The Johnson Homomorphism.

Birman's question: Is $[\mathcal{I}(S_g) : \mathcal{K}(S_g)] < \infty$?

Will now define

$$\tau : \mathcal{I}(S_g^1) \to \Lambda^3 H$$

where $H = H_1(S_g^1; \mathbb{Z})$.

- $\Lambda^3 H \cong \mathbb{Z}^{\binom{2g}{3}}$ = free abelian group.
- $\mathcal{K}(S_g^1) \leq \ker \tau$

This answers Birman's question.

Later:
- τ captures all of $H^1(\mathcal{I}(S_g); \mathbb{Z})$.
- $\mathcal{K}(S_g) = \ker \tau$
- answer negatively a question of Chillingworth: is $\mathcal{K}(S_g)$ the subgroup of $\mathcal{I}(S_g)$ preserving all winding numbers of curves.
- All $\mathbb{Z}HS^3$ come from $\mathcal{K}(S_g)$
- $\mathcal{I}(S_g)$ exp. distorted in $\mathrm{Mod}(S_g)$
- and much more...

Tensor & Wedge Products

$$V \otimes W = \langle v \otimes w : v \in V, w \in W \rangle / \text{bilinearity}$$

basis: $e_i \otimes f_j$ $\quad V = \langle e_i \rangle, W = \langle f_j \rangle$

$\rightsquigarrow (\dim V \cdot \dim W)$-dimensional.

$$\begin{aligned} V \wedge V &= (V \otimes V) / (v \otimes w = -w \otimes v) \\ &= (V \otimes V) / (v \otimes v). \\ &= \text{image of } V \otimes V \longrightarrow V \otimes V \\ &\qquad\qquad v \otimes w \longmapsto v \otimes w - w \otimes v \end{aligned}$$

basis: $e_i \wedge e_j$ $\quad i \neq j$ $\rightsquigarrow \binom{n}{2}$-dimensional.

So $\Lambda^k V = \langle v_1 \wedge \cdots \wedge v_k \rangle / \text{bilinearity}$

and...

$$v_1 \wedge \cdots \wedge v_k = \operatorname{sgn}(\sigma)\, v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(k)}.$$

$\rightsquigarrow \binom{n}{k}$ dimensional.

$$\Lambda^k V = \text{image of } \otimes^k V \to \otimes^k V$$

$$v_1 \otimes \cdots \otimes v_k \longmapsto \sum_{\sigma \in \Sigma_k} \operatorname{sgn}(\sigma)\, v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}$$

DEFINITION

Let $\Gamma = \pi_1(S_g^1) \cong F_{2g}$
$\Gamma' = [\Gamma, \Gamma]$

Consider:

$$1 \longrightarrow \frac{\Gamma'}{[\Gamma, \Gamma']} \longrightarrow \frac{\Gamma}{[\Gamma, \Gamma']} \longrightarrow \frac{\Gamma}{\Gamma'} \longrightarrow 1$$

or: $1 \to N \longrightarrow E \longrightarrow H \longrightarrow 1$

The Johnson homomorphism is

$$\tau : \mathcal{I}(S_g^1) \longrightarrow \mathrm{Hom}(H, N)$$

given by $\tau(f)(x) = f(e)e^{-1}$
where e is any lift of x to E.

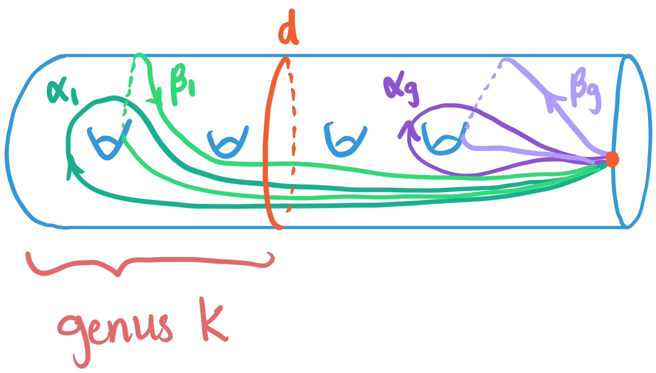
What about $\Lambda^3 H$?

