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A geometric invariant of discrete groups

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1. Introduction

We shall associate to each finitely generated group G a sphere $S(G) \cong S^{n-1}$, where $n = \operatorname{rank}(G/G')$ is the $\mathbb{Z}$-rank of the abelianization of G, and a subset $\Sigma \subseteq S(G)$ which, among other things, captures information about the finite generation of kernels of abelian quotients of G. More generally, to every finitely generated G-operator group A on which G' acts by inner automorphisms we associate a subset $\Sigma_A \subseteq S(G)$. The set Σ is the special case $\Sigma = \Sigma_{G'}$, with G acting on G' by conjugation. These sets Σ and Σ_A have previously been introduced for metabelian G and abelian A by Bieri and Strebel [B-S 1] (see also [B-S 2]), while Σ is closely related to the set $fg(G)$ of all finitely generated kernels of infinite cyclic quotients of G, studied by Neumann [N]. When G is a three-manifold fundamental group, Thurston defined a subset of $S(G)$ in [T], which we describe later and show to equal Σ.

The definitions are as follows. The set $\operatorname{Hom}(G, \mathbb{R})$ of homomorphisms of G to the additive group of the reals is an n-dimensional real vector space. The positive reals $\mathbb{R}_+$ act on $\operatorname{Hom}(G, \mathbb{R}) - \{0\}$ by multiplication and $S(G)$ is defined to be orbit space

$$S(G) = (\operatorname{Hom}(G, \mathbb{R}) - \{0\})/\mathbb{R}_+.$$

This is an $(n-1)$-sphere with the quotient topology inherited from the usual topology on $\operatorname{Hom}(G, \mathbb{R}) - \{0\}$. Thus an element of $S(G)$ is a ray $[\chi] = \{r\chi \mid r \in \mathbb{R}_+\}$. A non-trivial homomorphism $\chi: G \to \mathbb{R}$ with discrete (and hence infinite cyclic) image is said to be a *discrete* or *rank one* homomorphism. It represents a rational point of $S(G)$. The set

$$S\mathbb{Q}(G) = \{[\chi] \in S(G) \mid \chi \text{ is rank one}\}$$

of rational points is dense in $S(G)$.

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Each point $[\chi] \in S(G)$ gives rise to a submonoid

$$G_\chi = \chi^{-1}([0, \infty)) = \{g \in G \mid \chi(g) \geqq 0\}$$

of G. We define

$$\Sigma_A = \{[\chi] \in S(G) \mid A \text{ is finitely generated over some finitely generated submonoid of } G_\chi\}.$$

We shall give more practical "equational" criteria for $[\chi]$ to be in Σ_A in Sect. 2. As a direct consequence we obtain:

Theorem A. *Σ_A is an open subset of $S(G)$.*

Note that if, in the definition of Σ_A, $G_\chi = \chi^{-1}([0, \infty))$ were replaced by $\chi^{-1}((0, \infty))$, this theorem would be easy, since a finite subset of $\chi^{-1}((0, \infty))$ is also a subset of $\chi_1^{-1}((0, \infty))$ for any nearby χ_1. It will follow from the proof of Theorem A that this is in fact a valid alternate definition of Σ_A.

To describe the relation of Σ_A to finite generation we need the following definition. For any subgroup $H \leqq G$ let

$$S(G, H) = \{[\chi] \in S(G) \mid \chi(H) = 0\}.$$

This is a rationally defined great subsphere of $S(G)$ of codimension equal to rank (HG'/G').

Theorem B. *If H is a finitely generated subgroup of G then A is finitely generated as an H-group if and only if $S(G, H) \subseteq \Sigma_A$.*

Specializing to the case $A = G$, we obtain a criterion for finite generation of kernels of abelian quotients of G:

Theorem B1. *Let N be a normal subgroup of G with G/N abelian. Then N is finitely generated if and only if $S(G, N) \subseteq \Sigma$ (recall $\Sigma = \Sigma_{G'}$). In particular, G' is finitely generated if and only if $\Sigma = S(G)$.*

Indeed, we can choose a finitely generated subgroup $H \leqq G$ with $HG' = N$. Then N is finitely generated if and only if G' is finitely generated as an H-operator group, so Theorem B1 follows from Theorem B. Theorem B1 implies the main result of [N]: if N is finitely generated, then since Σ is open and $S(G, N)$ is compact, every normal subgroup N_1 for which $S(G, N_1)$ is sufficiently close to $S(G, N)$ will be finitely generated. Thus, *for any k, having finitely generated kernel is an "open condition" on epimorphisms of G to $\mathbb{Z}^k$.* It is worth remarking here that the problem, posed in [N], whether having finitely *presented* kernel is also an open condition, was answered affirmatively by D. Fried and R. Lee [F-L] (the papers [N] and [F-L] discuss the case $k = 1$, from which the general case follows easily).

The following theorem improves on a result of Bieri and Strebel [B-S 1, Theorem 3.1].

Theorem C. *If G is finitely presented and has no non-abelian free subgroups then*

$$\Sigma \cup -\Sigma = S(G)$$

where $-\Sigma$ is the image of Σ under the antipodal map.

Neither of the assumptions in this theorem is redundant, since if G is either free of free-metabelian of rank $\geqq 2$ then Σ is empty. However, as in [B−S 1], the condition that G be finitely presented can be weakened to the condition that G be of type $(FP)_2$ over some commutative ring with $1 \neq 0$.

Theorems A, B1, and C imply a curious result concerning the existence of finitely generated kernels of infinite cyclic quotients in finitely presented groups.

Theorem D. *Let G be a finitely presented group with no non-abelian free subgroups. Suppose also that* $\operatorname{rank}(G/G') \geqq 2$. *Then G contains a finitely generated normal subgroup $N \leqq G$ with infinite cyclic quotient G/N. In fact, every normal subgroup $L \leqq G$ with G/L free abelian of rank 2 is contained in such an N.*

Free-metabelian groups of rank $\geqq 2$, free groups of rank $\geqq 2$, and the group $G = \langle a, x \mid a^x = a^2 \rangle$ show that none of the assumptions on G is redundant in this theorem.

Proof of Theorem D. By Theorems A and C, Σ and $-\Sigma$ form an open cover of the connected space $S(G, L) \cong S^1$, so $\Sigma \cap -\Sigma$ has non-empty intersection with $S(G, L)$. Since $S(G, L)$ is rationally defined, its rational points are dense, so we can find a rational point $[\chi]$ in $(\Sigma \cap -\Sigma) \cap S(G, L)$. Then $N = \operatorname{Ker}(\chi)$ has infinite cyclic quotient and contains L. Also $\{[\chi], -[\chi]\}$ is the subsphere $S(G, N)$, so N is finitely generated by Theorem B1. (This "codimension one" case of Theorem B1 is much easier to prove than the general case; see Corollary 4.2.)

If G is the fundamental group of a compact connected smooth n-manifold Y then $\operatorname{Hom}(G, \mathbb{R}) = H^1(Y; \mathbb{R})$, so, using DeRham cohomology to compute $H^1(Y; R)$, we can define

$\Sigma(Y) = \{[\chi] \in S(G) \mid \chi$ can be represented by a no-where vanishing 1-form on Y whose restriction to ∂Y is also no-where vanishing$\}$.

The significance of $\Sigma(Y)$ lies in the observation of Tischler [Ti] that a rational point $[\chi]$ of $S(G)$ is in $\Sigma(Y)$ if and only if $\chi: G \to \mathbb{Z}$ can be represented by a smooth fibration $Y \to S^1$. It is quite easy to show that $\Sigma(Y) \subseteq \Sigma_{G'}$. We shall show:

Theorem E. *If Y is a smooth compact 3-manifold containing no fake cells then $\Sigma(Y)$ equals $\Sigma = \Sigma_{G'}$ except possibly if $\pi_1(Y) = \mathbb{Z} \oplus \mathbb{Z}/2$.*

The condition that Y have no fake cells is equivalent to the condition that no counterexample to the Poincaré conjecture occur among the prime summands of Y (this is a necessary technicality until the Poincaré conjecture is resolved: taking connected sum of Y with a counterexample would not affect G or Σ but would make $\Sigma(Y)$ empty). The exception $\mathbb{Z} \oplus \mathbb{Z}/2$ is because of the possible existence of an exotic homotopy $\mathbb{R}P^2 \times S^1$, which would also contradict the Poincaré conjecture.

Corollary F. *If G is the fundamental group of a compact 3-manifold then Σ satisfies $\Sigma = -\Sigma$ and is a disjoint union of finitely many open convex rational polyhedra in $S(G)$. (A convex rational polyhedron in $S(G)$ means an intersection of finitely many rationally defined hemispheres.)*

Indeed, that $\Sigma(Y) = -\Sigma(Y)$ is obvious while the polyhedral property was proved for $\Sigma(Y)$ by Thurston [T, Theorem 5]. He also shows that *any subset $\Sigma \subseteq S^{n-1}$ satisfying the conditions in Corollary F is the Σ of some 3-manifold.*

We do not have as strong results about the shape of Σ_A in general as in the above case, but in all cases in which we have computed Σ_A it is in fact a *polyhedral subset of* $S(G)$, that is, a finite union of finite intersections of open hemispheres. For many classes of groups it is even rationally polyhedral: a finite (not necessarily disjoint) union of convex rational polyhedra. This is true for instance if A is nilpotent (for then $\Sigma_A = \Sigma_{A/A'}$ and A/A' is a G/G'-module; and for abelian A and G the rational polyhedral property was shown by Bieri and Groves [B-G]).

If Σ_A is rationally polyhedral then the complement $\Sigma_A^c = S(G) - \Sigma_A$ is the closure of the subset of its rational points. This is a very useful property, which we had hoped would be general. However in Sect. 8 we give examples where it fails: $\Sigma^c = S(G) - \Sigma_{G'}$ consists of a pair of irrational points. These examples, which can be finitely presented, are certain groups of PL-homeomorphisms of the interval.

The proofs of Theorems C and E are based on the following topological interpretation of the invariant Σ for a finitely presented group G. Let Y be any connected finite CW-complex with fundamental group G and let $X \to Y$ be the universal abelian cover of Y. To any homomorphism $\chi: G \to \mathbb{R}$ we shall associate a continuous map $\chi': X \to \mathbb{R}$, well defined up to a bounded homotopy. Define

$$X_\chi = \chi'^{-1}([0, \infty)).$$

(Here is one description of χ'. Take a map of Y to the n-torus which induces an epimorphism of fundamental groups $G \to \mathbb{Z}^n$. Let $a: X \to \mathbb{R}^n$ be the induced map of universal abelian covers. This $\mathbb{R}^n$ can be functorially identified with $(G/G') \otimes_{\mathbb{Z}} \mathbb{R}$, so χ can be interpreted as a map $\mathbb{R}^n \to \mathbb{R}$; χ' is then the composition of this with the map a.)

Theorem G. *$[\chi] \in \Sigma$ if and only if the inclusion $X_\chi \subset X$ induces an epimorphism of fundamental groups. (If X_χ is not connected then use the component on which χ is unbounded, which exists by Lemma* 5.2.)

We actually prove a more general version, Theorem 5.1, which gives a similar interpretation of $\Sigma_{G'}$ for an arbitrary finitely generated group G. To sketch how Theorem C follows from Theorem G, suppose one has a χ such that neither $[\chi]$ nor $-[\chi]$ is in Σ. Assume that X_χ, $X_{-\chi}$, and $X_0 = X_\chi \cap X_{-\chi}$ are connected (this assumption is easily dispensed with). Then the Seifert-Van Kampen theorem expresses $G' = \pi_1(X)$ as a non-trivial amalgamated free product of the images in $\pi_1(X)$ of $\pi_1(X_\chi)$ and $\pi_1(X_{-\chi})$, amalgamated along the image of $\pi_1(X_0)$, and the existence of a non-abelian free subgroup in G' follows easily.

Another basic result of this paper is the following theorem describing the functorial nature of Σ_A. Let $f\colon H \to G$ be a homomorphism of finitely generated groups and let A be a G-operator group. Then A is also an H-group with H acting on A via f. We write $\Sigma_A(G)$ and $\Sigma_A^c(G) = S(G) - \Sigma_A(G)$ or $\Sigma_A(H)$ and $\Sigma_A^c(H) = S(H) - \Sigma_A(H)$ to distinguish the group of operators being considered. Let

$$f^*\colon S(G) - S(G, f(H)) \to S(H)$$

be the map $f^*[\chi] = [\chi \circ f]$.

Theorem H. *If A is finitely generated as an H-group then:*

$$\Sigma_A^c(G) \subset S(G) - S(G, f(H))$$

and

$$f^* \Sigma_A^c(G) = \Sigma_A^c(H).$$

The first equation of this theorem is immediate from the definition of Σ_A. It is second equation which is the real content of the theorem and is in fact technically the most difficult result of this paper. For metabelian G and abelian A a stronger result is available: by using the valuation theoretic description in [B-S 2] the set Σ_A^c can be defined for any (not necessarily finitely generated) module A over the group ring of G and the second equation of Theorem H still holds. The proof is given in [B-S 3] for rational points of Σ_A^c and this implies the general result, since Σ_A^c in a rational polyhedron [B-G].

