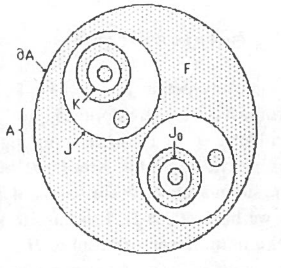

Fig. 10.

$\partial A_2, \ldots, \partial A_{2m}$ is a sequence of disjoint meridians of T which is in cyclic order on ∂T. It follows that $A_1, A_2, \ldots, A_{2m}$ is a sequence of pairwise disjoint meridianal disks of T in cyclic order on T.

It remains to verify the following assertion: if A is a disk component of $h(T) \cap P$, then A is a disk component of $T \cap P$. We proceed by contradiction. Suppose A is a disk component of $h(T) \cap P$ but not of $T \cap P$. Then ∂A is a component of $h(\partial T) \cap P$. Since $h(\partial T) \cap P \subset (\partial T) \cap P$, we conclude that ∂A is a component of $T \cap P$. Let F denote the component of $T \cap P$ which contains ∂A. F must be a proper subset of A, because A is not contained in $T \cap P$. Thus, F is not a disk. Also, the hypothesis of Case 2 prevents F from being an annulus. (This is the only point at which this hypothesis is used.) Hence, $(\partial F) \cap (\mathrm{int}(A))$ has at least two components. Suppose J is a component of $(\partial F) \cap (\mathrm{int}(A))$. Then $D(J)$ is not contained in T; but $\mathrm{int}(D(J))$ must intersect T because J, being a meridian of T, links a spine of T. Hence, J is of height ≥ 2. Thus, either $D(J_0) \subset D(J)$, or $D(J_0) \cap D(J) = \varnothing$. Since $(\partial F) \cap (\mathrm{int}(A))$ has more than one component, we can assume $D(J) \cap D(J_0) = \varnothing$. As J is of height ≥ 2, then $D(J)$ contains a component K of $(\partial T) \cap P$ of height exactly equal to 2. Our choice of J insures that $K \neq J_0$. (See Fig. 10.) In an earlier paragraph, we argued that J_0 is the only height 2 component of $(\partial T) \cap P$ that intersects C_0. So $K \cap C_0 = \varnothing$. Therefore, $K \cap C'_0 = \varnothing$. Hence, $K \subset ((\partial T) \cap P) - C'_0 = h(\partial T) \cap P$. But $K \subset \mathrm{int}(A) \subset h(\mathrm{int}(T))$. We have reached the desired contradiction. □

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CASSON'S INVARIANT FOR HOMOLOGY 3-SPHERES AND CHARACTERISTIC CLASSES OF SURFACE BUNDLES I

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INTRODUCTION

Recently Casson [6] succeeded to lift the classical Rohlin invariant, which is a $\mathbb{Z}/2$-valued invariant, to an integer valued invariant λ for oriented homology 3-spheres and by making use of it he obtained remarkable results concerning 3-manifolds and knots. Roughly speaking it is defined to be half of the algebraic sum of the number of conjugacy classes of irreducible representations of the fundamental group into SU(2).

Now by virtue of the classical theorem of the existence of the Heegaard splittings, the theory of 3-manifolds is closely related with that of the mapping class group of orientable surfaces. For example Birman–Craggs [5] and then Johnson [9] have clarified the effect of the Rohlin invariant on the algebraic structure of the Torelli group, which is the subgroup of the mapping class group consisting of all elements acting trivially on the homology. Johnson went on further with his extensive study of the structure of the Torelli group and obtained several fundamental results [11, 12, 13]. Now in this series of papers, keeping in mind the above results, we would like to clarify the relationship between the Casson invariant for homology 3-spheres and the structure of certain subgroups of the mapping class group. In doing so we make a crucial use of the methods of Johnson developed in [op. cit.]. However there is also an essentially new ingredient, namely we have found that the Casson invariant can be interpreted as a secondary invariant associated with the characteristic classes of surface bundles introduced in [18, 20, 27].

