,1})$

SURJECTIVITY OF THE Sp-REP

<u>Thm</u> $\Psi: \mathrm{Mod}(S_g) \longrightarrow Sp_{2g}(\mathbb{Z})$ is surjective.

$\boxed{\text{1}^{\text{st}} \text{ proof: transvections.}}$

A <u>transvection</u> in $Sp_{2g}(\mathbb{Z})$ is an elt whose 1-eigenspace is $(2g-1)$-dim.

$$\Updownarrow$$

$$\tau_v(u) = u + \omega(u,v)v \quad \text{(or a power)}$$

<u>Fact</u>. $Sp_{2g}(\mathbb{Z})$ is gen. by transvections.

<u>Pf of Thm</u>. Suffices to hit τ_v, v primitive.
Prop $\rightsquigarrow$ a s.t. $[a] = v$
$\Psi(T_a) = \tau_v$ $\square$

Want a proof that does not presuppose a genset for Sp.

2^{nd} proof: geometric symplectic bases

$\mathrm{Sp}_{2g}(\mathbb{Z}) \longleftrightarrow$ symplectic bases for $\mathbb{Z}^{2g}$

$I \longleftrightarrow$ standard symplectic basis.

Proof of Thm. Given $A \in \mathrm{Sp}_{2g}(\mathbb{Z})$, realize A as a geometric symplectic basis (supe up the proof of Prop above).

Realize I by standard geometric symplectic basis.

Apply change of coordinates: given two topologically equivalent configurations of curves, there is an element of $\mathrm{Mod}(S_g)$ taking one to the other. $\square$

Corollaries of the proof:

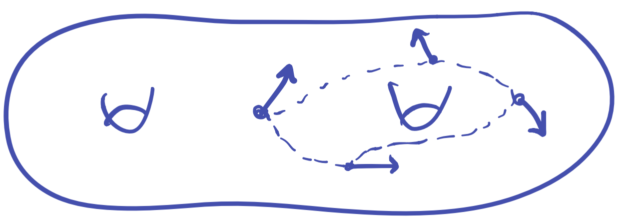
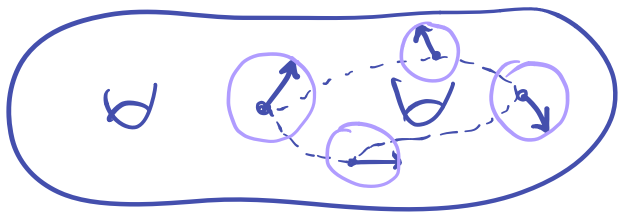
① $\mathcal{I}(S_g)$ acts transitively on s.c.c. reps of a homology class.

② $\mathcal{I}(S_g)$ acts trans. on sep curves/BPs inducing same splitting.

$\Longrightarrow$ conjugacy classification of such elts.

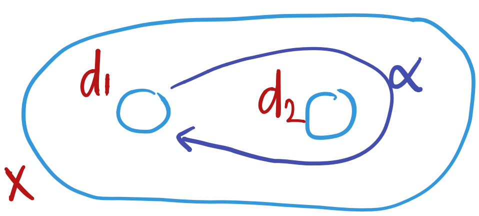
Unit Tangent Bundle Subgroups

A loop in $\pi_1 UT(S_g) \rightsquigarrow$ isotopy of a disk in S_g

$$
\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & & \\
 & \mathbb{Z} \arrow[d] & \mathbb{Z} \arrow[d] & & \\
1 \arrow[r] \arrow[d] & \pi_1(UT(S_g)) \arrow[r] \arrow[d] & \mathrm{Mod}(S_g^1) \arrow[r] \arrow[d] & \mathrm{Mod}(S_g) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \pi_1(S_g) \arrow[r] \arrow[d] & \mathrm{Mod}(S_{g,1}) \arrow[r] \arrow[d] & \mathrm{Mod}(S_g) \arrow[r] & 1 \\
 & 1 & 1 & &
\end{tikzcd}
$$