Here is a brief plan of the remainder of the paper. In Sect. 2 we discuss the "equational definition" of Σ_A; Theorem A (openness) is an immediate consequence. A related set Σ'_A is also briefly described, and its use for certain questions of finite generation, particularly finite normal generation, is discussed. Sections 3 and 4 describe some basic properties of the invariant, including useful characterizations of rational points in Σ_A in Sect. 4. In Sect. 5, the geometric definition of Σ (Theorem G) is deduced from an earlier characterization (Proposition 3.4) of Σ_N for a normal subgroup satisfying $G' \leqq N \leqq G$, and Theorems C and E are then derived from the geometric definition. In Sect. 6 we prove the functoriality theorem and use it to prove Theorem B.

Finally, Sect. 7 describes miscellaneous examples while Sect. 8 describes the *PL* homeomorphism group examples already mentioned. The basic result of Sect. 8 is that for "sufficiently general" finitely generated subgroups G of the group of orientation preserving *PL*-homeomorphisms of the interval $[0, 1]$, Σ^c consists of a pair of (in general irrational) points. As a corollary one obtains many subgroups of G which are finitely generated but not finitely presented.

Since the first draft of this paper there have been some further developments.

In [B] Kenneth S. Brown gives a description of $\Sigma_{G'}$ in terms of generalized ascending HNN-extensions. This relates Σ to actions of G on $\mathbb{R}$-trees. His results suggested to us another characterization of Σ_A^c which we have added as a final Sect. 9. This characterization could be used instead of the equational characterization (Proposition 2.1) at a couple of points in the paper with some conceptual advantage (but no significant shortening). For this reason, the reader may wish to read Sect. 9 directly after Sect. 2.

Gilbert Levitt has pointed out that if G is the fundamental group of a smooth closed orientable manifold M, then $\Sigma \cap -\Sigma$ is the set of χ represented by a "complete" 1-form on M, in the sense of [L]; moreover, he can give a similar description of Σ itself. He also brought to our attention an elegant homological characterization of Σ for a finitely presented group G, due to Jean-Claude Sikorav, one version of which follows (this characterization can also be deduced from Theorem G). Let $\mathbb{Z}[G]^r \to \mathbb{Z}[G]^g \to \mathbb{Z}[G]$ be the exact sequence associated to a presentation of G and let Λ_χ be the ring of formal series $\sum_{g\in G} n_g \cdot g$, $n_g \in \mathbb{Z}$, having only finitely many non-zero coefficients n_g with $\chi(g) < c$ for any c. Then $[\chi] \in \Sigma_{G'}$ if and only if the above sequence remains exact on tensoring with Λ_χ over $\mathbb{Z}[G]$.

Notation. Throughout the paper G will be a finitely generated group and A will be a finitely generated right G-group. The action of G on A will be written in exponential form: a^g is the result of applying the operator $g \in G$ to $a \in A$. We abbreviate $(a^{-1})^g$ to a^{-g}.

We shall always assume, without specific mention, that *the commutator subgroup G' of G acts on A by inner automorphisms.* We do not know to what extent this condition is needed for the results of this paper. Most of our arguments use it essentially, though some minor results clearly do not use it. In fact, with this assumption, *the invariant Σ_A only depends on the image of G/G' in the outer automorphism group* Out(A) (c.f. Lemma 3.1), so we could have formulated our investigations in terms of abelian subgroups of Out(A). Although such a formulation is maybe better conceptually, it would have led to extra notational complexity, so we did not do it.

If $\mathfrak{X}$ and $\mathfrak{R}$ are subsets of G and A then $\mathfrak{R}^{\mathfrak{X}}$ denotes the set of elements a^x with $a \in \mathfrak{R}$ and $x \in \mathfrak{X}$, $\mathfrak{X}^{-1}$ denotes the set of x^{-1} with $x \in \mathfrak{X}$, $\mathfrak{X}^{\pm 1}$ denotes $\mathfrak{X} \cup \mathfrak{X}^{-1}$, etc. Finally, for a subset $\mathfrak{B}$ of a group B, $\langle \mathfrak{B} \rangle$ and $\langle \mathfrak{B}]$ respectively denote the subgroup and submonoid generated by $\mathfrak{B}$.

2. Equational definition of Σ_A

If $\mathfrak{X}$ is a subset of the group G and $w = x_k x_{k-1} \ldots x_2 x_1$ is a word in the alphabet $\mathfrak{X}^{\pm 1} = \mathfrak{X} \cup \mathfrak{X}^{-1}$, then we will often use the same symbol for the word w and its value in G. The set of non-empty terminal segments

$$\{x_1, x_2 x_1, \ldots, x_k \ldots x_2 x_1\}$$

will be called the *trace* of w. It is empty for the empty word, which represents the identity in G. If $x: G \to \mathbb{R}$ is a homomorphism then the set of real numbers

$$\{\chi(x_1), \chi(x_2 x_1), \ldots, \chi(x_k \ldots x_2 x_1)\}$$

is called the *χ-track* of w. It is said to be *positive (non-negative)* if it is a set of positive (resp. non-negative) real numbers.

The following condition on finite subsets $\mathfrak{X} \subset G$ and $\mathfrak{R} \subset A$ will be very useful to commute the action of operators from $\mathfrak{X}$; for given $\mathfrak{X}$ it can always be achieved by adding finitely many elements to $\mathfrak{R}$ if necessary:

Condition C. *Each commutator $[x, y]$ of elements $x, y \in \mathfrak{X}^{\pm 1}$ acts on A by conjugation by some element $c(x, y) \in \langle \mathfrak{R} \rangle$.*

The following proposition gives an equational criterion for inclusion in Σ_A. A rather sharper result will be given Proposition 2.3, but this version suffices for most purposes.

Proposition 2.1. *Let $\mathfrak{X}$ and $\mathfrak{R}$ be finite generating sets of the group G and the G-group A respectively, such that $\mathfrak{X}$ and $\mathfrak{R}$ satisfy condition C. Then the following conditions are equivalent:*

(i) $[\chi] \in \Sigma_A$

(ii) *Each $a \in A$ has an expression*

$$a = r_1^{w_1} \dots r_f^{w_f} \tag{2.1}$$

with $r_1, \dots, r_f \in \langle \mathfrak{R} \rangle$ and where the w_i are $\mathfrak{X}^{\pm 1}$-words with positive nonempty χ-tracks.

(iii) *For each $r \in \mathfrak{R}$ and $x \in \mathfrak{X}^{\pm 1}$, $a = r^x$ has an expression as in* (ii).

(iv) *Same as* (iii), *but with just non-negative tracks for the w_i.*

Proof. Clearly (ii)$\Rightarrow$(iii)$\Rightarrow$(iv). To show that (iv)$\Rightarrow$(i), suppose we have an expression (2.1) for each r^x and let $\mathfrak{W}$ be the union of the traces of the w_i occurring in these finitely many equations. Note that any element u of the monoid $\langle \mathfrak{W}]$ is an $\mathfrak{X}^{\pm 1}$-word with trace contained in $\langle \mathfrak{W}]$. We shall show that $[\chi] \in \Sigma_A$ by showing that the subgroup $A_0 = \langle \mathfrak{R}^{\langle \mathfrak{W}]} \rangle$, generated by $\mathfrak{R}$ with operators in $\mathfrak{W}$, is stable under $\mathfrak{X}^{\pm 1}$ and hence equals A. It suffices to show, for $r \in \mathfrak{R}$ and $u \in \langle \mathfrak{W}]$ and $x \in \mathfrak{X}^{\pm 1}$, that r^{ux} is in A_0. We use for this the following Eq. (2.2) to pull x to the left in the exponent until it can be eliminated by use of the expression (2.1) for $a = r^x$.

For $\mathfrak{X}^{\pm 1}$-words v and w and $x, y \in \mathfrak{X}^{\pm 1}$:

$$r^{wyxv} = c^{-v} r^{wxyv} c^v, \tag{2.2}$$

where $c = c(x, y)^{-1}$ is the element of $\langle \mathfrak{R} \rangle$ given by Condition C.

Finally, to show (i)$\Rightarrow$(ii), assume that $[\chi] \in \Sigma_A$. Then we can find finite sets $\mathfrak{S} \subset A$ and $\mathfrak{Y} \subset G_\chi$ such that $\mathfrak{S}$ generates A using operators in $\mathfrak{Y}$. For each $s \in \mathfrak{S}$ choose a representation

$$s = r_1^{u_1} \dots r_f^{u_f} \tag{2.3}$$

with $r_i \in \mathfrak{R}$ and $\mathfrak{X}^{\pm 1}$-words u_i. Since this is a finite system of equations, there is a uniform lower bound $\alpha \in \mathbb{Z}$ on the tracks of the u_i. For each $y \in \mathfrak{Y}$ choose a representation of y as an $\mathfrak{X}^{\pm 1}$-word $w(y)$:

$$y = w(y). \tag{2.4}$$

We can assume that α is also a lower bound for the tracks of the $w(y)$. Now, any $b \in A$ has a representation

$$b = s_1^{v_1} \dots s_k^{v_k} \tag{2.5}$$

with $s_i \in \mathfrak{S}$ and the v_i being $\mathfrak{Y}$-words. If we use Eq. (2.4) to substitute in the exponents of (2.5) and (2.3) to substitute for the s_i we get a new equation

$$b = r_1'^{w_1} \ldots r_2'^{w_1}, \tag{2.6}$$

where the r_i' are in $\mathfrak{R}$ and the w_i are $\mathfrak{X}^{\pm 1}$-words with tracks bounded below by α (indeed, each w_i is a product of the form $u w(y_1) \ldots w(y_h)$ with u an exponent from an Eq. (2.3) and the $w(y_i)$ as in (2.4)). Choose $x \in \mathfrak{X}^{\pm 1}$ with $\chi(x) > 0$ and $\beta \in \mathbb{N}$ with $\beta \chi(x) + \alpha > 0$ and set $y = x^{-\beta}$. If, for any $a \in A$ one represents $b = a^y$ as in (2.6) and then applies x^β to this equation, one obtains the desired representation (2.1) of a.

Proof of Theorem A. Theorem A (openness) follows immediately from the above proposition since condition (iii) is clearly an open condition on $[\chi]$. The remark following Theorem A that Σ_A could also have been defined using $\chi^{-1}(0, \infty)$ in place of $G_\chi = \chi^{-1}[0, \infty)$ is also clear by applying the above proof that (iv)$\Rightarrow$(i) to (iii) instead of (iv).

We have seen the use of Property C for commuting the action of operators. The following simple lemma gives a concrete aspect of this which we will need later. We include it here for convenient reference.

Lemma 2.2. *Suppose $\mathfrak{X} \subset G$ and $\mathfrak{R} \subset A$ satisfy Property C and $\mathfrak{U}$ and $\mathfrak{V}$ are sets of $\mathfrak{X}^{\pm 1}$-words which are closed under formation of terminal segments of words (e.g. subsets of $\mathfrak{X}^{\pm 1}$). Then $\langle \mathfrak{R}^{\mathfrak{U}\mathfrak{V}} \rangle = \langle \mathfrak{R}^{\mathfrak{V}\mathfrak{U}} \rangle$ and $\langle \mathfrak{R}^{\langle \mathfrak{U} \cup \mathfrak{V}]} \rangle = \langle \mathfrak{R}^{\langle \mathfrak{U}] \langle \mathfrak{V}]} \rangle$.*

Proof. Given $u \in \mathfrak{U}$ and $v \in \mathfrak{V}$, to show $r^{uv} \in \langle \mathfrak{R}^{\mathfrak{V}\mathfrak{U}} \rangle$ we can assume inductively that this has been shown already for shorter words than uv and then use Eq. (2.2) to pull v letter by letter to the left in the exponent. For the second equation it suffices to show that $\langle \mathfrak{R}^{\langle \mathfrak{U}] \langle \mathfrak{V}]} \rangle$ is stable under both $\mathfrak{U}$ and $\mathfrak{V}$, which follows by applying the first equation to $\langle \mathfrak{U}]$ and $\langle \mathfrak{V}]$.

We close this section with two digressions. The first is a refinement of the "equational definition" of Σ_A (Proposition 2.1). It is rarely needed in practice; it can simplify computations for examples, but we include it mainly for its elegance. The second digression discusses a set Σ_A' related to Σ_A.

Proposition 2.3. *Let $\mathfrak{X}$ and $\mathfrak{R}$ be finite generating sets of the group G and the G-group A respectively, such that $\mathfrak{X}$ and $\mathfrak{R}$ satisfy condition C (or just the following weaker Condition $C(\chi)$). Then $[\chi] \in \Sigma_A$ if and only if each $r \in \mathfrak{R}$ has an expression*

$$r = r_1^{w_1} \ldots r_f^{w_f} \tag{2.7}$$

with $r_1, \ldots, r_f \in \langle \mathfrak{R} \rangle$ and where the w_i are $\mathfrak{X}^{\pm 1}$-words with non-negative χ-track and $\chi(w_i) > 0$.