Now we review the results of the present paper more precisely. Let Σ_g be a closed oriented surface of genus g and let $\mathcal{M}_g$ be its mapping class group, namely it is the group of all isotopy classes of orientation preserving diffeomorphisms of Σ_g. By technical reasons (which will be explained in §1), we also consider the mapping class group $\mathcal{M}_{g,1}$ of Σ_g *relative* to a fixed embedded disc $D^2 \subset \Sigma_g$. $\mathcal{M}_{g,1}$ is the group of all isotopy classes relative to D^2 of diffeomorphisms of Σ_g which restrict to the identity on D^2. Let $\mathcal{K}_g$ (resp. $\mathcal{K}_{g,1}$) be the subgroup of $\mathcal{M}_g$ (resp. $\mathcal{M}_{g,1}$) generated by all the Dehn twists on bounding simple closed curves on Σ_g (resp. $\Sigma_g \setminus D^2$). Now fix a Heegaard embedding $i: \Sigma_g \to S^3$ and for each element $\varphi \in \mathcal{K}_g$, consider the manifold M_φ which is obtained by first cutting S^3 along $i(\Sigma_g)$ and then regluing the resulting two pieces by the map φ (see [5]) and §2 for details). It is easy to see that M_φ is an oriented homology 3-sphere and conversely we prove in §2 that every homology 3-sphere can be obtained in this way. In view of this and also by a technical reason, the group $\mathcal{K}_g$ seems to be a better place to work in than the Torelli group. We have a mapping $\lambda^*: \mathcal{K}_g \to \mathbb{Z}$ defined by $\lambda^*(\varphi) = \lambda(M_\varphi)$. We also use the same symbol for the mapping $\lambda^*: \mathcal{K}_{g,1} \to \mathbb{Z}$ which is defined to be the composition of the natural surjective homomorphism $\mathcal{K}_{g,1} \to \mathcal{K}_g$ followed by the original λ^*.

On the other hand by making use of our previous results [20, 21, 22], in §5 we define a mapping $d: \mathcal{M}_{g,1} \to \mathbb{Z}$ whose restriction to $\mathcal{K}_{g,1}$ should be considered as the secondary invariant associated with the signature of oriented surface bundles over surfaces. Now our main result (Theorem 6.1) may be stated that the two integer valued maps λ^* and d on $\mathcal{K}_{g,1}$ are essentially equal each other. The precise statement however takes somewhat complicated form because we have to correct the mapping d by adding a "normalizing term" which depends on how we choose a Heegaard embedding $i: \Sigma_g \to S^3$. Also the actual proof of the above main result goes in the "opposite direction" in the following sense. Namely motivated by a formula which expresses the Casson invariant of a knot as a homogeneous polynomial of degree two on the entries of its Seifert matrix (Proposition 3.2), in §4 we try to express the mapping λ^* in terms of Johnson's homomorphism which we study in §1. It turns out that except for one single factor the latter contains all the informations about λ^* and we collect them to define the normalizing term. Then we prove in §5 that the restriction of the mapping d to $\mathcal{K}_{g,1}$ "realizes" the missing factor which is in some sense the core of the Casson invariant. Since there is no canonical way to choose a Heegaard embedding $i: \Sigma_g \to S^3$, the above complication is inevitable. But if we restrict the two maps to a yet smaller subgroup $\mathcal{L}_{g,1}$ of $\mathcal{M}_{g,1}$, then the normalizing term vanishes and the result takes the following simple form: $\lambda^* = \frac{1}{24} d$ (see Proposition 6.4 and Remark 6.6).

Although the definition of the mapping d will be given purely in the framework of cohomology of groups, we believe that it can be given a more geometrical meaning related to the geometry of the Teichmüller space. More precisely it should be explained as a secondary invariant associated with the fact that the pull back of a certain non-trivial holomorphic line bundle over the moduli space of compact Riemann surfaces to its infinite ramified covering space corresponding to the Torelli group is trivial. We would like to pursue this point in a near future. Also in the second paper of this series [25], we will prove the well-definedness of the Casson invariant entirely in the framework of the mapping class groups (see Remark 6.2). The main results of this paper have been announced in [23, 24].

1. JOHNSON'S HOMOMORPHISM

In this section first we recall Johnson's method of investigating the structure of the mapping class groups (see [10, 11] for details) and then we make a detailed study of one particular case which extends Johnson's earlier works.

As in the introduction, let $\mathcal{M}_g$ be the mapping class group of Σ_g and let $\mathcal{M}_{g,1}$, be the mapping class group of Σ_g relative to a fixed embedded disc $D^2 \subset \Sigma_g$. We have a short exact sequence

$$1 \to \pi_1(T_1\Sigma_g) \to \mathcal{M}_{g,1} \to \mathcal{M}_g \to 1$$

where $T_1\Sigma_g$ is the unit tangent bundle of Σ_g and the mapping $\mathcal{M}_{g,1} \to \mathcal{M}_g$ is the natural forgetful homomorphism (see [20] §3). We will be mainly concerned with (subgroups of) $\mathcal{M}_{g,1}$ rather than $\mathcal{M}_g$ because of the following technical reasons. One is that, since $\mathcal{M}_{g,1}$ can be considered as a certain subgroup of the automorphism group of a *free group* $\pi_1(\Sigma_g \setminus \mathring{D}^2)$ by a classical result of Nielsen, we can apply various techniques of combinatorial group theory directly, while $\mathcal{M}_g$ is only isomorphic to the proper outer automorphism group of $\pi_1(\Sigma_g)$. Another reason is that since there is a canonical homomorphism $\mathcal{M}_{g,1} \to \mathcal{M}_{g+1,1}$, we can consider various direct systems consisting of subgroups of $\mathcal{M}_{g,1}$ naturally (see §2).