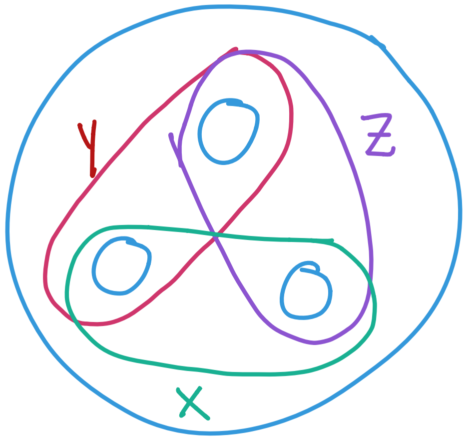
Example

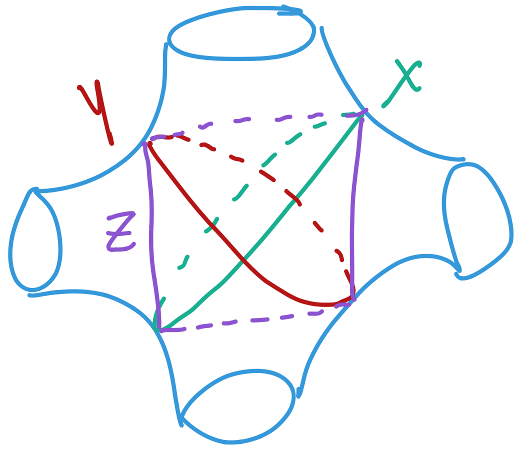


$\mathrm{Push}(\alpha) = T_x T_{d_1}^{-1} T_{d_2}^{-1}$

"do si do"

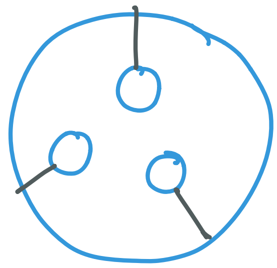
THE LANTERN RELATION.





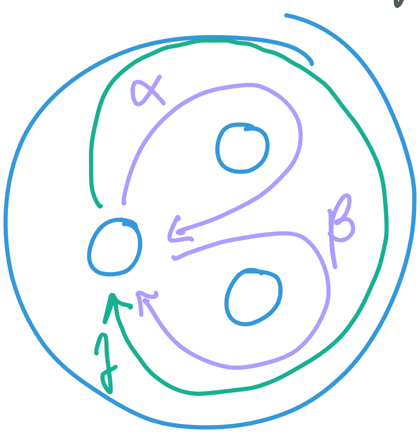
$$T_x T_y T_z = \prod_{i=1}^{4} T_{\partial_i}$$

<u>Pf #1</u>. Check action on



By symmetry, enough to check one (why?). Apply Alexander trick.

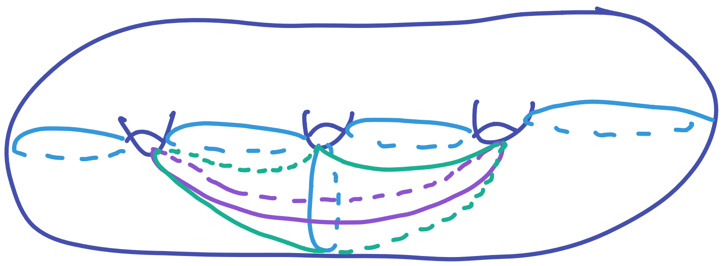
<u>Pf #2</u>. Consider push maps:



Note $\alpha\beta = \gamma$. Reinterpret as lantern relation.

Consequences of Lantern Relation.