Condition $C(\chi)$. *For each $x \in \mathfrak{X}^{\pm 1}$ with $\chi(x) < 0$ and each $y \in \mathfrak{X}$, one of the commutators $[x, y]$ and $[x, y^{-1}]$ acts on A by conjugation by some element $c \in \langle \mathfrak{R} \rangle$.*

Proof. We first assume that $\mathfrak{X}$ and $\mathfrak{R}$ satisfy condition C. Clearly any one of the conditions (ii) to (iv) of Proposition 2.1 implies the condition of Proposition 2.3. Assume the condition of Proposition 2.3; we shall deduce condition (iv) of Proposition 2.1. If we take the finite system of Eq. (2.7) and substitute it into itself, we obtain a new such system where each exponent u is a product $u=vw$ of two exponents v and w from the original system. By iterating this process we may thus make the minimum of $\chi(w_i)$, taken over all exponents in the system, as large as we wish. We may assume therefore that for each Eq. (2.7) we have:

$$\chi(w_i) \geqq \chi(x) \qquad \text{for each } w_i \text{ and each } x \in \mathfrak{X}^{\pm 1}.$$

Choose $x \in \mathfrak{X}^{\pm 1}$ and $r \in \mathfrak{R}$ and rewrite each exponent in Eq. (2.7) as $x x^{-1} w_i$. Then use the following Eq. (2.8) to pull the x^{-1} to the right in the exponents until eventually one has a new equation in which each exponent is either of the form $x w_i x^{-1}$ for some exponent w_i of the original equation or of the form $u x^{-1}$ where u is a terminal segment of one of the w_i. Applying x to this new equation gives us an equation for r^x as in part (iv) of Proposition 2.1, completing the proof.

For $\mathfrak{X}^{\pm 1}$-words w and v and $x, y \in \mathfrak{X}^{\pm 1}$:

$$r^{wx^{-1}yv} = c^{-x^{-1}v} r^{wyx^{-1}v} c^{x^{-1}v}. \tag{2.8}$$

where $c = c(x, y)^{-1}$ is the element of $\langle \mathfrak{R} \rangle$ given by Condition C.

We now describe how the above proofs of both Propositions 2.1 and 2.3 must be modified if one only has the weaker condition $C(\chi)$. In the proof of Proposition 2.1, if one replaces $\mathfrak{W}$ by the monoid generated by $\mathfrak{W} \cup (\mathfrak{X}^{\pm 1} \cap G_\chi)$, then one only needs to move elements $x \in \mathfrak{X}^{\pm 1}$ with $\chi(x) < 0$ to the left in exponents, and one can use a choice of equation (2.2) and the following (2.9) to do so:

$$r^{wyxv} = c(x, y^{-1})^{-yv} r^{wxyv} c(x, y^{-1})^{yv}. \tag{2.9}$$

Similarly, in the proof of Proposition 2.3 one need only consider $x \in \mathfrak{X}^{\pm 1}$ with $\chi(x) < 0$ and one can use a choice of (2.8) and the following (2.10):

$$r^{wx^{-1}yv} = c(x, y^{-1})^{-yx^{-1}v} r^{wyx^{-1}v} c(x, y^{-1})^{yx^{-1}} v. \tag{2.10}$$

Comment on the definition of Σ_A. By definition, a point $[\chi]$ of $S(G)$ belongs to Σ_A if A is finitely generated over a *finitely generated* submonoid M of G_χ. If we dispense with the requirement that M be finitely generated we obtain the closely related invariant

$$\Sigma'_A = \{[\chi] \in S(G) \mid A \text{ is finitely generated over } G_\chi\}.$$

Obviously $\Sigma_A \subseteq \Sigma'_A$, but the two sets do not in general coincide, as later examples will show. However we have the following Proposition which shows, in particular, that the Σ_A defined in this paper is a true generalization of the one introduced in [B-S 1] for abelian A.

Proposition 2.4. (i) *Σ'_A is open in $S(G)$.*

(ii) *If the image of G' in* Aut(A) *is finitely generated (e.g. if G' is finitely generated or if A is abelian) then $\Sigma_A = \Sigma'_A$.*

Proof. Let $\mathfrak{R} \subset A$ be a finite G-generating set of A. We first show.

$[\chi] \in \Sigma'_A$ *if and only if the following condition* $(*)$ *holds:* $(*)$ *each* $r \in \mathfrak{R}$ *has an expression:*

$$r = r_1^{g_1} \dots r_f^{g_f} \tag{2.11}$$

with the r_i *in* $\langle \mathfrak{R} \rangle$ *and the* g_i *in* G *with* $\chi(g_i) > 0$.

This is clearly an open condition on $[\chi]$, so (i) then follows. To prove it, suppose $[\chi] \in \Sigma'_A$. Choose $h \in G$ with $\chi(h) < 0$. Any element of A, in particular the element r^h, has an expression

$$r^h = r_1^{h_1} \dots r_f^{h_f}$$

with the r_i in $\langle \mathfrak{R} \rangle$ and with $\chi(h_i) \geqq 0$. Applying h^{-1} to this gives the desired representation of r. Conversely, if the condition $(*)$ holds, let $h \in G$ be arbitrary. By substituting the system of Eq. (2.11) into itself sufficiently often we can replace it by one in which the exponents g_i all satisfy $\chi(g_i) + \chi(h) > 0$. Then applying h to this system shows that r^h is in $\langle \mathfrak{R}^{G_\chi} \rangle$ for each $r \in \mathfrak{R}$. Since h was arbitrary, $\langle \mathfrak{R}^{G_\chi} \rangle = A$.

To prove part (ii) of the Proposition let A_0 be a finitely generated subgroup of A such that each element of G' acts on A by conjugation by an element of A_0. Choose $\mathfrak{R}$ to include a generating set of A_0. Then Condition C is satisfied for $\mathfrak{R}$ and *any* set $\mathfrak{X} \subset G$. Thus if condition $(*)$ is satisfied we can choose $\mathfrak{X}$ to include the exponents occurring in (2.11) and the condition of Proposition 2.3 is then trivially satisfied. The converse is trivial.

The significance of Σ'_A. Most results for Σ_A have analogs for Σ'_A with finite generation with respect to a set of operators always replaced by finite generation with respect to the same operator set union G'. In particular, the analog of the functoriality theorem (Theorem H) holds for Σ'_A and the analogs of Theorems B and B1 are:

Proposition 2.5. *Suppose* $G' \leqq N \leqq G$. *Then* A *is finitely generated as an* N*-group if and only if* $S(G, N) \subseteq \Sigma'_A$. *In particular,* N *is finitely generated as an* N*-group (that is,* N *can be killed by finitely many relations), if and only if* $S(G, N) \subseteq \Sigma'_{G'}$.

We omit proofs; they parallel the corresponding proofs for Σ_A, but are much easier in that less care need be taken with commutation of operators. Alternatively, note that everything in this paper can be carried out with an additional group H of operators always acting on A if the images of H and G in Aut(A) satisfy $[\bar{H}, \bar{G}] \subseteq \bar{H}$. The case that $H = G'$ gives Σ'_A.

3. Elementary properties of Σ_A

In this section we discuss how Σ_A changes under change of A or G. A useful concept for this discussion will be the concept of a subset $\mathfrak{X}$ of G being *sufficient* for A, by which we shall mean that A is finitely generated as a group with operators $\mathfrak{X}$. The set $\Sigma_A \subset S(G)$ might be considered to be a first step in an analysis of the structure of the set of finite subsets of G which are sufficient for A.

Lemma 3.1. (i) *Let $N \leqq G$ be a normal subgroup which acts on A by inner automorphisms. Then whether a finite subset $\mathfrak{X} \subset G$ is sufficient for A only depends on the image of $\mathfrak{X}$ in G/N.*

(ii) *If $\mathfrak{X} = \mathfrak{X}_1 \cup \{x\}$ is sufficient for A then so is $\mathfrak{X}_1 \cup \{x^p\}$ for any $p \in \mathbb{N}$.*

Proof. (i) Suppose $\mathfrak{X}_1$ be sufficient for A and let $\mathfrak{R} \subset A$ be a finite $\mathfrak{X}_1$-generating set for A. If $\mathfrak{X} \subset G$ is a finite subset with the same image in G/N as $\mathfrak{X}_1$, then each element of $\mathfrak{X}_1$ can be written in the form $n(x)x$ with $n(x) \in N$ and $x \in \mathfrak{X}$. Moreover $n(x)$ acts on A by conjugation by some element $a(x)$. Clearly, anything one can generate from $\mathfrak{R}$ using operators in $\mathfrak{X}_1$ can also be generated from $\mathfrak{R} \cup \{a(x) \mid x \in \mathfrak{X}\}$ using operators in $\mathfrak{X}$.

(ii) Let $\mathfrak{R}$ be a finite $\mathfrak{X}$-generating set for A such that the pair $\mathfrak{X}$, $\mathfrak{R}$ satisfies Property C (Sect. 2). By Lemma 2.2, A is generated by elements of the form r^u with $r \in \mathfrak{R}$ and u an $\mathfrak{X}$-word of the form $x^k v$ with v an $\mathfrak{X}_1$-word. This r^u is in the subgroup generated by

$$\mathfrak{S} = \mathfrak{R} \cup \mathfrak{R}^x \cup \ldots \cup \mathfrak{R}^{x^{p-1}}$$

using operators in $(\mathfrak{X}_1 \cup \{x^p\})$, so $\mathfrak{S}$ is a $(\mathfrak{X}_1 \cup \{x^p\})$-generating set of A.

If a group A has different groups of operators G, G_1, etc., acting on it, recall that we write $\Sigma_A(G)$, $\Sigma_A(G_1)$, etc., to distinguish the various Σ_A's. The above lemma has the following immediate consequences for Σ_A.

Proposition 3.2. (i) *If $N \leqq G$ is a normal subgroup which acts on A by inner automorphisms then $S(G) - S(G, N) \subseteq \Sigma_A$.*

(ii) *If $G_1 \leqq G$ is a subgroup of finite index then*

$$[\chi \mid G_1] \in \Sigma_A(G_1) \Leftrightarrow [\chi] \in \Sigma_A(G).$$

Proof. (i) Choose a finite set $\mathfrak{X} \subset G$ which is sufficient for A. Then $[\chi] \in S(G) - S(G, N)$ means that $\chi \mid N \neq 0$, so we can find a finite set $\mathfrak{X}_1 \subset G_\chi$ such that $\mathfrak{X}$ and $\mathfrak{X}_1$ have the same image in G/N. Then by Lemma 3.1, $\mathfrak{X}_1$ is sufficient for A, so $[\chi] \in \Sigma_A$.

(ii) The implication "$\Rightarrow$" is trivial. For the converse, suppose that $\chi \in \Sigma_A(G)$. Let $\mathfrak{X} \subset G_\chi$ be a finite sufficient set for A and let p be the index $[G:G_1]$. Then, by Lemma 3.1 (ii), the set $\mathfrak{X}^p$ of p-th powers of elements of $\mathfrak{X}$ will still be sufficient for A. But $\mathfrak{X}^p \subset (G_1)_\chi$, so $[\chi \mid G_1] \in \Sigma_A(G_1)$.

In the following Proposition recall that the notation Σ_A^c represents the complement of Σ_A in $S(G)$. Part (ii) is a special case of the functoriality Theorem H.

Proposition 3.3. (i) *Suppose that A and B are finitely generated G- and H-groups respectively and $\pi: H \twoheadrightarrow G$ and $\rho: B \twoheadrightarrow A$ are epimorphisms compatible with the group actions. Then the induced embedding*

$$\pi^*: S(G) \rightarrowtail S(H),$$

with image $S(H, \operatorname{Ker} \pi)$, maps Σ_A^c into Σ_B^c.

(ii) *If ρ is an isomorphism in* (i) *then $\pi^*(\Sigma_A^c) = \Sigma_B^c$.*

Proofs. Part (i) can be reformulated: $[\chi\circ\pi]\in\Sigma_B\Rightarrow[\chi]\in\Sigma_A$. It is then immediate from the fact that if B is generated by $\mathfrak{R}$ using operators $\mathfrak{X}\subset H$ then A will be generated over $\pi(\mathfrak{X})$ by $\rho(\mathfrak{R})$. Given (i), the conclusion of part (ii) can be reformulated as the two statements:

(a) $\pi^*(\Sigma_A)\subseteq\Sigma_B$;

(b) $S(H)-\pi^*(S(G))\subseteq\Sigma_B$.

(a) is trivial, while (b) is Proposition 3.2 (i) with $N=\operatorname{Ker}\pi$.

We now discuss the case that $A=N$ is a normal subgroup of G, containing G', and acted on by conjugation. The set Σ_N is then completely determined by $\Sigma=\Sigma_{G'}$ and we can give a new characterization of it. Let $\mathfrak{X}$ be a finite generating set of the group G.

Proposition 3.4. *If* $G'\leqq N\leqq G$ *then:*

(i) $\Sigma_N=\Sigma_{G'}\cup(S(G)-S(G,N))$.

(ii) *Choose* $c\leqq 0$; *then* $[\chi]\in\Sigma_N\cap S(G,N)$ *if and only if each element of* N *can be represented by an* $\mathfrak{X}^{\pm1}$*-word with* χ*-track bounded below by* c.

