Proposition 1. The only \( n \)-gons in the complex \( B_x(S_g) \) have \( n = 3, 4, 5, 6 \).

Proof. Let \( C \) be a reduced multicurve which represents \( x \) with some weight. Since we want 2-cells, we assume \( S_g \setminus C \) has 3 components. We can ignore curves which are not part of the boundary of some component of \( S_g \setminus C \), since the weights cannot be shifted to other curves.

Consider the dual graph of \( C \). Since \( C \) is reduced, the graph is recurrent. If we ignore the directions and multiplicities of edges, the new graph is connected. There are two possibilities.

Case 1: \( \cdots \). The surface is homeomorphic to the picture below.

\[
\begin{array}{c}
\includegraphics[width=1in]{diagram.png}
\end{array}
\]

Here the curves on the left are labeled \( a_1, \ldots, a_n \), not necessarily in order. Similarly, the curves on the right are labeled \( b_1, \ldots, b_m \). The components of \( S_g \setminus C \) provides two relations

\[
\sum_{i=1}^{n} s_i [a_i] = 0, \quad \sum_{j=1}^{m} t_j [b_j] = 0,
\]

where \( s_i, t_j = \pm 1 \) depend on the orientations of the curves. By relabeling the curves, we can assume

\[
\begin{align*}
s_1 = \cdots = s_k &= 1, & s_{k+1} = \cdots = s_n &= -1, \\
t_1 = \cdots = t_l &= 1, & t_{l+1} = \cdots = t_m &= -1
\end{align*}
\]
for some $1 \leq k < n$ and $1 \leq l < m$. This is possible since $C$ is reduced. Let $p = \sum_{i=1}^{n} \alpha_i a_i + \sum_{j=1}^{m} \beta_j b_j$ be a point in the 2-cell. Using the relations, we can eliminate $[a_1]$ and $[b_1]$ in $[p]$ to get

$$[p] = \sum_{i=2}^{n}(\alpha_i - s_i \alpha_1)[a_i] + \sum_{j=2}^{m}(\beta_j - t_j \beta_1)[b_j].$$

Let $x = \sum_{i=2}^{n} u_i[a_i] + \sum_{j=2}^{m} v_j[b_j]$. Comparing the coefficients with $x$, we see

$$\alpha_i - s_i \alpha_1 = u_i, \quad i = 2, \cdots, n,$$

$$\beta_j - t_j \beta_1 = v_j, \quad j = 2, \cdots, m.$$

Thus every coefficient is determined by $\alpha_1$ and $\beta_1$. Since all the coefficients are non-negative, we get the constraints on $\alpha_1$.

$$\alpha_1 \geq 0,$$

$$\alpha_1 \geq -u_i, \quad i = 2, \cdots, k,$$

$$\alpha_1 \leq u_i, \quad i = k + 1, \cdots, n$$

and similar constraints on $\beta_1$. This shows $\alpha_1$ and $\beta_1$ take values in intervals independently. Thus the 2-cell is a rectangle.

Case 2: The surface is homeomorphic to the picture below.

As before, the three families of curves are labeled $\{a_i\}_{i=1}^{m}$, $\{b_j\}_{j=1}^{n}$, $\{c_k\}_{k=1}^{l}$. The components provide relations

$$\sum_{i=1}^{n} s_i[a_i] + \sum_{j=1}^{m} t_j[b_j] = 0, \quad \sum_{i=1}^{n} s_i[a_i] + \sum_{k=1}^{l} r_k[c_k] = 0,$$

where $s_i, t_j, r_k = \pm 1$. Next we show we can assume $s_1 = -1$ and $t_1 = r_1 = 1$ by relabeling or changing the signs of both equation simultaneously. If all
the curves on each prong are pointing in the same way, i.e., \( s_1 = \cdots = s_m, \) \( t_1 = \cdots = t_n, \) \( r_1 = \cdots = r_l, \) the only possibility where \( C \) is reduced is \( t_1 = r_1 = -s_1. \) Thus we only need to adjust the sign. If there is a prong on which the curves point in different directions, we label that prong \( a. \) The only way we cannot arrange \( t_1 = r_1 \) is if \( t_j = -r_k \) for all \( j, k, \) and that implies \( b \cup c \) is trivial in homology, which is forbidden. By our choice of \( a, \) we can make \( s_1 = -1. \)

As before, let 
\[
p = \sum_{i=1}^{m} \alpha_i a_i + \sum_{j=2}^{n} \beta_j b_j + \sum_{k=2}^{l} \gamma_k c_k
\]
be a point in the 2-cell. Then
\[
[p] = \sum_{i=1}^{m} (\alpha_i - s_i \beta_1 - s_i \gamma_1) [a_i] + \sum_{j=2}^{n} (\beta_j - t_j \beta_1) [b_j] + \sum_{k=2}^{l} (\gamma_k - r_k \gamma_1) [c_k].
\]

Let 
\[
x = \sum_{i=1}^{m} u_i [a_i] + \sum_{j=2}^{n} v_j [b_j] + \sum_{k=2}^{l} w_k [c_k].
\]
Comparing the coefficients with \( x, \) we see
\[
\begin{align*}
\alpha_i - s_i \beta_1 - s_i \gamma_1 &= u_i, \quad i = 1, \cdots, m, \\
\beta_j - t_j \beta_1 &= v_j, \quad j = 2, \cdots, n, \\
\gamma_k - r_k \gamma_1 &= w_k, \quad k = 2, \cdots, l.
\end{align*}
\]
When \( i = 1, \) we get \( \alpha_1 + \beta_1 + \gamma_1 = u_1. \) We can rewrite the first set of equations as
\[
\alpha_i + s_i \alpha_1 = u_i + s_i u_1, \quad i = 2, \cdots, m.
\]
Similar to Case 1, all coefficients are determined by \( \alpha_1, \beta_1, \gamma_1. \) The other equations may provide cutoff for \( \alpha_1, \beta_1, \gamma_1 \) individually. Thus we can only get \( n \)-gons up to \( n = 6. \)

With this information, it is not hard to construct examples to realize all of these possibilities. \( \square \)