Now we write Σ_g^0 for $\Sigma_g \setminus \operatorname{Int} D^2$ and put $\Gamma_1 = \pi_1(\Sigma_g^0)$, which is a free group of rank $2g$. For later use choose a system of free generators $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$ of Γ_1 as is illustrated in Fig. 1a. We successively define $\Gamma_{k+1} = [\Gamma_k, \Gamma_1]$ $(k = 1, 2, \ldots)$ and set $N_k = \Gamma_1/\Gamma_k$. We may call the group N_k the k-th nilpotent quotient of Γ_1. For simplicity we write H for $N_2 = H_1(\Sigma_g, \mathbb{Z})$ and let x_i (resp. y_i) be the homology class of α_i (resp. β_i) so that $x_1, \ldots, x_g, y_1, \ldots, y_g$ is a symplectic basis of it. Now let $\mathcal{L}$ be the free graded Lie algebra (over $\mathbb{Z}$) on H and let $\mathcal{L}_k$ be the submodule of $\mathcal{L}$ consisting of all homogeneous elements of degree k so that in particular we have $\mathcal{L}_1 \cong H$. It is easy to see that the correspondence $u \wedge v \to [u, v]$ $(u, v \in H)$ defines a natural isomorphism $\Lambda^2 H \cong \mathcal{L}_2$ and also the correspondence $(u \wedge v) \otimes w \to [[u, v], w]$ defines a short exact sequence $0 \to \Lambda^3 H \to \Lambda^2 H \otimes H \to \mathcal{L}_3 \to 0$ so that we have a natural isomorphism $\mathcal{L}_3 \cong \Lambda^2 H \otimes H / \Lambda^3 H$, where an element $u \wedge v \wedge w \in \Lambda^3 H$ goes to $(u \wedge v) \otimes w + (v \wedge w) \otimes u + (w \wedge u) \otimes v \in \Lambda^2 H \otimes H$ and vanishes in $\mathcal{L}_3$ because of the Jacobi's identity. Now it is a classical result that there exists a natural isomorphism $\Gamma_k/\Gamma_{k+1} \cong \mathcal{L}_k$ (see [19]) and we have a central extension $0 \to \mathcal{L}_k \to N_{k+1} \to N_k \to 1$. The mapping class group $\mathcal{M}_{g,1}$ acts on N_k and let $\mathcal{M}(k)$ be the subgroup of it consisting of all the elements which act on N_k trivially. Now Johnson's homomorphism

$$\tau_k: \mathcal{M}(k) \to \mathcal{L}_k \otimes H$$

is defined as follows (see [11]). Let φ be an element of $\mathcal{M}(k)$. For each element $\gamma \in \Gamma_1$, the element $\varphi(\gamma)\gamma^{-1}$ is contained in Γ_k because of the assumption that φ acts on the k-th nilpotent quotient N_k trivially. Let $\tau_\varphi(\gamma)$ be the image of that element in $\mathcal{L}_k$. Johnson proved that $\tau_\varphi(\gamma\gamma') = \tau_\varphi(\gamma) + \tau_\varphi(\gamma')$ and considered τ_φ as an element of $\operatorname{Hom}(H, \mathcal{L}_k) = \mathcal{L}_k \otimes H$. Here recall that the intersection pairing on H defines a selfdual structure on it. Explicitly we have $\tau_\varphi = \Sigma \tau_\varphi(x_i) \otimes y_i - \Sigma \tau_\varphi(y_i) \otimes x_i$. Finally Johnson defined $\tau_k: \mathcal{M}(k) \to \mathcal{L}_k \otimes H$ by $\tau_k(\varphi) = \tau_\varphi$ and proved that it is actually a homomorphism. The homomorphism $\tau_1: \mathcal{M}_{g,1} \to H \otimes H$ is nothing but the classical representation of $\mathcal{M}_{g,1}$ as the group of symplectic automorphisms of H. Namely with respect to our symplectic basis of H we can write $\tau_1: \mathcal{M}_{g,1} \to \operatorname{Sp}(2g; \mathbb{Z})$, where $\operatorname{Sp}(2g; \mathbb{Z})$ is the Siegel modular group of degree g. The group $\mathcal{M}(2)$ is nothing but the Torelli group $\mathcal{I}_{g,1}$ and Johnson [10] proved that the homomorphism $\tau_2: \mathcal{I}_{g,1} \to \Lambda^3 H$ is surjective. He has also proved that the group $\mathcal{M}(3)$ which is equal to $\operatorname{Ker} \tau_2$ is the subgroup of $\mathcal{M}_{g,1}$ generated by all the Dehn twists on bounding simple closed curves on Σ_g^0 (see [13]). Henceforth we write $\mathcal{K}_{g,1}$ for this group which will play a central role in this paper.