Proof of (i). Since N/G is finitely generated, $\Sigma_N\supseteq\Sigma_{G'}$, and by Proposition 3.2 (i), $\Sigma_N\supseteq(S(G)-S(G,N))$. It remains to show $\Sigma_N\cap S(G,N)\subseteq\Sigma_{G'}$. If $[\chi]\in\Sigma_N\cap S(G,N)$, note that $N\subset G_\chi$. If $\mathfrak{X}\subset G_\chi$ is sufficient for N, the following lemma (with $A=G'$) shows that we can enlarge $\mathfrak{X}$ by a finite subset of N to make it sufficient for G'.

Lemma 3.5. *Let* $A\rightarrowtail N\twoheadrightarrow Q$ *be a short exact sequence of* $\mathfrak{X}$*-operator-groups, where* $\mathfrak{X}$ *is a finite set, such that* Q *is finitely presented as a group without operators.* $\mathfrak{Y}\subset N$ *is a finite set which generates* N *as an* $\mathfrak{X}$*-group and whose image in* Q *generates* Q *as a group. Then* A *is a finitely generated* $(\mathfrak{X}\cup\mathfrak{Y}^{\pm1})$*-group.*

Proof. If $\mathfrak{X}=\emptyset$ this lemma is standard. Namely, choose a finitely generated free group F with an epimorphism $F\twoheadrightarrow N$. If $R=\operatorname{Ker}(F\twoheadrightarrow N\twoheadrightarrow Q)$, one can form a commutative diagram with exact rows:

$$\begin{array}{ccccc} R & \rightarrowtail & F & \twoheadrightarrow & Q \\ \downarrow & & \downarrow & & \| \\ A & \rightarrowtail & N & \twoheadrightarrow & Q. \end{array}$$

Since Q is finitely presented, R is finitely generated as an F-group, so A is finitely generated as an N-group.

This same proof applies with operator set $\mathfrak{X}\neq\emptyset$ to show that A is finitely generated as an $(\mathfrak{X}\cup N)$-group; let $\mathfrak{R}\subset A$ be a finite generating set. We must show that A is finitely generated as an $(\mathfrak{X}\cup\mathfrak{Y}^{\pm1})$-group. Since the image of $\mathfrak{Y}$ generates Q as a group, for each $y\in\mathfrak{Y}^{\pm1}$ and $x\in\mathfrak{X}$ the element of Q represented by y^x can also be represented by a $\mathfrak{Y}^{\pm1}$-word $v(x,y)$. Thus in N we have $y^x=b(x,y)\,v(x,y)$ with $b(x,y)\in A$. Let A_0 be the $(\mathfrak{X}\cup\mathfrak{Y}^{\pm1})$-group generated by $\mathfrak{S}=\mathfrak{R}\cup\{b(x,y)\mid x\in\mathfrak{X},\,y\in\mathfrak{Y}^{\pm1}\}$; we claim that $A_0=A$. It suffices to show that $a^n\in A_0$ for any $a\in A_0$ and $n\in N$. We need only show this for n of the form y^w with $y\in\mathfrak{Y}^{\pm1}$ and $w=x_1\ldots x_k$ an $\mathfrak{X}$-word, since such elements n generate

N. We use induction on k. For $k=0$ we have $a^n = a^y \in A_0$. Assume a^n is in A_0 for any $a \in A_0$ and any $n = y^w$ with $y \in \mathfrak{Y}^{\pm 1}$ and w an $\mathfrak{X}$-word of length $k-1$. Then if $w = x_1 \dots x_k$ we have $y^w = b v^{x_2 \dots x_k}$ with $b = b(x_1, y)^{x_2 \dots x_k} \in A_0$ and $v = v(x_1, y)$ a $\mathfrak{Y}^{\pm 1}$-word. Now $a^n = b^{-m} a^m b^m$ with $m = v^{x_2 \dots x_k}$. By induction assumption, each of a^m and b^m is in A_0, so a^n is in A_0, completing the proof.

Proof of 3.4 (ii). Suppose $[\chi] \in \Sigma_N \cap S(G, N)$. Let $\mathfrak{R} \subset N$ be a finite set such that $\mathfrak{R}$ generates N as an $\mathfrak{X}$-group and $\mathfrak{X}$ and $\mathfrak{R}$ satisfy Property C. Choose a representation of each $r \in \mathfrak{R}^{\pm 1}$ as an $\mathfrak{X}^{\pm 1}$-word $u(r)$ and let α be a lower bound on the tracks of these words. By Proposition 2.1 (ii) any element $a \in N$ has a representation

$$a = r_1^{w_1} \dots r_f^{w_f}$$

with the r_i in $\mathfrak{R}^{\pm 1}$ and w_i being $\mathfrak{X}^{\pm 1}$-words with positive χ-track. Thus a can be represented by the $\mathfrak{X}^{\pm 1}$-word

$$w = w_1^{-1} u(r_1) w_1 \dots w_f^{-1} u(r_f) w_f$$

which has track bounded below by α. Choose an $\mathfrak{X}^{\pm 1}$-word v with non-negative track and $\chi(v) > \alpha$. Then $v^{-1} a v$ is represented by the word $v^{-1} w v$ with non-negative track. Since $v^{-1} a v$ is arbitrary in N, this suffices.

Conversely, suppose each element of N has a representation as in the proposition. By conjugating by a suitable word v as above, we may assume $c=0$. Then no element of N has negative χ-value, so certainly $[\chi] \in S(G, N)$. Let F be the free group on $\mathfrak{X}$ and let S be the preimage of N under the obvious epimorphism from F to G. Let $\mathfrak{S}$ be any finite subset S which generates S modulo $[F, F]$. We assume each $s \in \mathfrak{S}$ has non-positive χ-track; since χ vanishes on N, we may reorder s to achieve this. We shall show that condition (iv) of Proposition 2.1 holds for the finite set $\mathfrak{R} = \mathfrak{S} \cup [\mathfrak{X}^{\pm 1}, \mathfrak{X}^{\pm 1}]$.

We begin with a preliminary calculation. Suppose w is an $\mathfrak{X}^{\pm 1}$-word with non-negative χ-track which has exponent sum 0 with respect to each element of $\mathfrak{X}$. We claim that w is freely equal to a product of commutators

$$c_1^{v_1} c_2^{v_2} \dots c_f^{v_f} \qquad (*)$$

with each c_i in $[\mathfrak{X}^{\pm 1}, \mathfrak{X}^{\pm 1}]$ and each v_i being an $\mathfrak{X}^{\pm 1}$-word with non-negative χ-track. The proof will be by induction on the number m of letters in w. The claim is obvious if $m=0$. If $m>0$ we choose the rightmost letter y in w with $\chi(y) \leqq 0$ ("letter" means element of $\mathfrak{X}^{\pm 1}$) and choose also an occurrence of y^{-1} in w. Putting x equal to the rightmost of this y and y^{-1} lets us write w as a product

$$w = w_1 x^{-1} w_2 x w_3,$$

where the w_i are subwords of w and either $\chi(x) \leqq 0$ or w_2 has positive x-track. Then w is freely equal to $w_1 w_2 [w_2, x] w_3$ and hence to

$$w_1 w_2 w_3 [w_2, x]^{w_3},$$

and $w_1 w_2 w_3$ has non-negative χ-track and is shorter than w, so it has a representation of the desired form $(*)$ by induction assumption. On the other hand

$[w_2, x]^{w_3}$ has a representation of the form $(*)$, since, if w_2 is the word $x_k \dots x_2 x_1$, then

$$[w_2, x]^{w_3} = [x_k, x]^{x_{k-1} \dots x_1 w_3} \dots [x_2, x]^{x_1 w_3} [x_1, x]^{w_3}.$$

This proves the claim.

Now, given $r \in \mathfrak{R}$ and $x \in \mathfrak{X}^{\pm 1}$, there exists, by assumption, an $\mathfrak{X}^{\pm 1}$-word v with non-negative χ-track that represents r^x. Find an $\mathfrak{S}^{\pm 1}$-word $s_1 s_2 \dots s_m$ which equals v modulo $[F, F]$ and set $w = (s_1 \dots s_m)^{-1} v$. Then the above calculation applies to w, so r^x has a representation

$$r^x = s_1 s_2 \dots s_m c_1^{v_1} c_2^{v_2} \dots c_f^{v_f},$$

which has the form required by Proposition 2.1.

4. Rational points

For rational points we can give a different characterization of Σ_A. Recall that a *rational point* $[\chi] \in S\mathbb{Q}(G)$ is one for which the homomorphism $\chi\colon G \to \mathbb{R}$ is rank one, i.e. has infinite cyclic image. By multiplying χ be a positive real number, we may assume that the image of χ is $\mathbb{Z} \subset \mathbb{R}$.

Proposition 4.1. *Let $\chi\colon G \twoheadrightarrow \mathbb{Z} \subset \mathbb{R}$ be a rank one homomorphism and let $t \in G$ be an element with $\chi(t) = 1$. Then $[\chi] \in \Sigma_A$ if and only if there is a finitely generated subgroup $Y \subseteq \operatorname{Ker} \chi$ and a finitely generated Y-subgroup $B \leqq A$ such that*

$$\dots \subseteq B^{t^{-1}} \subseteq B \subseteq B^t \subseteq \dots \quad \text{and} \quad \bigcup_{j>0} B^{t^j} = A.$$

Proof. The "if" is trivial; to show "only if" assume $[\chi] \in \Sigma_A$. Let $\mathfrak{X} \subset G_\chi$ be sufficient for A. Since $G = \langle \operatorname{Ker} \chi, t \rangle$, we may assume $\mathfrak{X} = \mathfrak{Y}^{\pm 1} \cup \{t\}$ with $\mathfrak{Y} \subset \operatorname{Ker} \chi$. Let $\mathfrak{R}$ be a finite $\mathfrak{X}$-generating set of A such that $\mathfrak{X}$ and $\mathfrak{R}$ satisfy Property C (Sect. 2). Let $Y = \langle \mathfrak{Y} \rangle$ and $B_0 = \langle \mathfrak{R}^Y \rangle$. Then by Lemma 2.2, A is generated by $B_0^{\langle t \rangle}$, so, since $\mathfrak{R}$ is finite, there exists a $k \in \mathbb{N}$ such that $\mathfrak{R}^{t^{-1}}$ is contained in

$$B = \langle B_0 \cup B_0^t \cup \dots \cup B_0^{t^k} \rangle.$$

Lemma 2.2 with $\mathfrak{U} = \{1, t, \dots, t^k\}$ and $\mathfrak{B} = Y$ shows that B is generated as a Y-group by the finite set $\mathfrak{R}^{\mathfrak{U}}$. Also, since B is Y-stable, $\mathfrak{R}^{t^{-1} Y} \subset B$, but by Lemma 2.2 again, $\langle \mathfrak{R}^{t^{-1} Y} \rangle = B_0^{t^{-1}}$. Hence $B^{t^{-1}} \subseteq B$, so $B^{t^j} \subseteq B^{t^{j+1}}$ for all j and $\bigcup_{j>0} B^{t^j} = A$.

As an easy corollary we obtain the special case of Theorem B which we used in the proof of Theorem D:

Corollary 4.2. *If $[\chi]$ is a rational point and both $[\chi]$ and $-[\chi]$ are in Σ_A then A is finitely generated over a finitely generated subgroup of* $\operatorname{Ker} \chi$.

Proof. In the above proof of 4.1 we may assume that $\mathfrak{R}$ also generates A using operators in $\mathfrak{Y}^{\pm 1} \cup \{t^{-1}\}$. Then B_0 generates A as a $\{t^{-1}\}$-group, so B also does, which clearly implies that $B = A$.

Proposition 4.1 leads to characterizations of the rational points in $\Sigma=\Sigma_{G'}$ in terms of HNN-extensions.

Proposition 4.3. *If* $\chi\colon G\twoheadrightarrow\mathbb{Z}\subset\mathbb{R}$ *is a rank one homomorphism and* $N=\operatorname{Ker}\chi$ *and* $t\in G$ has $\chi(t)=1$ *then the following are equivalent:*

(i) $[\chi]\in\Sigma$;

(ii) *N is finitely generated over* $\langle t]$;

(iii) *G is an ascending HNN-extension* $G=\langle B,t;B_1^t=B\rangle$ *with finitely generated base group* $B\subseteq N$. *That is:*

$$\ldots\subseteq B\subseteq B^t\subseteq\ldots \quad \textit{and} \quad \bigcup_{j>0} B^{t^j}=N.$$

(iv) *If G is an ascending HNN-extension* $G=\langle C,s;C_1^s=C\rangle$ *with* $s=t^{-1}$ *and* $C\subseteq N$ *then this extension is trivial:* $C=N$.