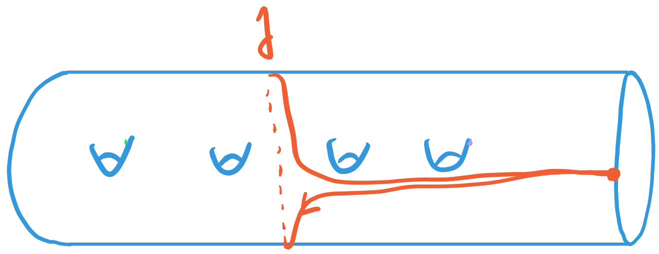
- $\Lambda^2 H \cong N$ via $a \wedge b \longmapsto [\tilde{a}, \tilde{b}]$
  where $\tilde{a}, \tilde{b}$ are lifts to E — see Putman's lecture notes
- $\mathrm{Hom}(H, \Lambda^2 H) \cong H^* \otimes \Lambda^2 H \cong H \otimes \Lambda^2 H$
- Will show $\mathrm{im}\,\tau$ is im $\Lambda^3 H \longrightarrow H \otimes \Lambda^2 H$

$$a \wedge b \wedge c \longmapsto a \otimes (b \wedge c) + b \otimes (c \wedge a) + c \otimes (a \wedge b)$$

The Image of a Dehn Twist

Consider T_d





Action on Γ: $T_d(x) = x \quad x \in \{\alpha_{k+1}, \beta_{k+1}, \dots \alpha_g, \beta_g\}$

$T_d(x) = \delta x \delta^{-1} \quad x \in \{\alpha_1, \beta_1, \dots, \alpha_k, \beta_k\}$.

So: $\tau(T_d)([x]) = 1$ or $[x, \delta]$ in N

but both equal 1 in N.

$\Longrightarrow \tau(T_d) = 0$.

This is just <u>one</u> Dehn twist, but:

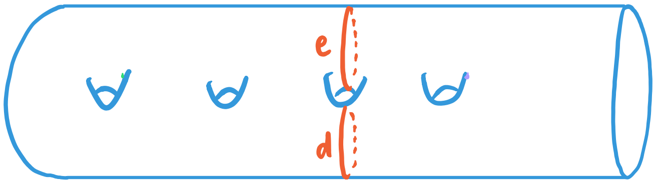
Naturality: If $h \in Mod(S_g^1)$, $f \in \mathcal{I}(S_g^1)$

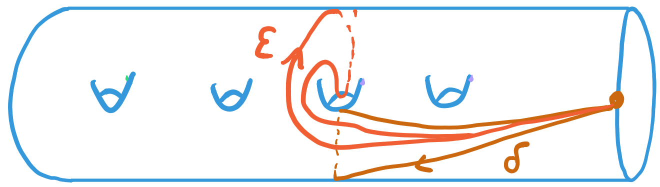
then $\tau(hfh^{-1}) = h\,\tau(f)$.

(straightforward)

Thus $\mathcal{K}(S_g^1) \leq \ker \tau$.

The Image of a Bounding Pair Map





Let $f = T_d T_e^{-1}$

Compute: $f(\alpha_i) = \delta \alpha_i \delta^{-1} \quad i \le k \qquad f(\alpha_i) = \alpha_i \quad i \ge k+1$

$f(\beta_i) = \delta \beta_i \delta^{-1} \quad i \le k \qquad f(\beta_i) = \beta_i \quad i \ge k+1$

$f(\alpha_{k+1}) = \delta \varepsilon^{-1} \alpha_{k+1}$

Now write down all $f(x)x^{-1} \in N \cong \Lambda^2 H \qquad x \in \{\alpha_i, \beta_i\}$

$$f(\alpha_i)\alpha_i^{-1} = [\delta, \alpha_i] \longleftrightarrow [\beta_{k+1}] \wedge [\alpha_i]$$
$$f(\beta_i)\beta_i^{-1} = [\delta, \beta_i] \longleftrightarrow [\beta_{k+1}] \wedge [\beta_i]$$
$$f(\alpha_{k+1})\alpha_{k+1}^{-1} = \delta\varepsilon^{-1} \longleftrightarrow \sum [\alpha_i] \wedge [\beta_i]$$
$$f(x)x^{-1} = id \longleftrightarrow 0 \quad \text{otherwise.}$$