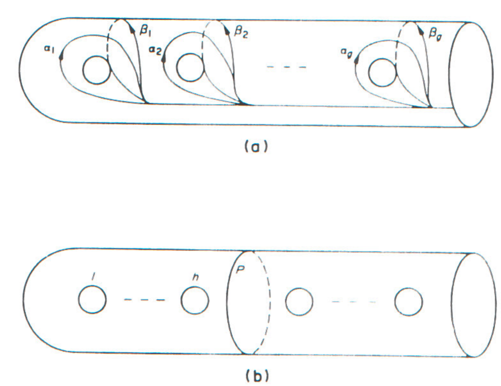

Fig. 1

Now we describe the homomorphism $\tau_3\colon \mathscr{K}_{g,1} \to \Lambda^2 H \otimes H/\Lambda^3 H$. Consider the basis $x_i \wedge x_j$ $(i<j)$, $y_i \wedge y_j$ $(i<j)$, $x_i \wedge y_j$ of $\Lambda^2 H$ and write t_i $(i = 1, \ldots, \binom{2g}{2})$ for these elements in any order. Let T be the submodule of $\Lambda^2 H \otimes \Lambda^2 H$ generated by the elements $t_i \otimes t_i$ and $t_i \otimes t_j + t_j \otimes t_i$ $(i \neq j)$. For simplicity hereafter we write $t_i^{\otimes 2}$ (resp. $t_i \leftrightarrow t_j$) for the element $t_i \otimes t_i$ (resp. $t_i \otimes t_j + t_j \otimes t_i$). Also let $\bar{T}$ be the image of T in $\mathscr{L}_3 \otimes H = \Lambda^2 H \otimes H^2/\Lambda^3 H \otimes H$ under the projection $\Lambda^2 H \otimes \Lambda^2 H \subset \Lambda^2 H \otimes H^2 \to \Lambda^2 H \otimes H^2/\Lambda^3 H \otimes H$. Now let p be a bounding simple closed curve on $\operatorname{Int} \Sigma_g^0$ and let $\psi \in \mathscr{K}_{g,1}$ be the right handed Dehn twist on p. Let F be the subsurface of Σ_g^0 which p bounds and choose a symplectic basis $u_1, \ldots, u_h, v_1, \ldots, v_h$ of $H_1(F; \mathbb{Z})$ (h is the genus of F). Then we have

PROPOSITION 1.1. *$\tau_3(\psi)$ is equal to the image of the element $-(u_1 \wedge v_1 + \cdots + u_h \wedge v_h)^{\otimes 2}$ in $\bar{T}$ under the projection $T \to \bar{T}$. In particular* $\operatorname{Im} \tau_3$ *is contained in $\bar{T}$.*

Proof. First we recall an important property of Johnson's homomorphism, namely for any element $\varphi \in \mathscr{M}_{g,1}$ we have $\tau_k(\varphi\psi\varphi^{-1}) = \varphi_*(\tau_k(\psi))$ where φ_* is the natural automorphism of $\mathscr{L}_k \otimes H$ induced from the symplectic automorphism $\varphi_*\colon H \to H$. Hence to prove our Proposition it is enough to prove the assertion for a particular bounding simple closed curve p illustrated in Fig. 1b. It is easy to see that the action of the corresponding Dehn twist ψ on Γ_1 is given by $\psi(\gamma) = [\beta_h, \alpha_h] \cdots [\beta_1, \alpha_1]\gamma[\alpha_1, \beta_1] \cdots [\alpha_h, \beta_h]$ for $\gamma = \alpha_1, \ldots, \alpha_h, \beta_1, \cdots, \beta_h$ and it acts on the elements $\alpha_{h+1}, \ldots, \alpha_g, \beta_{h+1}, \ldots, \beta_g$ trivially. Hence we have

$$\tau_\psi(x_i) = \sum_{j=1}^{h} [[y_j, x_j], x_i] \in \mathscr{L}_3 \qquad (i = 1, \ldots, h)$$

$$\tau_\psi(y_i) = \sum_{j=1}^{h} [[y_j, x_j], y_i] \in \mathscr{L}_3 \qquad (i = 1, \ldots, h)$$

$$\tau_\psi(x_i) = \tau_\psi(y_i) = 0 \qquad (i = h+1, \ldots, g).$$

Therefore

$$\tau_3(\psi) = \sum_{i=1}^{h} \sum_{j=1}^{h} y_j \wedge x_j \otimes x_i \wedge y_i = -(x_1 \wedge y_1 + \ldots + x_h \wedge y_h)^{\otimes 2}$$

as required.