Proof. By Proposition 3.4 $[\chi]\in\Sigma$ if and only if $[\chi]\in\Sigma_N$ and by Lemma 3.1, this is equivalent to (ii). To see the equivalence of (i) and (iii) one need just observe that in the proof of Proposition 4.1 applied to $A=\operatorname{Ker}\chi$ one can choose $\mathfrak{R}$ to include $\mathfrak{Y}$ and then B is finitely generated. Next, if (iii) holds and G is an HNN-extension as in (iv) then $B\subseteq C^{s^k}$ for some k so $B^{t^i}\subseteq C^{s^k t^i}=C^{s^{k-i}}\subseteq C^{s^k}$ for all $i\geqq 0$, so $C=N$. Finally if (iv) holds and $\mathfrak{X}\subset N$ is a finite $\langle t\rangle$-generating set, put $C=\langle\mathfrak{X}^{\langle t]}\rangle$. Then $\ldots\subseteq C\subseteq C^s\subseteq\ldots$ and $\bigcup C^{s^k}=N$, so by assumption $C=N$, that is (ii) holds.

There is an interesting partial converse to Proposition 4.3.

Proposition 4.4. *Let G be an HNN-extension* $\langle B,t;B_1^t=B_2\rangle$ *over a finitely generated base group B with associated subgroups* B_1, B_2 *and stable letter t. If* $\chi\colon G\twoheadrightarrow\mathbb{Z}\subset\mathbb{R}$ *is the homomorphism with* $\chi(B)=0$ *and* $\chi(t)=1$ *then* $[\chi]$ *is in* Σ *if and only if* $B=B_2$.

Proof. Assume $[\chi]\in\Sigma$. Then $N=\operatorname{Ker}\chi$ is finitely generated over $\langle t]$; let $\mathfrak{R}$ be a finite generating set. Since $B^{\langle t\rangle}$ generates N there exists m_0 such that $\mathfrak{R}$ is contained in

$$L_+=\langle B^{t^j}\,|\,j>m_0\rangle.$$

There is no loss in assuming $m_0=0$. Then $N=\langle\mathfrak{R}^{\langle t]}\rangle\subseteq\langle L_+^{\langle t]}\rangle=L_+$ so $N=L_+$. On the other hand, N is the amalgamated free product

$$L_- *_{B_2^{t^{-1}}=B_1} B *_{B_2=B_1^t} L_+,$$

where $L_-=\langle B^{t^j}\,|\,j\langle 0\rangle$, so $L_+=N$ implies $B_2=B$, as asserted.

5. Geometric interpretation of Σ

In this section we describe a geometric interpretation of Σ and use it to prove theorems C and E.

Choose an epimorphism $\psi\colon H\twoheadrightarrow G$ of a finitely presented group H to G and let Y be a finite connected CW-complex with fundamental group $\pi_1(Y)=H$.

Let $X \to Y$ be the covering corresponding to the subgroup $K = \psi^{-1}(G') \subset H$. It is a regular covering with covering transformation group $H/K = G/G'$. Given a homomorphism $\chi: G \to \mathbb{R}$, we get an induced action of G/G' on $\mathbb{R}$ by translations: $r^{[g]} = r + \chi(g)$.

Construction. *To any $\chi: G \to \Re$ we shall construct a continuous map $\chi': X \to \mathbb{R}$ which is compatible with the G/G' actions on X and $\mathbb{R}$ and is well defined up to a (G/G')-equivariant homotopy.*

We postpone the details of this construction to give its application. Let $X_\chi = \chi'^{-1}([0, \infty))$. This X_χ may not be connected, but we will see later (Lemma 5.2) that it has a unique component, which we call $X(\chi)$, on which χ' is unbounded.

Theorem 5.1. $[\chi] \in \Sigma_{G'}$ *if and only if* $\psi \circ i_{\#}: \pi_1(X(\chi)) \to G'$ *is an epimorphism, where* $i: X(\chi) \rightarrowtail X$ *is the inclusion.*

One way of constructing χ' was described in the introduction. We use here a different approach which is more convenient for our purposes. To describe it, we first describe the simple case that $G = H = \mathbb{Z}^n$ and Y is the n-torus T^n. Then X is the universal cover $\mathbb{R}^n$ of T^n. If χ_0 is a rank one homomorphism, $\chi_0: \mathbb{Z}^n \to \mathbb{Z}$, then we can represent it by a map $T^n \to S^1$ and lift this map to the universal covers $\mathbb{R}^n$ and $\mathbb{R}$ to obtain χ'_0. However, if χ_0 is not rank one, this construction needs modification. We can identify $\mathrm{Hom}(\mathbb{Z}^n, \mathbb{R}) = H^1(T^n; \mathbb{R})$. Using DeRham cohomology to compute $H^1(T^n; \mathbb{R})$, we can thus associate to χ_0 a closed 1-form ω on T^n. If we lift ω to $\mathbb{R}^n$ it becomes exact, that is, ω lifts to $d(\chi'_0)$ for some map $\chi'_0: \mathbb{R}^n \to \mathbb{R}$, which is the desired map.

The compatibility of χ'_0 with the $\mathbb{Z}^n$-actions on $\mathbb{R}^n$ and $\mathbb{R}$ follows from following the equation, in which $x \in \mathbb{R}^n$, $g \in \mathbb{Z}^n$ is a covering transformation, and γ_g is a closed loop in T^n representing the element g considered as an element of $\pi_1(T^n)$:

$$\chi'_0(x^g) - \chi'_0(x) = \int_x^{x^g} d\chi'_0 = \int_{\gamma_g} \omega = \chi(g).$$

Here the last equality is the definition of the identification $\mathrm{Hom}(\mathbb{Z}^n, \mathbb{R}) = H^1(T^n; \mathbb{R})$. The map χ'_0 depends on the choice of ω, but a different choice would differ from ω by an exact form dg on T^n, and a deformation of the real-valued function g on T^n to the constant function would induce a deformation between the two forms ω and hence a (G/G')-equivariant deformation between the two maps χ'_0.

In the general case, if G/G' has rank n, let $G \twoheadrightarrow \mathbb{Z}^n$ be an epimorphism and let $\varphi: H \twoheadrightarrow \mathbb{Z}^n$ be its composition with ψ. Since the n-torus T^n is a classifying space for $\mathbb{Z}^n$, we can find a continuous map $Y \to T^n$, unique up to homotopy, which realizes the homomorphism φ. This map lifts to the covers X and $\mathbb{R}^n$ of Y and T^n to give

$$\begin{array}{ccc} X & \xrightarrow{a} & \mathbb{R}^n \\ \downarrow & & \downarrow \\ Y & \longrightarrow & T^n, \end{array}$$

with $a: X \to \mathbb{R}^n$ unique up to equivariant homotopy. Any homomorphism $\chi: G \to \mathbb{R}$ factors over a homomorphism $\chi_0: \mathbb{Z}^n \to \mathbb{R}$ and, as above, this then induces a map $\chi'_0: \mathbb{R}^n \to \mathbb{R}$. The desired map χ' is the composition $\chi'_0 \circ a$.

Note that an equivariant homotopy of χ' is bounded, since the amount a point x of X moves under the homotopy only depends on the image of x in Y, which is compact. Thus, although the subspace $X_\chi \subset X$ depends on the choice of χ', any two such subspaces can be mapped inside each other by suitable covering transformations of X, and it follows that the same holds also for $X(\chi)$. Thus the epimorphism condition of Theorem 5.1 does not depend on these choices. We must still prove the existence of the unique component $X(\chi)$ of X_χ on which χ' is unbounded.

Lemma 5.2. *There exists a constant $C \geqq 0$ such that $\chi'^{-1}([r+C, \infty))$ is contained in a single component of $x'^{-1}([r, \infty))$ for any $r \in \mathbb{R}$. In particular, $X(\chi)$ exists, and it contains $\chi'^{-1}([C, \infty))$.*

Proof. For any path $\gamma: [0, 1] \to Y$, choose a lift $\bar{\gamma}: [0, 1] \to X$ of γ and define

$$|\gamma| = \max\{\chi' \bar{\gamma}(e) - \chi' \bar{\gamma}(t) \mid e = 0 \text{ or } 1, 0 \leqq t \leqq 1\};$$

this does not depend on the choice of lift $\bar{\gamma}$. Choose a basepoint $* \in Y$. For any $y \in Y$ define

$$|y| = \min\{|\gamma| \mid \gamma: [0, 1] \to Y \text{ is a path from } * \text{ to } y\}.$$

This is a continuous function on the compact space Y so it has a maximum

$$C_1 = \max\{|y| \mid y \in Y\}.$$

Let Γ be a finite set of closed loops $\gamma: [0, 1] \to Y$ based at $*$ which represent a generating set for $H = \pi_1(Y, *)$ and put

$$C_2 = \max\{|\gamma| \mid \gamma \in \Gamma\}.$$

Now any point $x \in X$ can be connected to a lift of the basepoint by a path whose χ'-image never goes below $\chi'(x) - C_1$, and any two lifts $*_1$ and $*_2$ of the basepoint can be connected by a path whose χ'-image never goes below $\min(\chi'(*_1), \chi'(*_2)) - C_2$. Thus $C = C_1 + C_2$ satisfies the Lemma.

Proof of Theorem 5.1. We show first that the condition of Theorem 5.1 only depends on the homotopy type of Y. Suppose Y' is a finite CW-complex homotopy equivalent to Y. Let X' and X'_χ be constructed analogously to X and X_χ. Let $h: X' \to X$ be a lift of a homotopy equivalence $Y' \to Y$. Then $a' = a \circ h$ can play the role for X' that a plays for X, so, assuming this choice of a', we have $h(X'_\chi) \subseteq X_\chi$ and hence $h(X'(\chi)) \subseteq X(\chi)$. Thus the condition of Theorem 5.1 for Y' implies the condition for Y. By symmetry, the reverse is also true.

By collapsing a maximal tree in the 1-skeleton of Y, we can replace Y by a homotopy equivalent complex with just one 0-cell. We therefore assume that Y has just one 0-cell, which we denote $*$. The 1-cells of Y then form a finite collection Γ of closed paths in Y which represent a generating set of $H = \pi_1(Y, *)$. Let $\mathfrak{X}$ be the image of this set of generators in G. Choose a lift of the basepoint

$* \in Y$ to a basepoint, also called $*$, in X. We may assume $\chi'(*)=0$. Any $\mathfrak{X}^{\pm 1}$-word w is represented by a cellular path in Y (obtained by concatenating the corresponding elements of Γ) and this path lifts to a cellular path γ_w in X starting at the basepoint $*$. Using the constant C_2 from the proof of Lemma 5.2 we have:

$$\min(\chi' \mid \gamma_w) \leqq \min(\chi\text{-track of } w) \leqq \min(\chi' \mid \gamma_w) + C_2.$$

Thus, if we apply a suitable covering transformation to X to move the χ-image of $X(\chi)$ down by at least C, then $* \in X(\chi)$ and every $\mathfrak{X}^{\pm 1}$-word w with non-negative χ-track is represented by a path in $X(\chi)$. The "only if" of Theorem 5.1 then follows directly from Proposition 3.4 (ii). The "if" would also follow if we knew that every element of $\pi_1(X(\chi), *)$ were represented by a cellular path. This would be so if $X(\chi)$ were a subcomplex of X, but this may not be true: $X(\chi)$ may include some incomplete cells of X. Let X_χ^0 be the largest subcomplex of X contained in X_χ. Then, since the diameters of cells of X_χ with respect to χ' are bounded, X_χ^0 contains some translate of X_χ by a covering transformation. Thus X_χ^0 has a unique component $X^0(\chi)$ on which χ' is unbounded, and this component contains some translate of $X(\chi)$. We can thus replace $X(\chi)$ by $X^0(\chi)$ without changing the epimorphism condition of the Theorem, and the proof is then complete.

Theorem G of the Introduction is immediate: if G itself is finitely presented then we can take $\pi_1(Y)=G$ so $\psi\colon H \twoheadrightarrow G$ is the identity map. A proof of Theorem C can now be completed just as in [B-S 1]. It was sketched in the Introduction; since it is short we give a precise version here for completeness. Let Y be chosen as above with $\pi_1(Y)=G$. Suppose we can find a χ such that neither $[\chi]$ nor $[-\chi]$ is in Σ. Then neither of the inclusions $X(\chi) \rightarrowtail X$ or $X(-\chi) \rightarrowtail X$ induces an epimorphism of fundamental groups. On the other hand, by replacing $X(-\chi)$ by a suitable translate we may assume that $X(-\chi) \cup X(\chi) = X$ and $X(-\chi) \cap X(\chi)$ is connected, so by the Seifert-Van Kampen theorem, $G' = \pi_1(X)$ is the free product of $N_- = \operatorname{Im}(\pi_1(X(-\chi)) \to \pi_1(X))$ and $N_+ = \operatorname{Im}(\pi_1(X(\chi)) \to \pi_1(X))$, amalgamated along $N_0 = \operatorname{Im}(\pi_1(X(-\chi) \cap X(\chi)) \to \pi_1(X))$. We claim N_0 has infinite index in each of N_- and N_+, whence G' has non-abelian free subgroups. Indeed, for each $c>0$ choose a translate $X(-\chi)_c$ of $X(-\chi)$ which contains $\chi'^{-1}((-\infty, c])$. Then any element of N_+ is in the image of $\pi_1(X(\chi) \cap X(-\chi)_c)$ for some sufficiently large c. Thus if N_0 had finite index in N_+, we could replace $X(-\chi)$ by $X(-\chi)_c$ with c large enough that the new $N_0 = \operatorname{Im}(\pi_1(X(\chi) \cap X(-\chi)_c) \to \pi_1(X))$ would contain a complete set of coset representatives of the old N_0 in N_+. It would hence equal N_+. So $\pi_1(X)$ would equal N_-, contradicting the fact that $[-\chi]$ is not in Σ.