$$\Rightarrow \tau(f) = \sum_{i=1}^{k} \Big([\beta_i] \otimes ([\beta_{k+1}] \wedge [\alpha_i]) - [\alpha_i] \otimes ([\beta_{k+1}] \wedge [\beta_i]) \Big) + [\beta_{k+1}] \otimes \Big(\sum_{i=1}^{k} [\alpha_i \wedge \beta_i] \Big)$$
$$= \sum_{i=1}^{k} ([\alpha_i] \wedge [\beta_i]) \wedge [\beta_{k+1}]$$

$\Rightarrow |\mathrm{Im}\, \tau| = \infty \Rightarrow$ Birman question.

The Image of Tau

Recall $\Lambda^3 H \hookrightarrow H \otimes \Lambda^2 H$
via $a \wedge b \wedge c \longmapsto a \otimes (b \wedge c) + b \otimes (c \wedge a) + c \otimes (a \wedge b)$

<u>Prop.</u> $\operatorname{im} \tau = \Lambda^3 H$.

By the naturality, $\operatorname{Im} \tau$ is an Sp-rep. And $\Lambda^3 H$ is one of two GL-irreps in $\Lambda^2 H \otimes H$.

<u>Pf.</u> <u>for</u> <u>$g=2$</u> Already have: $\boxed{x_1 \wedge y_1 \wedge y_2} \in \operatorname{Im} \tau$
(consider the "first" bounding pair).
Now use Sp to move this around.

Apply Sp-map $x_2 \longrightarrow y_2 \longrightarrow -x_2$
(other basis elements fixed).
to $\boxed{x_1 \wedge y_1 \wedge y_2}$ to get $\boxed{-x_1 \wedge y_1 \wedge x_2}$

Apply $x_1 \leftrightarrow x_2$, $y_1 \leftrightarrow y_2$ to get
$\boxed{x_2 \wedge y_2 \wedge y_1}$ $\boxed{-x_2 \wedge y_2 \wedge x_1}$ $\square$

Higher genus similar.

WHY WOULD JOHNSON THINK OF THIS?

$\mathcal{I}(S_g^1)$ is by definition the kernel of

$$\mathrm{Mod}(S_g^1) \longrightarrow \mathrm{Aut}(\Gamma/\Gamma').$$

Then you notice that twists about sep curves conjugate by <u>commutators</u> and BP maps conjugate by non-commutators.

In other words the former acts trivially on $\Gamma/[\Gamma,\Gamma']$ and the latter does not.

The map τ measures this action.

The Johnson Homomorphism via Mapping Tori

$f \in \mathrm{Mod}(S_g^1) \rightsquigarrow$ mapping torus M_f.

$f \in \mathcal{I}(S_g^1) \Longrightarrow H^*(M_f) \cong H^*(S_g^1 \times S^1)$.

<u>but</u> ring structure (or, intersection theory) can be different.

Given f, want $\tau(f) \in \Lambda^3 H$.

By taking duals, this is the same as an alt. linear function that takes as input 3 elts of H and outputs a number