According to [8], the group $\mathscr{K}_{g,1}$ is generated by Dehn twists on bounding simple closed curves of genus one and two. Hence $\operatorname{Im} \tau_3$ is equal to the submodule of $\bar{T}$ generated by the $\mathrm{Sp}(2g; \mathbb{Z})$-orbits of the following two elements: $(x_1 \wedge y_1)^{\otimes 2}$ and $(x_1 \wedge y_1 + x_2 \wedge y_2)^{\otimes 2}$. Using this fact we can determine $\operatorname{Im} \tau_3$ explicitly. However to avoid unnecessary complicated expressions, here we only prove the following.

PROPOSITION 1.2. *The image of the homomorphism $\tau_3\colon \mathscr{K}_{g,1} \to \bar{T}$ is a submodule of $\bar{T}$ of index a power of two.*

Proof. Recall that T is the free abelian group generated by the elements $t_i^{\otimes 2}$ and $t_i \leftrightarrow t_j$ $(i \neq j)$. It is easy to see that any such element is $\mathrm{Sp}(2g; \mathbb{Z})$-equivalent to one and only one of the following ten elements (or negatives of them).

(I) $(x_1 \wedge y_1)^{\otimes 2}$ (II) $(x_1 \wedge x_2)^{\otimes 2}$ (III) $x_1 \wedge y_1 \leftrightarrow x_2 \wedge y_2$

(IV) $x_1 \wedge y_1 \leftrightarrow x_1 \wedge x_2$ (V) $x_1 \wedge x_2 \leftrightarrow y_1 \wedge x_2$ (VI) $x_1 \wedge x_2 \leftrightarrow y_1 \wedge y_2$

(VII) $x_1 \wedge y_1 \leftrightarrow x_2 \wedge x_3$ (VIII) $x_1 \wedge x_2 \leftrightarrow x_2 \wedge x_3$ (IX) $x_1 \wedge x_2 \leftrightarrow y_1 \wedge x_3$

(X) $x_1 \wedge x_2 \leftrightarrow x_3 \wedge x_4$.

Since $\operatorname{Im} \tau_3$ is an $\mathrm{Sp}(2g; \mathbb{Z})$-invariant submodule of $\bar{T}$, to prove the required assertion it is enough to show that twice of any of the above elements is contained in $\operatorname{Im} \tau_3$. First of all type (I) is clearly contained in $\operatorname{Im} \tau_3$. Next since $(x_1 \wedge y_1 + x_2 \wedge y_2)^{\otimes 2} = (\mathrm{I}) + (x_2 \wedge y_2)^{\otimes 2} + (\mathrm{III})$ and the second element in this expression is equivalent to (I), we have $(\mathrm{III}) \in \operatorname{Im} \tau_3$. Now suppose that two elements $u, v \in H$ satisfy the condition $u \cdot v = 1$, then it is well known that there exists a symplectic automorphism A of H such that $A(x_1) = u$ and $A(y_1) = v$. Therefore $(u \wedge v)^{\otimes 2}$ is contained in $\operatorname{Im} \tau_3$ (here we have used the classical result that the representation $\tau_1\colon \mathscr{M}_{g,1} \to \mathrm{Sp}(2g; \mathbb{Z})$ is surjective). Now we have

$$\{x_1 \wedge (y_1 + x_2)\}^{\otimes 2} = (\mathrm{I}) + (\mathrm{II}) + (\mathrm{IV})$$
$$\{x_1 \wedge (y_1 - x_2)\}^{\otimes 2} = (\mathrm{I}) + (\mathrm{II}) - (\mathrm{IV}).$$

Therefore both of (II) + (IV) and (II) − (IV) are contained in $\operatorname{Im} \tau_3$ and hence $2 \times (\mathrm{II})$, $2 \times (\mathrm{IV}) \in \operatorname{Im} \tau_3$. Next we have

$$\{(x_1 + x_2) \wedge (y_1 + x_2)\}^{\otimes 2} - \{(x_1 + x_2) \wedge y_1\}^{\otimes 2} = (\mathrm{II}) + (\mathrm{IV}) - (\mathrm{V}).$$

Hence $(\mathrm{V}) \in \operatorname{Im} \tau_3$. Similarly we compute

$$\{(2x_1 + y_2) \wedge (x_1 + y_1 + x_2)\}^{\otimes 2} - \{(x_1 + x_2) \wedge y_2\}^{\otimes 2} - \{(x_2 + y_1) \wedge y_2\}^{\otimes 2}$$
$$= 4 \times (\mathrm{I}) + 4 \times (\mathrm{II}) - (x_2 \wedge y_2)^{\otimes 2} - 2 \times (\mathrm{III}) + 2 \times (\mathrm{IV})^* + (\mathrm{V})^* - 2 \times (\mathrm{VI}),$$

where (IV)* (resp. (V)*) stands for a linear combination of elements which are $\mathrm{Sp}(2g; \mathbb{Z})$-equivalent to (IV) (resp. (V)). It follows that $2 \times (\mathrm{VI}) \in \operatorname{Im} \tau_3$. If we apply the symplectic automorphism $y_2 \to y_2 + x_2 + x_3$, $y_3 \to y_3 + x_2 + x_3$ (other basis elements are fixed) to (III), we obtain

$$(\mathrm{III})^*\colon x_1 \wedge y_1 \leftrightarrow x_2 \wedge (y_2 + x_2 + x_3) = (\mathrm{III}) + (\mathrm{VII}).$$