We now finally come to the proof of Theorem E. Recall that Y is a compact connected smooth 3-manifold containing no fake cells with $\pi_1(Y)=G$ and $\Sigma(Y)$ is the set of $[\chi] \in \operatorname{Hom}(G, \mathbb{R}) = H^1(Y; \mathbb{R})$ which can be represented by a closed 1-form ω which is nowhere vanishing on Y and ∂Y. We first show: $\Sigma(Y) \subseteq \Sigma = \Sigma_{G'}$. We shall apply Theorem 5.1 using this Y and its universal abelian cover X. Let $[\chi] \in \Sigma(Y)$ and represent χ by a closed 1-form ω as above on Y. This form lifts to an exact form, df say, on X and, rather than going through T^n, we can use the real valued function $f\colon X \to \mathbb{R}$ directly to construct X_χ as $X_\chi = f^{-1}([0, \infty))$. Let ζ_0 be any vector field on Y, tangent to ∂Y, with $\omega(\zeta_0)=1$

and let ζ be its lift to X. ζ_0 is complete since Y is compact so ζ is complete. Moreover, by construction, ζ is a lift via f of the constant vector field d/dt on $\mathbb{R}$. For any $x \in X$ let $g(x) \in f^{-1}(0)$ be the intersection of the ζ-flow line through χ with $f^{-1}(0)$. Then $(f, g): X \to \mathbb{R} \times f^{-1}(0)$ is a diffeomorphism which maps X_χ to $[0, \infty) \times f^{-1}(0)$, so the inclusion $X_\chi \rightarrowtail X$ induces an isomorphism on fundamental groups, and $[\chi] \in \Sigma$ by Theorem 5.1.

As described in the Introduction, Thurston showed in [Th] that $\Sigma(Y)$ is a rational polyhedral subset of $S(G)$. In particular, rational points are dense in the complement of $\Sigma(Y)$. To finish the proof of Theorem E it therefore suffices to show that no rational point of Σ is in the complement of $\Sigma(Y)$, that is: $\Sigma \cap S\mathbb{Q}(G) \subset \Sigma(Y)$, where $S\mathbb{Q}(G)$ is the set of rational points of $S(G)$.

Let $[\chi] \in \Sigma$ be a rational point. We can take the representative $\chi \in \operatorname{Hom}(G, \mathbb{Z})$ to be an epimorphism. Now $\operatorname{Hom}(G, \mathbb{Z}) = [Y, S^1] = H^1(Y; \mathbb{Z})$. Let $(F, \partial F) \subset (Y, \partial Y)$ be a properly embedded incompressible surface dual to χ and having the least number of components among such surfaces (F being dual to χ is equivalent to $F = \varphi^{-1}(t)$ for some map $\varphi: Y \to S^1$ representing χ with regular value t). Cutting Y along F represents Y as a union of pieces Y_i pasted along components F_j of F. Thus $G = \pi_1(Y)$ is the fundamental group of a graph of groups $(\Gamma, \{\pi_1(Y_i)\}, \{\pi_1(F_j)\})$ and χ factors as $\pi_1(Y) \twoheadrightarrow \pi_1(\Gamma) \xrightarrow{\xi} \mathbb{Z}$. Since $[\chi] \in \Sigma_{G'}$, Lemma 3.3 (i) shows that $[\xi] \in \Sigma_{\pi_1(\Gamma)'}$. But $\pi_1(\Gamma)$ is free, so $\Sigma_{\pi_1(\Gamma)'}$ would be empty unless $\pi_1(\Gamma) = \mathbb{Z}$. Since χ was an epimorphism it follows that some component of F is already dual to χ and since F had as few components as possible, F is connected. Cutting Y along F gives a manifold Y_0 with two copies F_0 and F_1 of F in ∂Y, such that Y is obtained from Y_0 by identifying F_0 to F_1. By incompressibility of F, the fundamental groups of F_0 and F_1 inject into $\pi_1(Y_0)$. Hence $G = \pi_1(Y)$ is an HNN-extension $\langle \pi_1(Y_0), t \mid \pi_1(F_0)^t = \pi_1(F_1) \rangle$ with $\chi: G \to \mathbb{Z}$ equal to the homomorphism which is trivial on $\pi_1(Y_0)$ and maps t to 1. By Proposition 4.4 this HNN-extension must be ascending, that is, $\pi_1(Y_0)$ coincides with $\pi_1(F_1)$. By Stallings [St], this implies that (Y_0, F_1) is homeomorphic to $(F \times I, F \times \{1\})$ (except possibly if $f \cong \mathbb{R}P^2$, in which case we are in the excluded case $\pi_1(Y) = \mathbb{Z} \oplus \mathbb{Z}/2$). In particular, $F_2 := \partial Y_0 - \operatorname{int}(F_1)$ is homeomorphic to F_1. The inclusion $F_0 \subseteq F_2$ is an inclusion of mutually homeomorphic compact connected surfaces, and it induces an injection on π_1 since $\pi_1(F_2) = \pi_1(Y_0)$. Now $F_2 - \operatorname{int}(F_0)$ has no closed components since F_2 is connected and no disc components since $\pi_1(F_0) \to \pi_1(F_2)$ in injective, so it is a (possibly empty) union of components with non-positive euler characteristic. but it has euler characteristic zero since F_0 is homeomorphic to F_2, so its components have zero euler characteristic; that is, they are annuli or möbius bands. The occurrence of möbius bands would give F_0 more boundary components than F_2, so $F_2 - \operatorname{int}(F_0)$ is a union of annuli. It follows that $(Y_0, F_0 \cup F_1)$ is homeomorphic to $(F \times I, F \times \{0, 1\})$. Thus Y is homeomorphic, and hence diffeomorphic, to the total space of a fibration over S^1, and χ can be represented by the fibration map. Thus $[\chi] \in \Sigma(Y)$, completing the proof.

6. Functoriality

In this section we first show how Theorem B follows using Theorem H (functoriality of Σ_A) and then prove Theorem H.

Proof of Theorem B. Let $H \leqq G$ be a finitely generated subgroup. If the group A is finitely generated over H then certainly $S(G, H) \subseteq \Sigma_A(G)$, since $H \subseteq G_\chi$ for all $[\chi] = S(G, H)$.

To prove the converse suppose $S(G, H) \subseteq \Sigma_A(G)$. For any subgroup $K \leqq G$ let K^0 be the largest subgroup of K which acts on A by inner automorphisms. In particular $G' \leqq G^0$ and $K^0 = K \cap G^0$ for any K. We shall show that A is finitely generated over H by induction on $k = \mathrm{rank}_{\mathbb{Z}}(G/HG^0)$. If $k = 0$ then A is finitely generated over HG^0 by Proposition 3.2 (ii) and hence finitely generated over H by Lemma 3.1 (i). If $k \geqq 1$ choose a finitely generated N with $H \leqq N < G$ and $G/NG^0 = \mathbb{Z}$. Then $S(G, N) \subseteq S(G, H) \subseteq \Sigma_A(G)$, so A is finitely generated over a finite subset of NG^0 by Corollary 4.2, and hence over N by Lemma 3.1 (i). We claim:

(i) $S(N, H) \subseteq \Sigma_A(N)$;

(ii) $\mathrm{rank}_{\mathbb{Z}}(N/HN^0) = k - 1$;

so then the induction hypothesis applies to show A is finitely generated over H, as desired. Indeed for (i) consider the map

$$f^*: S(G) - S(G, N) \to S(N)$$

induced by the inclusion $N \subset G$. Now $(f^*)^{-1} S(N, H) \subseteq S(G, H)$, which is disjoint from $\Sigma_A^c(G)$ by assumption. Thus $S(N, H)$ is disjoint from $f^* \Sigma_A^c(G)$, but $f^* \Sigma_A^c(G) = \Sigma_A^c(N)$ by functoriality, so this proves (i). For (ii) note that $N/HN^0 \cong NG^0/HG^0$ which has free abelian rank $k-1$.

Proof of Theorem H (functoriality). Functoriality has already been proved for epimorphisms (Proposition 3.3), so, since any homomorphism is the composition of an epimorphism and an inclusion, it suffices to prove it for an inclusion $H \rightarrowtail G$. We may assume therefore that $H \leqq G$ is a finitely generated subgroup. We can also reduce to the case of an inclusion $H \leqq G$ with G/HG^0 finite or infinite cyclic: replace the inclusion $H \leqq G$ by a sequence of inclusions $H = H_1 \leqq H_2 \leqq \ldots \leqq H_m = G$ with $H_{i+1} G^0/H_i G^0$ finite or infinite cyclic for each i; since $H_{i+1} G^0/H_i G^0 \cong H_{i+1}/H_i(H_{i+1})^0$, each inclusion $H_i \leqq H_{i+1}$ is of the desired type. Now if G/HG^0 is finite then the functoriality statement follows easily from Proposition 3.2. We will assume therefore from now on that $G/HG^0 \cong \mathbb{Z}$.

The map $f^*: S(G) - S(G, H) \to S(H)$ induced by the inclusion $f: H \rightarrowtail G$ is given by $f^*[\chi] = [\chi \circ f] = [\chi | H]$. Proving the inclusion $f^* \Sigma_A^c(G) \subseteq \Sigma_A^c(H)$ is thus easy: if A is not finitely generated over a finite subset of G_χ then it is certainly not finitely generated over a finite subset of $H_{\chi|H} = G_\chi \cap H$. The reverse inclusion $f^* \Sigma_A^c(G) \supseteq \Sigma_A^c(H)$ is equivalent to the two statements:

(a) $S(H) - \mathrm{Im}(f^*) \subseteq \Sigma_A(H)$;

(b) if $[\chi_0] \in \mathrm{Im}(f^*)$ and $(f^*)^{-1} [\chi_0] \subseteq \Sigma_A(G)$ then $[\chi_0] \in \Sigma_A(H)$.

Statement (a) is a special case of Proposition 3.2 (i), since $\mathrm{Im}(f^*) = S(H, G' \cap H)$. The crux of the proof is thus to prove (b).

Statement (b) can be rewritten:

(b)′ If $[\bar{\chi}] \in \Sigma_A(G)$ for every $[\bar{\chi}] \in S(G)$ with $\bar{\chi} | H = \chi_0$, then $[\chi_0] \in \Sigma_A(H)$.

By Proposition 3.2, any $\bar{\chi}: G \to \mathbb{R}$ or $\chi_0: H \to \mathbb{R}$ which does not vanish on G^0 or H^0 is in $\Sigma_A(G)$ or $\Sigma_A(H)$ respectively, so in (b)′ we may restrict to homomorphisms $\bar{\chi}$ and χ_0 which do vanish on G^0 and H^0. Denote by φ and ϑ the projections

$$\varphi: G \to G/HG^0 \xrightarrow{\cong} \mathbb{Z} \quad \text{and} \quad \vartheta: G \to (G/G^0)/\text{Torsion} \xrightarrow{\cong} \mathbb{Z}^n.$$

Choose $t \in G$ with $\varphi(t) = 1$ and then choose $\mathfrak{X} = \{x_1, \ldots, x_{n-1}\} \subset H$ such that $\mathfrak{X} \cup \{t\}$ maps to a basis of $\mathbb{Z}^n$ under ϑ. By Lemma 3.1, we can replace H and G by the groups $\langle \mathfrak{X} \rangle$ and $\langle \mathfrak{X} \cup \{t\} \rangle$ without changing the truth or falsity of (b)′, so from now on we assume that $H = \langle \mathfrak{X} \rangle$ and $G = \langle \mathfrak{X} \cup \{t\} \rangle$. A is finitely generated over $\mathfrak{X}^{\pm 1}$, so choose a finite $\mathfrak{X}^{\pm 1}$-generating set $\mathfrak{S}$ of A with $[(\mathfrak{X} \cup \{t\})^{\pm 1}, (\mathfrak{X} \cup \{t\})^{\pm 1}] \subset \mathfrak{S}$.

As already remarked, we may assume that $\chi_0(H^0) = 0$. Since $H/H^0 = HG^0/G^0$, there is a unique extension of χ_0 to $\chi: HG^0 \to \mathbb{R}$ with $\chi(G^0) = 0$. Assume the premise of (b)′. Since $S(G, HG^0)$ is by assumption contained in $\Sigma_A(G)$, the set

$$D(\chi) := \{[\bar{\chi}] \in S(G) \mid \bar{\chi} \mid HG^0 = \chi\} \cup S(G, HG^0)$$

is contained $\Sigma_A(G)$. This set is a closed great half-circle with boundary $S(G, HG^0)$.