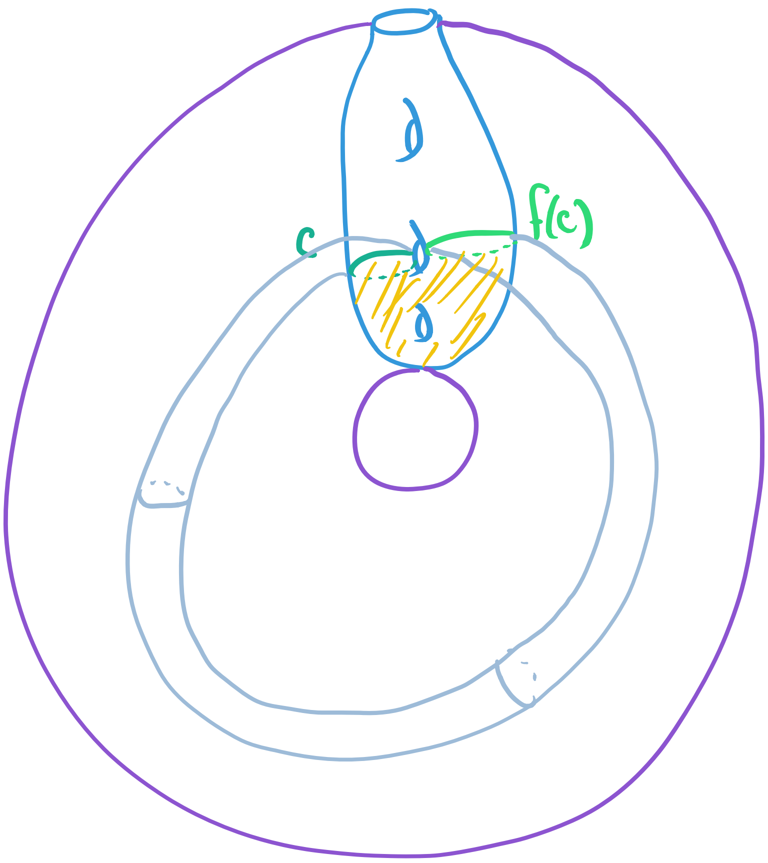
As above, there is an inclusion

$$H^* \cong H^1(S_g^1) \xrightarrow{i} H^1(M_f) \cong H_2(M_f, \partial)$$

The desired function is triple cup product or, dually, triple intersection.

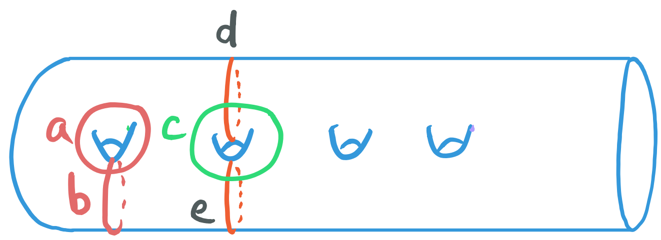
Let's check this does the right thing on BP maps.

First, let's give an explicit description of i:



If we "flow" c we get a tube in M_f from c to f(c) in one fiber. But c & f(c) are homologous so we can fill in the homology to get a closed surface, i.e. elt. of $H_2(M_f)$.

Now fix a BP map $T_d T_e^{-1}$



Only one curve from std geom. symplectic basis gets moved, namely c. The curves a & b give tori in M_f, intersecting at $(a \cap b) \times S^1$. The curve c gives a surface of genus 2 as in above picture, intersecting $(a \cap b) \times S^1$ in one point. This gives the term $[a] \wedge [b] \wedge [c]$, as desired. All other terms are zero.

The Johnson Homomorphism via the Jacobian

Starting point: $\Lambda^3 H \cong H_3(T^{2g}; \mathbb{Z})$.

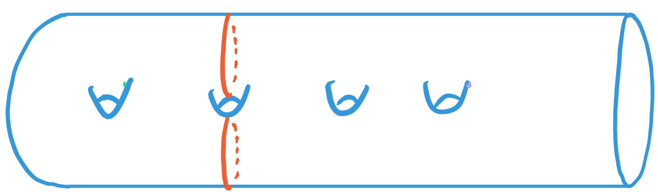
and we have a map $j : S_g^1 \longrightarrow T^{2g}$ corresponding to $\pi_1(S_g^1) \longrightarrow H_1(S_g^1)$.

By $K(G,1)$ theory, the map is unique up to homotopy.

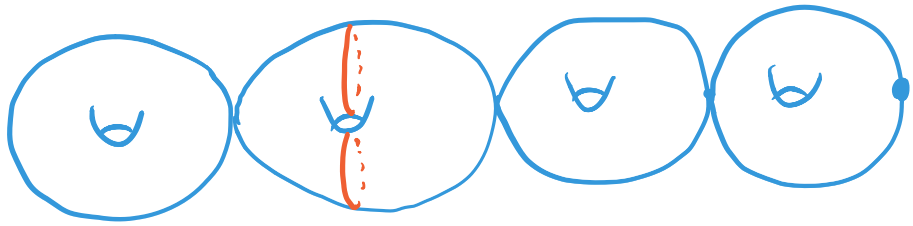
Let $f \in \mathcal{I}(S_g^1)$. Then $j \circ f$ is homotopic to j and $\operatorname{im} j \circ f = \operatorname{im} j$.