Hence $(\mathrm{VII}) \in \operatorname{Im} \tau_3$. Similarly we have

$$(\mathrm{V})^*\colon \quad x_1 \wedge x_2 \leftrightarrow (y_1 + x_1 + x_3) \wedge x_2 = (\mathrm{V}) + 2 \times (\mathrm{II}) - (\mathrm{VIII})$$
$$(\mathrm{VI})^*\colon \quad x_1 \wedge x_2 \leftrightarrow y_1 \wedge (y_2 + x_2 + x_3) = (\mathrm{VI}) + (\mathrm{V}) + (\mathrm{IX})$$
$$(\mathrm{VII})^*\colon \quad x_1 \wedge (y_1 + x_1 + x_4) \leftrightarrow x_2 \wedge x_3 = (\mathrm{VII}) + x_1 \wedge x_4 \leftrightarrow x_2 \wedge x_3.$$

From these equations we can conclude that the elements (VIII), $2 \times (\mathrm{IX})$ and (X) are all contained in $\operatorname{Im} \tau_3$. This completes the proof.

2. HOMOLOGY 3-SPHERES AND THE TORELLI GROUP

Let $\mathscr{H}(3)$ be the set of all orientation preserving diffeomorphism classes of oriented homology 3-spheres. Then by virtue of the classical theorem of the existence and stable uniqueness of the Heegaard splittings of 3-manifolds (see [3] for example), we can identify $\mathscr{H}(3)$ as the limit of a certain direct system, which is constructed by using subgroups of the mapping class groups $\mathscr{M}_{g,1}$, as follows.

First of all we fix a handlebody which Σ_g bounds as follows. Let $V = S^1 \times D^2$ be a framed solid torus and choose two disjoint embedded discs D_- and D_+ in ∂V. Also choose a meridian (resp. longitude) curve m (resp. l) on ∂V which is disjoint from the two discs $D_\pm$. Now define an oriented 3-dimensional handlebody H_g of genus g by $H_g = V_1 \natural \ldots \natural V_g$

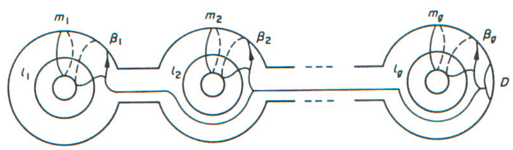

Fig. 2.

(boundary connected sum of g-copies of V), where D_+ of V_i is attached to V_- of V_{i+1} $(i = 1, \ldots, g-1)$. We identify Σ_g with ∂H_g which has an embedded disc $D = D_+$ of V_g so that the homology class x_i (resp. y_i) is represented by the meridian curve m_i (resp. longitude curve l_i) of V_i and the element β_i of Γ_1 $(= \pi_1(\Sigma_g^0))$ is null homotopic in H_g for all i (see Fig. 2). The compact surface $\Sigma_g^0 = \Sigma_g \setminus \mathring{D}$ is contained in Σ_{g+1}^0 and hence we have a homomorphism $\mathcal{M}_{g,1} \to \mathcal{M}_{g+1,1}$ which is actually an injection because, as was already mentioned in §1, $\mathcal{M}_{g,1}$ can be considered as a certain subgroup of the automorphism group of the free group $\pi_1(\Sigma_g^0)$ by the classical theorem of Nielsen. Now let ι_g be an element of $\mathcal{M}_{g,1}$ defined as $\iota_g = \Pi_{i=1}^g \rho_i$, where $\rho_i = \lambda_i \mu_i \lambda_i$ (λ_i and μ_i being respectively the Dehn twist on the longitude and meridian curves on ∂V_i) is the "rotation" of the i-th handle of Σ_g by 90°. It is easy to see that the manifold $H_g \cup_{\iota_g} (-H_g)$, which is obtained from the disjoint union of H_g and $-H_g$ by identifying the boundaries by the map ι_g, is the 3-sphere S^3. Henceforth we identify them. In another word, we consider Σ_g as a fixed Heegaard surface of S^3. Now for each element φ of the Torelli group $\mathcal{I}_{g,1}$ we denote M_φ for $H_g \cup_{\iota_g \varphi} (-H_g)$, here we understand φ as (the isotopy class of) a diffeomorphism of Σ_g which is the identity on the embedded disc $D \subset \Sigma_g$. Namely M_φ is the 3-manifold obtained by cutting S^3 along the Heegaard surface Σ_g and then regluei ng the resulting two pieces by the map φ. Since φ acts on the homology H of Σ_g trivially M_φ is an oriented homology 3-sphere. Now let $\mathcal{N}_{g,1}$ be the subgroup of $\mathcal{M}_{g,1}$ consisting of all elements which can be extended to a diffeomorphism of H_g which is the identity on D.