We shall think of $\mathbb{Z}^n$ as embedded in $\mathbb{R}^n$ and we endow $\mathbb{R}^n$ with the euclidean metric having $\vartheta(\mathfrak{X} \cup \{t\})$ as orthonormal basis. Thus ϑ is a map from G to $\mathbb{R}^n$. Any $\bar{\chi}: G \to \mathbb{R}$ factors as $\bar{\chi}_{\mathbb{R}} \circ \vartheta$ for a unique map $\bar{\chi}_{\mathbb{R}}: \mathbb{R}^n \to \mathbb{R}$. Let $(\mathbb{R}^n)_{\bar{\chi}}$ denote the half space on which $\bar{\chi}_{\mathbb{R}} \geqq 0$. Similarly χ induces a map $\chi_{\mathbb{R}}: \mathbb{R}^{n-1} \to \mathbb{R}$. By exchanging x_i and x_i^{-1} if necessary, we may assume that $\chi(x_i) \leqq 0$ for each i. Put $\chi_i = -\chi(x_i)$. By multiplying χ by a positive constant if necessary, we may assume that $\chi_1^2 + \ldots + \chi_{n-1}^2 = 1$, so $(\chi_1, \ldots, \chi_{n-1})$ is the point on the unit sphere in $\mathbb{R}^{n-1}$ with minimal $\chi_{\mathbb{R}}$-value.

For $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n$ we denote by x^m the lexicographically ordered word

$$x^m = x_1^{m_1} \ldots x_{n-1}^{m_{n-1}} t^{m_n}.$$

For subsets B, C, etc. of $\mathbb{R}^n$ we denote by $\mathfrak{B}$, $\mathfrak{C}$, etc. the set of x^m with $m \in B$, C, etc. In particular, $B(r)$ will denote the closed ball of radius r in $\mathbb{R}^n$, so $\mathfrak{B}(r)$ is the set of lexicographically ordered representative words for lattice points in $B(r)$. Finally, if $u = y_k y_{k-1} \cdots y_1$ is a $(\mathfrak{X} \cup \{t\})^{\pm 1}$-word, we call the subset of $\mathbb{R}^n$

$$\{\vartheta(y_1), \vartheta(y_2 y_1), \ldots, \vartheta(y_k \cdots y_2 y_1)\}$$

the *ϑ-track* of u.

Lemma 6.1. *If u is a $(\mathfrak{X} \cup \{t\})^{\pm 1}$-word with ϑ-track in $B(r)$, then $s^u \in \langle \mathfrak{S}^{\mathfrak{B}(r)} \rangle$ for any $s \in \mathfrak{S}$.*

Proof. If one assumes the Lemma proven for shorter words than u or words of the same length which are closer to lexicographic ordering, then the proof is an easy induction using Eq. (2.2) to reorder the exponent u.

For each $\bar{\chi} \in \Sigma_A(G)$ we can find a ball $B(\bar{\chi}) \subset \mathbb{R}^n$ which is tangent at $0 \in \mathbb{R}^n$ to the plane $\operatorname{Ker}(\bar{\chi}_{\mathbb{R}})$ and contained in the half-space $(\mathbb{R}^n)_{\bar{\chi}}$ and such that the corresponding set $\mathfrak{B}(\bar{\chi})$ of words satisfies:

$$\mathfrak{S}^{\mathfrak{X} x_1 \ldots x_{n-1}} \subseteq \langle \mathfrak{S}^{B(\bar{\chi})} \rangle. \qquad (*)$$

The minimal radius of $B(\bar{\chi})$ is a continuous function of $\bar{\chi}$, so it has an upper bound r say on the compact set $D(\chi)$. Let $B=B(2r)$ be the ball centered at the origin of radius $2r$ and $\mathfrak{B}$ the corresponding set of representative words. We shall show that $\mathfrak{X}$ and $\mathfrak{R}=\mathfrak{S}^{\mathfrak{B}}$ satisfy the criterion of Proposition 2.1 (iv) to complete the proof.

For any $s\in\mathfrak{S}$, $w\in\mathfrak{B}$, and $x\in\mathfrak{X}^{\pm 1}$ we must show that s^{wx} is in $A_0=\langle\langle\mathfrak{S}^{\mathfrak{B}}\rangle^{\mathfrak{P}}\rangle$, where $\mathfrak{P}$ is the set of $\mathfrak{X}^{\pm 1}$-words with non-negative χ-track. If $x\in\mathfrak{X}^{-1}$ then $\chi(x)\geqq 0$ and s^{wx} is obviously in A_0. So assume $x\in\mathfrak{X}$. Suppose we have shown that s^{xw} is in A_0 and that, inductively, $s^{xw'}$ is in A_0 for every final segment w' of w. Then we can use Eq. 2.2 to pull x across w to get the desired result that s^{wx} is in A_0.

To show that s^{xw} is in A_0 we proceed as follows. First we construct an element $v\in\mathfrak{B}$ having the same φ-value, α say, as w and whose χ-value is close to the minimum of χ on

$$(\varphi^{-1}(\alpha)\cap\mathfrak{B})\cdot t^{-\alpha}.$$

Specifically, let $y_i=(4r^2-\alpha^2)^{1/2}\chi_i$ for $i=1, \ldots, n-1$, and put $y=(y_1, \ldots, y_{n-1}, \alpha)$, so y is the point on the boundary of $\varphi^{-1}(\alpha)\cap\mathfrak{B}$ for which $(y_1, \ldots, y_{n-1})$ has least $\chi_{\mathbf{R}}$-value. Let $m_i=\operatorname{int}(y_i)$ for each i and put

$$v=x_1^{m_1}\ldots x_{n-1}^{m_{n-1}}t^{\alpha}$$

and

$$v_+=x_1\ldots x_{n-1}\,v.$$

From the fact that $\chi_{\mathbf{R}}(y_1, \ldots, y_{n-1})\leqq\chi(w't^{-\alpha})$ for all w' in $\varphi^{-1}(\alpha)\cap\mathfrak{B}$ and the construction of v and v_+ it follows that

$$\chi(w'v_+^{-1})\geqq 0 \qquad \text{for all } w'\in\varphi^{-1}(\alpha)\cap\mathfrak{B}.$$

We exploit this inequality as follows. Let p be the unique lexicographically ordered word with $\vartheta(w)=\vartheta(vp)$. Then p is an $\mathfrak{X}^{\pm 1}$-word and the inequality shows that $\chi(px_{n-1}^{-1}\ldots x_2^{-1}x_1^{-1})\geqq 0$. Now $px_{n-1}^{-1}\ldots x_2^{-1}x_1^{-1}$ may not have non-negative χ-track, but the letters making up p can be reordered so that the resulting word q satisfies:

$$q_-=qx_{n-1}^{-1}\ldots x_2^{-1}x_1^{-1} \qquad \text{has non-negative } \chi\text{-track.}$$

Of course w is still the lexicographically ordered representative of vq.

Our next aim is to prove that $s^{xv_+q_-}$ is in $A_0=\langle\langle\mathfrak{S}^{\mathfrak{B}}\rangle^{\mathfrak{P}}\rangle$. Since q_- has non-negative χ-track, it suffices to show that s^{xv_+} is in A_0. This follows from the following Lemma.

Lemma 6.2. *There exists a ball $B(\bar{\chi})$ as above, such that*

$$(m_1, \ldots, m_{n-1}, \alpha)+B(\bar{\chi})\subset B.$$

Indeed, granting this Lemma, we then have

$$s^{xv_+}=s^{xx_1\ldots x_{n-1}v}\in\mathfrak{S}^{\mathfrak{X}x_1\ldots x_{n-1}v}\subseteq\langle\mathfrak{S}^{\mathfrak{B}(\bar{\chi})}\rangle^{v}\subseteq\langle\mathfrak{S}^{\mathfrak{B}}\rangle,$$

where the first inclusion is by property $(*)$ of $B(\bar{\chi})$ and the second follows by using Lemma 6.1 to straighten exponents into lexicographic order, noting that Lemma 6.2 implies that the ϑ-track of any element uv with $u \in \mathfrak{B}(\bar{\chi})$ is in B.

The proof is then easily completed. Knowing that s^{xv_+} and hence $s^{xv_+q_-}$ is in A_0, use Eq. 2.2 to reorder the vq in the exponent $xv_+q_- = xx_1 \ldots x_{n-1} vqx_{n-1}^{-1} \ldots x_1^{-1}$ to lexicographic order by pulling letters off the left of q to the left. This introduces extra factors of the form $c^{v'q'}$ where v' and q' are final segments of v and of $q_- = qx_{n-1}^{-1} \ldots x_1^{-1}$. These factors are in $\langle \mathfrak{S}^{\mathfrak{B}} \rangle^{\mathfrak{P}} \subset A_0$, so

$$s^{xx_1 \ldots x_{n-1} w x_{n-1}^{-1} \ldots x_1^{-1}} \in A_0.$$

Now again use Eq. (2.2) to pull $x_{n-1}^{-1}, \ldots, x_1^{-1}$ one at a time through w to cancel the $x_1, \ldots, x_{n-1}$ on the left in the exponent. This introduces extra factors which are again in A_0, so $s^{xw} \in A_0$. As explained earlier, further applications of Eq. (2.2) then imply the desired result that $s^{wx} \in A_0$.

It thus only remains to prove Lemma 6.2. Let $\bar{\chi} = (4r^2 - \alpha^2)^{1/2} \chi + \alpha \varphi$ and let $B(\bar{\chi})$ be the ball of radius r and center $-(y_1/2, \ldots, y_{n-1}/2, \alpha/2)$. Then $B(\bar{\chi})$ is tangent at $0 \in \mathbb{R}^n$ to the plane $\operatorname{Ker}(\bar{\chi}_{\mathfrak{R}})$ and is contained in the half space $(\mathbb{R}^n)_{\bar{\chi}}$. The center of the ball $(m_1, \ldots, m_{n-1}, \alpha) + B(\bar{\chi})$ is at $(m_1 - y_1/2, \ldots, m_{n-1} - y_{n-1}/2, \alpha/2)$. Since $|m_i - y_i/2| \leqq y_i/2$ for each i, this point is within distance r of the origin, so the ball is in B as required.

7. Miscellaneous examples

Groups with many automorphisms. If G is a finitely generated group then $\operatorname{Aut}(G)$ acts on $S(G)$, taking $\Sigma^c = S(G) - \Sigma_{G'}$ to itself. Hence Σ^c is the closure of a union of orbits of the action of $\operatorname{Aut}(G)$ on $S(G)$, a fact which can considerably restrict the possibilities for Σ^c. In particular, if the orbits of $\operatorname{Aut}(G)$ on $S(G)$ are dense – for example if G is relatively free (i.e. of the form F/R with R a fully invariant subgroup of a free group F) – then Σ^c must be either empty or all of $S(G)$, so we have the following dichotomy:

Theorem 7.1. *In the above situation either:*
(i) *G' is finitely generated, or*
(ii) *no normal subgroup of G with infinite cyclic quotient is finitely generated and G either contains non-abelian free subgroups or G is not finitely presented.*

In particular, this Theorem implies the non finite presentability of many relatively free groups, see also [*S*, p. 269, Theorem 8].

Groups of defect at least 2. Recall that a finitely presented group has defect $\geqq 2$ if it has a presentation with at least two more generators than relations.

Theorem 7.2. *If G has defect at least 2 then $\Sigma_{G'} = \emptyset$.*

Corollary 7.3. *G contains non-abelian free subgroups.*

Proofs. The Corollary is immediate by Theorem C. For the Theorem, if $\chi: G \to \mathbb{Z} \subset \mathbb{R}$ is a rank one homomorphism with kernel N and if $t \in G$ has $\chi(t) = 1$

then, as in the proof of the proposition in [B-S 4], $A = N/N'N^p$ is the quotient of a free $\mathbb{Z}_p\langle t\rangle$-module of rank $m-1$ (m = number of generators of G) by a submodule generated by at most $m-2$ elements. Since $\mathbb{Z}_p\langle t\rangle$ is a PID this implies that A contains $\mathbb{Z}_p\langle t\rangle$ as a submodule. Using that A is noetherian it follows that neither $[\chi]$ nor $-[\chi]$ is in Σ_A. But by Propositions 3.3 and 3.4 (i), $\Sigma_{G'} \subseteq \Sigma_A$. Thus $\Sigma_{G'}$ contains no rational points, so, since it is open, it is empty.

Cartesian products. If A_1 and A_2 are finitely generated G_1- and G_2-operator groups respectively then $A_1 \times A_2$ is a finitely generated $G_1 \times G_2$-group. The projections $\pi_i\colon G_1 \times G_2 \to G_i$ induce disjoint embeddings

$$\pi_i^*\colon S(G_i) \to S(G_1 \times G_2).$$

Theorem 7.4. $\Sigma^c_{A_1\times A_2}(G_1 \times G_2) = \pi_1^* \Sigma^c_{A_1}(G_1) \cup \pi_2^* \Sigma^c_{A_2}(G_2)$.