The homotopy gives a mapping torus in T^{2g}, which represents an elt of $H^3(T^{2g})$!

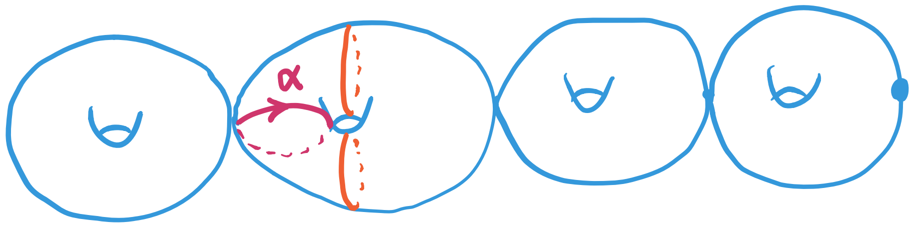
Again, lets check this does the right thing on BP maps. Consider:



Observe that j factors through:



Each torus maps to a coordinate 2-torus $x_i \wedge y_i$ in T^{2g}.



The BP map is obtained by pushing the leftmost torus along α.

What is happening in T^{2g}? The $x_1 \wedge y_1$ torus is moving in the y_2-direction, tracing out the 3-torus $x_1 \wedge y_1 \wedge y_2$.

But this is exactly the image of the BP map under τ!

Another definition: Moriyama (J. London Math Soc.) shows that $K(S_g^1)$ is the kernel of the action of $\mathrm{Mod}(S_g^1)$ on the compactly supp. cohomology group of the config. space of two points in S_g^1.

Can you see directly that this action is τ?

What about the Closed Case?

Since $\mathcal{I}(S_g) \cong \mathcal{I}(S_g^1)/\pi_1 UT(S_g)$
we have a map
$$\tau : \mathcal{I}(S_g) \longrightarrow \Lambda^3 H / \tau(\pi_1 UT(S_g))$$

Claim: $\tau(\pi_1 UT(S_g))$ = image of $H \longrightarrow \Lambda^3 H$
given by $x \longmapsto \omega \wedge x$
where $\omega = x_1 \wedge y_1 + \cdots + x_g \wedge y_g$

Now, $\pi_1 UT(S_g)$ is gen. by BP maps of genus $g-1$:

$$\longmapsto (x_1 \wedge y_1 + \cdots + x_{g-1} \wedge y_{g-1}) \wedge y_g = \omega \wedge y_g$$

Now apply Sp-action. Note Sp fixes ω, and acts transitively on basis elements of H. That does it!

Put another way, the ambiguity is because a BP bounds <u>two</u> 2-chains in S_g, the difference being S_g.

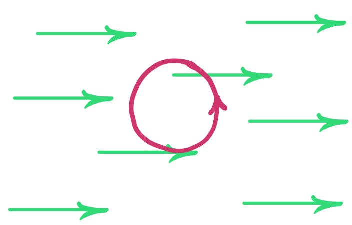
The Chillingworth Class

X = nonsingular vector field on S_g'.

$\omega_X(\gamma)$ = # times X rotates along γ.
well def on homotopy classes

X_1, X_2 two nonsingular vector fields
$\rightsquigarrow \omega_{X_2}(\gamma) - \omega_{X_1}(\gamma)$ is a cohomology class, denoted $d(X_1, X_2)$.

A cohomology class is a function on cycles that is zero on boundaries $\Longleftrightarrow$ it is zero on small cycles (cocycle condition).
So $\omega_X(\gamma)$ is not a cocycle because it evaluates to ± 1 on small loops, but $d(X_1, X_2)$ is.



Let $f \in \mathcal{I}(S_g^1)$. Consider the function

$$e_{f,x}(\gamma) = \omega_x(f(\gamma)) - \omega_x(\gamma)$$

This is the change of winding number induced by f.

Two facts: ① $e_{f,x} = d(X, f^{-1}(X))$
② $e_{f,x}$ is indep. of X!