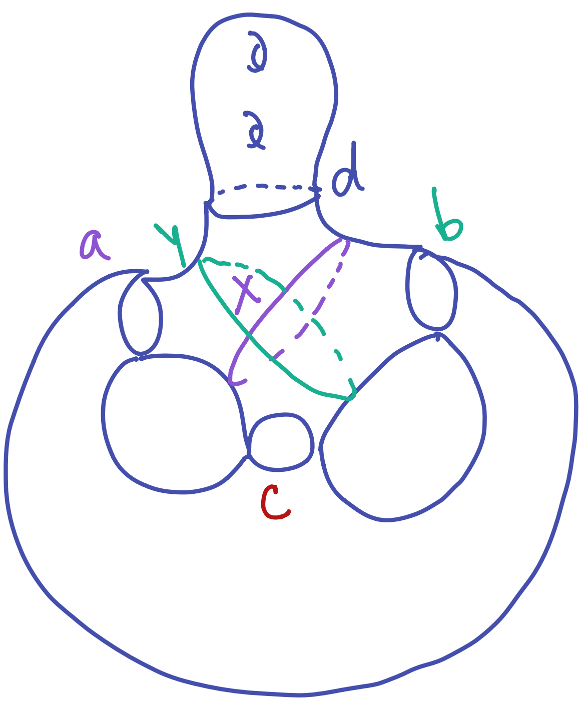
1. <u>Theorem</u> $\mathrm{Mod}(S_g)^{ab} = 1 \qquad g \geq 3$



<u>Pf.</u> All 7 twists are conjugate
$\Rightarrow$ have same image t in $\mathrm{Mod}(S_g)^{ab}$
Lantern $\Rightarrow t^3 = t^4 \Rightarrow t = 1$.

2. <u>Prop</u> Dehn twists about sep curves are products of BP maps.

<u>Pf.</u>



Lantern relation
$\Rightarrow$

$$T_d = (T_x T_a^{-1})(T_y T_b^{-1})(T_z T_c^{-1})$$

Torelli Groups Are Torsion Free

$\boxed{\text{BLACK BOX}}$ If $f \in \mathrm{Mod}(S)$ has finite order, then $f = [\varphi]$ st φ = isometry

Theorem $\mathcal{I}(S_g)$ is torsion free

Pf. #1 Say $f \in \mathcal{I}(S_g)$ has finite order
Black box $\rightsquigarrow$ φ = isometry. $\Rightarrow$ iso. fix pts.
Lefschetz FPT:

$$\sum_{p = \text{fix pt}} \mathrm{ind}_p(\varphi) = \sum (-1)^i \,\mathrm{trace}\left(\varphi_* : H_i(S_g)^{\circlearrowleft}\right)$$

$$\#\text{ fixed pts} = 1 - 2g + 1 < 0 \qquad \square$$

Pf #2 Fact: $G \curvearrowright X \Rightarrow$

$$H_1(X/G;\mathbb{Q}) \cong H_1(X;\mathbb{Q})^G$$

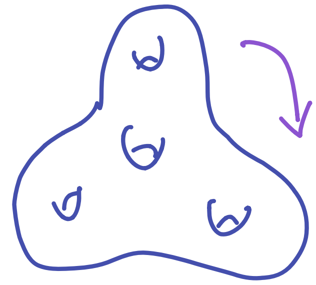
Assume now $g \geq 2$. As above $f \rightsquigarrow \varphi$
Take $G = \langle \varphi \rangle$. Have:

$$\mathrm{genus}(S_g/G) < g$$

$$\Longrightarrow H_1(S_g/G) \not\cong H_1(S_g)$$

$$\Longrightarrow G \not\subseteq \mathcal{I}(S_g). \qquad \square$$

Pf #3 Fact: f is either
- deck transf.
- rotation of a polygon

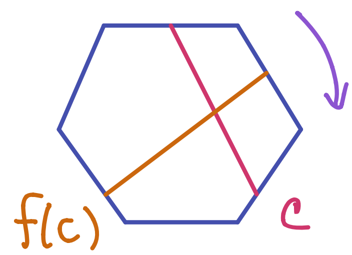


Deck transformations

clearly not in $\mathcal{I}(S_g)$

Rotations of polygons

Choose a diag. that is a s.c.c. c in S_g



Note $i(c, f(c)) \leq 1$.

Case 1. c sep $\Rightarrow i(c, f(c)) = 0 \Rightarrow$ f not in Torelli ✓

Case 2. c nonsep, $i(c, f(c)) = 1$ ✓

Case 3. $|f| \geq 3 \Rightarrow c, f(c), f^2(c)$ cannot all be homologous (consider the labels they cut off) ✓

Case 4. $|f| = 2$, all diags are nonsep, all taken to homologous curves $\Rightarrow f =$ hyperelliptic involution ✓