Proof. It is immediate from the definition of Σ_A that

$$\Sigma_{A_1\times A_2}(G_1 \times G_2) = \Sigma_{A_1}(G_1 \times G_2) \cap \Sigma_{A_2}(G_1 \times G_2).$$

On the other hand, $\Sigma^c_{A_i}(G_1 \times G_2) = \pi_i^* \Sigma^c_{A_i}(G_1)$ for $i = 1, 2$ by functoriality (Theorem H; here we only need its easy special case, Proposition 3.3).

Corollary 7.5. *If G_1 and G_2 are finitely generated and $G_1' \times G_2' \leqq H \leqq G_1 \times G_2$ and $\pi_i^* H = G_i$ for $i = 1, 2$, then H is finitely generated. If, moreover, A_i is a finitely generated G_i-group for $i = 1, 2$, then $A_1 \times A_2$ is finitely generated over H, and $\Sigma^c_{A_1\times A_2}(H)$ is the union of $p_1^* \Sigma^c_{A_1}(G_1)$ and $p_2^* \Sigma^c_{A_2}(G_2)$, where $p_i = \pi_i | H$ for $i = 1, 2$.*

Proof. The conditions imply that $S(G_1 \times G_2, H)$ is disjoint from each $\pi_i^* S(G_i)$, so $S(G_1 \times G_2, H) \subseteq \Sigma_{A_1\times A_2}(G_1 \times G_2)$ by the above Theorem. Hence $A_1 \times A_2$ is finitely generated over a finitely generated subgroup of H by Theorem B. The fact that H itself is finitely generated is a special case of this: take $A_i = G_i'$ for $i = 1, 2$. Finally, the computation of $\Sigma^c_{A_1\times A_2}(H)$ is by Theorem 7.4 and functoriality (Theorem H).

This Corollary allows one to construct complicated subsets Σ^c of S^{n-1} which are realizable as a $\Sigma^c_A(G)$. For example, if two such subsets Σ^c_1 and Σ^c_2 have been realized, then their union can be realized (embedded in a possibly higher dimensional sphere: if $a_i\colon G_i \to \mathbb{Z}^n$ are the mod torsion abelianization then take H in the Corollary to be $\operatorname{Ker}(a_1 - a_2\colon G_1 \times G_2 \to \mathbb{Z}^n)$); also if Σ^c has been realized in S^{n-1} then the same Σ^c can be realized in a higher sphere S^{n+k-1} (replace G by $G \times \mathbb{Z}^k$). In particular, one can apply this to show:

Corollary 7.6. *Any closed rational polyhedron in S^{n-1} is realizable as $\Sigma^c_A(G) \subseteq S(G, A)$ for some G and $G' \leqq A \leqq G$.*

Proof sketch. Since a polyhedron can be built up out of simplices, it suffices to realize any $(k-1)$-simplex in S^{k-1} for each k. This can be done as $\Sigma^c_{G'}(G)$ for the following group G (we omit the proof): Let R be the normal subgroup of $\mathbb{Z}^k * \langle x\rangle$ generated by the k elements $x^{-1}[x^{v_i}, x^{2v_i}]$, $i = 1, \ldots, k$, where $v_1, \ldots, v_k$ are linearly independent elements of $\mathbb{Z}^k$, and put $G = (\mathbb{Z}^k * \langle x\rangle)/R$.

The invariant Σ'_A. All the results of this section hold also for the invariant Σ'_A introduced at the end of Sect. 2. In particular, one can easily give examples where Σ and Σ' fall on opposite sides of the dichotomy described in the first paragraph. For example, let H be the group of permutations of $\mathbb{Z}^n$ with finite support (i.e. fixing all but finitely many points of $\mathbb{Z}^n$); then $\mathbb{Z}^n$ acts in an obvious way on H and we let G be the corresponding split extension. G can also be described as the group of permutations of $\mathbb{Z}^n$ which are translations off a finite set. H is finitely generated over $\mathbb{Z}^n$, so G is finitely generated. On the other hand, $G'=H$ is not finitely generated, although it is finitely generated as a G'-group (since it has a subgroup of index 2 which is simple). Since $GL(n, \mathbb{Z})$ acts on G and has dense orbits on $S(G)\cong S^{n-1}$, we see that $\Sigma=\emptyset$ but $\Sigma'=S(G)$.

8. Groups of PL homeomorphisms of the interval

Let $\mathfrak{H}$ denote the group of all orientation preserving PL homeomorphisms of the unit interval [0, 1]. We shall investigate $\Sigma(G)$ for finitely generated subgroups G of $\mathfrak{H}$. We shall assume that G is *irreducible*, that is, G has no fixed points on the open interval (0, 1). This is no real loss of generality, since a reducible G is a subdirect product of irreducible groups of PL homeomorphisms of subintervals of [0, 1] and can be analyzed in terms of these "irreducible components".

For any subgroup G of $\mathfrak{H}$ there are two homomorphisms λ and ρ to the multiplicative group $\mathbb{R}^+$ of positive reals given by derivatives at the left and right end-points of [0, 1]:

$$\lambda: G\to\mathbb{R}^+, \qquad \lambda(f)=\frac{df}{dt}(0);$$

$$\rho: G\to\mathbb{R}^+, \qquad \rho(f)=\frac{df}{dt}(1).$$

We shall say that λ and ρ are *independent* if $\lambda(G)=\lambda(\mathrm{Ker}\,\rho)$ *and* $\rho(G)=\rho(\mathrm{Ker}\,\lambda)$.

Theorem 8.1. *If G is finitely generated and irreducible and λ and ρ are independent then* $\Sigma^c(G)=\{[\log\lambda], [\log\rho]\}$.

We shall need the following

Lemma 8.2. *If G is irreducible then for any $0<a<b<1$ there exists $f\in G$ with $bf<a$.*

Proof. If no such f existed then $\inf\{bf \mid f\in G\}$ would be a fixed point of G in the open interval (0, 1), contradicting irreducibility.

Proof of Theorem. Suppose $[\chi]\in S(G)-\{[\log\lambda], [\log\rho]\}$. The independence of λ and ρ implies that χ cannot vanish on both $\mathrm{Ker}\,\lambda$ and $\mathrm{Ker}\,\rho$. We assume $\chi(\mathrm{Ker}\,\rho)\neq 0$, the proof for the other case is similar. Then we can find $h\in\mathrm{Ker}\,\rho$ with $\chi(h)>0$. We denote by $\mathrm{Supp}(h)$ the *support* of h, that is, the closure of $[0, 1]-\mathrm{Fix}(h)$, where $\mathrm{Fix}(h)$ is the set of fixed points. Since $\rho(h)=0$, there is some $b<1$ with $\mathrm{supp}(h)\subseteq[0, b]$. Now since $[\chi]\neq[\log\lambda]$, there exists $g\in G$ with

$\lambda(g)<1$ and $\chi(g)>0$. Then $g(x)<x$ for all x in some interval $[0, a]$. We may assume $b<a$ by conjugating h by the element given by Lemma 8.2 if necessary.

Let $\mathfrak{X}$ be a finite generating set of G which includes g and h and let $\mathfrak{R}=[\mathfrak{X}^{\pm 1}, \mathfrak{X}^{\pm 1}]$. There exists an $\varepsilon>0$ such that $\operatorname{Supp}(r)\subseteq[\varepsilon, 1-\varepsilon]$ for all $r\in\mathfrak{R}$. Choose k sufficiently large that $bg^k\leqq\varepsilon$. Then $w=g^{-k}hg^k$ is an $\mathfrak{X}$-word with positive χ-track and, moreover, $\operatorname{Supp}(w)\subseteq[0, \varepsilon]$, so

$$r=r^w \quad \text{for } r\in\mathfrak{R}.$$

By Proposition 2.3, the existence of this set of equations shows that $[\chi]\in\Sigma$.

Conversely, suppose $[\chi]=[\log\lambda]$ say. The assumptions on G imply G' is non-trivial. Suppose G' were generated over the finite subset $\mathfrak{X}\subset G_\chi$ by the finite set $\mathfrak{R}\subset G'$. Choose $a>0$ such that $\operatorname{Supp}(r)\subset[a, 1]$ for all $r\in\mathfrak{R}$ and $xg\geqq x$ for all $g\in\mathfrak{X}$ and $x\in[0, a]$. Then $\operatorname{Supp}(r^w)\subset[a, 1]$ for any $\mathfrak{X}$-word w and any $r\in\mathfrak{R}$, so, since these elements r^w generate G', we have $\operatorname{Supp}(h)\subset[a, 1]$ for any $h\in G'$. But if we choose any $h\in G'$ and $b\in[0, 1]$ with $bh\neq b$ and then choose f by Lemma 8.2, then $bf\in\operatorname{Supp}(f^{-1}hf)$ and $bf\notin[a, 1]$, which is the desired contradiction.

Proposition 8.3. *Under the conditions of Theorem* 8.1, *the set* Σ' *described at the end of Sect.* 2 *satisfies* $\Sigma'=S(G)$.

Proof. We must show $[\log\lambda]$ and $[\log\rho]$ are in Σ'. Let $\mathfrak{R}$ be a finite G-generating set of G'. If $[\chi]=[\log\lambda]$ then we can find $g\in G$ with $\chi(g)>0$ and $\rho(g)=0$. By conjugating g by an element given by Lemma 8.2 we can make its support disjoint from the supports of all $r\in\mathfrak{R}$. Then $r=r^g$ for all $r\in\mathfrak{R}$, so by (2.11) we have $[\chi]\in\Sigma'$. The argument for $[\chi]=[\log\rho]$ is analogous.

Discussion

In general $[\log\lambda]$ and $[\log\rho]$ will be irrational points of $S(G)$, so $\Sigma(G)$ is not rationally defined. For an explicit finitely *presented* example one can take the group $G(\{2, 3\})$ of homeomorphisms h with slopes in $\{2^j 3^k \mid j, k\in\mathbb{Z}\}$ and discontinuities of h' in $\mathbb{Z}[1/6]$. Finite presentation has been shown by Ken Brown. More generally, he has shown that the analogously defined group $G(\{n_1, \dots, n_k\})$ is finitely presented for any integers $n_1, \dots, n_k$ (private communication).

By [B-Sq], $\mathfrak{H}$ contains no non-abelian free subgroups (one can restrict to irreducible subgroups, and, as we have seen, they have many commuting elements and therefore cannot be free), so Theorem C can be applied to subgroups of $\mathfrak{H}$. For example, let G be as in Theorem 8.1 and suppose $G'<H$ with $H/G'\cong\mathbb{Z}$. Then by Theorem B, H is finitely generated if and only if $S(G, H)$ misses $[\log\lambda]$ and $[\log\rho]$, i.e. λ and ρ are non-trivial on H. Moreover, $S(G, H)$ is a codimension one sphere which cuts $S(G)$ into two hemispheres. If $[\log\lambda]$ and $[\log\rho]$ are in opposite hemispheres, then $\Sigma_{G'}(H)=\emptyset$ by functoriality (Theorem *H*), so $S(H, G')\subseteq\Sigma^c_{H'}(H)$ (Proposition 3.4 (i)), so Theorem C implies that H *is not finitely presented.* In some cases one can show that if $[\log\lambda]$ and $[\log\rho]$ *are* both in the same hemisphere then H *is* finitely presented. This is true for example for the group $G(\{2\})$ of homeomorphisms h with slopes in $\{2^k \mid k\in\mathbb{Z}\}$ and discontinuities of h' in $\mathbb{Z}[1/2]$; this came out of a discussion with Ross Geoghegan.

9. Another characterization of Σ_A

After this paper was completed, the work of Ken Brown [B] suggested to us another characterization of the complementary set $\Sigma_A^c(G) = S(G) - \Sigma_A$.

Proposition 9.1. *$[\chi] \in \Sigma_A^c$ if and only if there exists an increasing chain $A_0 \subseteq A_1 \subseteq \ldots$ of proper subgroups of A with $A = \bigcup_{j \geqq 0} A_j$ and such that for each $g \in G_\chi$ there exists $j(g) \in \mathbb{N}$ with $A_j^g \subseteq A_j$ for $j \geqq j(g)$.*

Proof. If such a chain $\{A_j\}$ exists then any subgroup of A which is finitely generated over a finite subset of G_χ will be in A_j for sufficiently large j, so $[\chi] \notin \Sigma_A$. Conversely, suppose no such chain exists. Choose finite $\mathfrak{X}$ and $\mathfrak{R}$ as in Proposition 2.1 and for $j \geqq 0$ define

$$A_j = \langle \{a^w \mid a \in \mathfrak{R} \text{ and } w \text{ is an } \mathfrak{X}^{\pm 1}\text{-word with } \chi\text{-track } \geqq -j\} \rangle.$$

Clearly $\{A_j\}$ is an increasing sequence of subgroups with union A. Moreover, if $g \in G_\chi$ is represented by an $\mathfrak{X}^{\pm 1}$-word with χ-track $\geqq -j_0$ then $A_j^g \subseteq A_j$ for $j \geqq j_0$. Hence, by assumption, $A_k = A$ for some k. Choose an $\mathfrak{X}^{\pm 1}$-word u with positive χ-track and with $\chi(u) > k$. The equation $A_k^u = A$ together with the definition of A_k implies condition (ii) of Proposition 2.1, so $[\chi] \in \Sigma_A$.

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