Proof of ①:
$$\begin{aligned} e_{f,x}(\gamma) &= \omega_x(f(\gamma)) - \omega_x(\gamma) \\ &= \omega_{f^{-1}(x)}(\gamma) - \omega_x(\gamma) \\ &= \langle d(f^{-1}(X), X), \gamma \rangle \end{aligned}$$

Proof of ②:
$$\begin{aligned} e_{f,x_2}(\gamma) - e_{f,x_1}(\gamma) &= \omega_{x_2}(f(\gamma)) \\ &- \omega_{x_2}(\gamma) - \omega_{x_1}(f(\gamma)) - \omega_{x_1}(\gamma) \\ &= \langle d(X_1, X_2), f(\gamma) \rangle - \langle d(X_1, X_2), \gamma \rangle \\ &= \langle d(X_1, X_2), f(\gamma) - \gamma \rangle. \end{aligned}$$
$\square$

So we may write e_f. This is an elt of $H^* \cong H$.

Fact. $e_{fh} = e_f + e_h$ for $f, h \in \mathcal{I}(S_g^1)$.

i.e. $e : \mathcal{I}(S_g^1) \longrightarrow H$ is a homomorphism.

Pf.
$$\begin{aligned} e_{fh}(\gamma) &= \omega_x(fh(\gamma)) - \omega_x(\gamma) \\ &= \omega_x(f(h(\gamma))) - \omega_x(h(\gamma)) \\ &\quad + \omega_x(h(\gamma)) - \omega_x(\gamma) \\ &= e_f(h(\gamma)) + e_h(\gamma) \\ &= e_f(\gamma) + e_h(\gamma). \qquad \square \end{aligned}$$

Let $t(f) = e_f^*$. This is the Chillingworth class.

The contraction $C : \Lambda^3 H \longrightarrow H$

$$x \wedge y \wedge z \longmapsto 2\left[\langle x, y\rangle z + \langle y, z\rangle x + \langle z, x\rangle y\right]$$

Theorem. $t(f) = C(\tau(f))$.

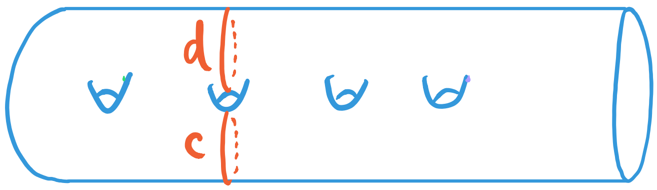
Need to check two things

① Naturality: $t(hfh^{-1}) = h\, t(f)$.

② t & $C \circ \tau$ agree on a BP map of genus 1.

IMAGE OF A BP MAP UNDER τ.

Consider:



$$C(\tau(T_c T_d^{-1})) = C(x_1 \wedge y_1 \wedge y_2) = 2y_2$$

$T_c T_d^{-1}$ leaves all the a_i, b_i (std geom. symp. basis) fixed except a_2. It and its image a_2' bound a torus with two holes, call it N.

Fix a nonsingular vector field X on S_g^1, hence N. Cap N with disks, D & D', bounded by α_2, α_2'. Extend $X|_N$ to capped N minus 2 pts.

Fix nonsing. vector fields Y, Y' on D, D'.

Have $\omega_Y(a_2) = \omega_{Y'}(a_2') = +1$.

and $\omega_Y(a_2) - \omega_X(a_2) =$ index of singularity

$\omega_{Y'}(a_2') - \omega_X(a_2') =$ index of singularity.

But Σ indices $= \chi(T^2) = 0$.

$\Rightarrow \omega_X(a_2') - \omega_X(a_2) = 2$ if you get the signs right. $\square$

Signed Stable Lengths – Irmer's Theorem

We may consider $C_x(S_g^1)$ as a directed graph: given two vertices of an edge, one lies to the left of the homology & one to the right.

$\rightsquigarrow$ signed distance d_s on vertices.

Signed stable length:

$$\phi_x(f) = \lim_{n\to\infty} \frac{d_s(v, f^n(v))}{n}$$

for $f \in \mathcal{I}(S_g^1)$. This is indep. of basept v.

$\rightsquigarrow$ $\quad SSL : \mathcal{I}(S_g^1) \longrightarrow H$.

<u>Theorem.</u> $SSL = t/2$.