Definition 2.1. *Two elements $\varphi, \psi \in \mathcal{I}_{g,1}$ are said to be equivalent, denoted by $\varphi \sim \psi$, if there are elements $\xi_1, \xi_2 \in \mathcal{N}_{g,1}$ such that*

$$\psi = \iota_g^{-1} \xi_1 \iota_g \varphi \xi_2.$$

It is easy to see that the above relation is in fact an equivalence relation and also if $\varphi \sim \psi$, then M_φ is orientation preserving diffeomorphic to M_ψ. Moreover if two elements of $\mathcal{I}_{g,1}$ are equivalent, then they remain to be so even if we consider them as elements of $\mathcal{I}_{g+1,1}$. Hence if we denote $\mathcal{I}_{g,1}/\sim$ for the quotient set with respect to the above equivalence relation, then we have a direct system $\{\mathcal{I}_{g,1}/\sim\}_g$ and we obtain a natural map $\lim_{g \to \infty} \mathcal{I}_{g,1}/\sim \;\to \mathcal{H}(3)$ defined by the correspondence $\mathcal{I}_{g,1} \ni \varphi \to M_\varphi \in \mathcal{H}(3)$.

Theorem 2.2. *The correspondence $\lim_{g \to \infty} \mathcal{I}_{g,1}/\sim \;\to \mathcal{H}(3)$ is a bijection.*

Proof. This is just the classical theorem on the existence and stable uniqueness of the Heegaard splittings of 3-manifolds adapted to the subclass of oriented homology 3-spheres.

Proposition 2.3. *Any element $\varphi \in \mathcal{I}_{g,1}$ is equivalent to an element $\psi \in \mathcal{K}_{g,1}$ so that we have a bijection*

$$\lim_{g \to \infty} \mathcal{K}_{g,1}/\sim \;\cong \mathcal{H}(3).$$

Remark 2.4. If we define an equivalence relation $\sim$ on the Torelli group $\mathcal{I}_g$ similarly as above, then it can be shown that the natural forgetful mappings $\mathcal{I}_{g,1}/\sim \;\to \mathcal{I}_g/\sim$ and $\mathcal{K}_{g,1}/\sim \;\to \mathcal{K}_g/\sim$ are both *bijective*. However since there is no canonical map $\mathcal{I}_g \to \mathcal{I}_{g+1}$ and also since Johnson's homomorphisms are defined (at least primarily) on $\mathcal{I}_{g,1}$ and $\mathcal{K}_{g,1}$, it seems to be better to use the groups $\mathcal{I}_{g,1}$ and $\mathcal{K}_{g,1}$ to consider the direct systems with respect to g.

To prove Proposition 2.3, we consider Johnson's homomorphism $\tau_2 : \mathcal{I}_{g,1} \to \Lambda^3 H$ (see [10] and §1). Let W_y be the submodule of $\Lambda^3 H$ generated by the elements $x_i \wedge x_j \wedge y_k$, $x_i \wedge y_j \wedge y_k$ and $y_i \wedge y_j \wedge y_k$. Then we have

Lemma 2.5. $\tau_2(\mathcal{I}_{g,1} \cap \mathcal{N}_{g,1}) = W_y$.

Proof. Let Γ_1^β be the normal subgroup of $\Gamma_1 = \pi_1(\Sigma_g^0)$ generated by the elements $\beta_1, \ldots, \beta_g$ and let φ be any element of $\mathcal{N}_{g,1}$. Then it is easy to see that the automorphism φ_* of Γ_1 induced by φ preserves the subgroup Γ_1^β. Hence if $\varphi \in \mathcal{I}_{g,1} \cap \mathcal{N}_{g,1}$, then $\varphi_*(\beta_i)\beta_i^{-1}$ is a product of elements of the form $[\gamma_1, \gamma_2 \beta_j^{\pm 1} \gamma_2^{-1}]$ $(\gamma_1, \gamma_2 \in \Gamma_1)$. Hence in the terminology of §1 we have $\tau_\varphi(y_i) = \Sigma \pm [\gamma_1] \otimes y_j$. It is easy to deduce from this fact that $\tau_2(\varphi) \in W_y$. Next observe that the symplectic automorphism $x_i \leftrightarrow x_j$, $y_i \leftrightarrow y_j$ ($i \neq j$, other basis elements are fixed) can be realized by an element of $\mathcal{N}_{g,1}$. For example the interchanging of the i-th and the j-th handles (cf. [28]). Since the equality $\tau_2(\varphi \psi \varphi^{-1}) = \varphi_* \tau_2(\psi)$ holds for any $\varphi \in \mathcal{M}_{g,1}$, $\psi \in \mathcal{I}_{g,1}$, to prove the claim we have only to show that the following five elements belong to $\tau_2(\mathcal{I} \cap \mathcal{N})$:

(i) $x_1 \wedge y_1 \wedge x_2$ (ii) $x_1 \wedge y_1 \wedge y_2$ (iii) $x_1 \wedge x_2 \wedge y_3$

(iv) $x_1 \wedge y_2 \wedge y_3$ (v) $y_1 \wedge y_2 \wedge y_3$.

The first two elements can be realized by the homeomorphisms indicated in Figs 3a, 3b respectively, here the + (or −) sign in the figures means that we take the right (or left) handed Dehn twist on the corresponding simple closed curve. It is easy to see that these two elements belong to $\mathcal{I} \cap \mathcal{N}$. Hence (i), (ii) $\in \tau_2(\mathcal{I} \cap \mathcal{N})$. Next consider the homeomorphism

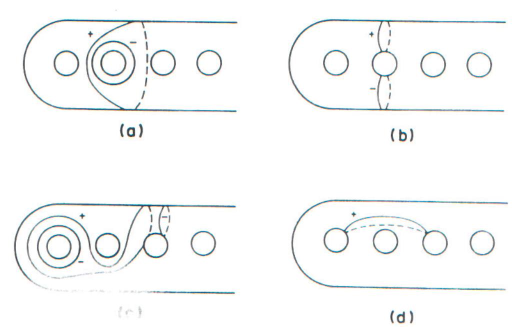

Fig. 3

σ_1 indicated in Fig. 3c. This is one of the homeomorphisms which are called spins or slidings in [4, 28], and it is easy to see that σ_1 belongs to $\mathcal{M}_{g,1}$. The effect of σ_1 on the homology H is given by $y_1 \to y_1 + y_3, x_3 \to x_3 - x_1$. Now if we apply this automorphism to the elements (i), (ii), we can conclude that (iii) and (iv) are in $\tau_2(\mathcal{I} \cap \mathcal{M})$. Finally let $\sigma_2 \in \mathcal{M}_{g,1}$ be the homeomorphism indicated in Fig. 3d. Then $\sigma_2(x_1 \wedge y_1 \wedge y_2) = x_1 \wedge y_1 \wedge y_2 + y_1 \wedge y_2 \wedge y_3$. Hence (v)$\in \tau_2(\mathcal{I} \cap \mathcal{M})$. This completes the proof.

Proof of Proposition 2.3. According to Johnson [10, 13], we have an exact sequence

$$1 \to \mathcal{K}_{g,1} \to \mathcal{I}_{g,1} \xrightarrow{\tau_2} \Lambda^3 H \to 0.$$

Now for each element $\varphi \in \mathcal{I}_{g,1}$, write $\tau_2(\varphi) = w_x + w_y$, where w_x is a linear combination of $x_i \wedge x_j \wedge x_k$'s and $w_y \in W_y$. Observe here that the effect of the homeomorphism ι_g on the homology is given by $x_i \to -y_i, y_i \to x_i$. Hence in view of Lemma 2.4, we can conclude that there are elements $\xi_1, \xi_2 \in \mathcal{I}_{g,1} \cap \mathcal{M}_{g,1}$ such that $\tau_2(\iota_g^{-1}\xi_1\iota_g) = -w_x$ and $\tau_2(\xi_2) = -w_y$. It follows that $\tau_2(\iota_g^{-1}\xi_1\iota_g\varphi\xi_2) = 0$. Hence the element $\iota_g^{-1}\xi_1\iota_g\varphi\xi_2$, which is equivalent to the given one φ, belongs to $\mathcal{K}_{g,1}$. This completes the proof.

3. CASSON INVARIANT FOR HOMOLOGY 3-SPHERES AND KNOTS

In this section we briefly recall Casson's result [6] on his invariants and prove a few properties of them. In particular we prove a formula which expresses Casson invariant of a knot as a homogeneous polynomial of degree two on the entries of its Seifert matrix (see Proposition 3.2).

Recall first that Casson defined an invariant λ' of knots in homology 3-spheres as follows. Let K be a knot in an oriented homology 3-sphere and let $\Delta_K(t) = a_0 + a_1(t + t^{-1}) + a_2(t^2 + t^{-2}) + \ldots$ be its normalized Alexander polynomial with $\Delta_K(1) = 1$. Then $\lambda'(K) = \frac{1}{2}\Delta''(1) = \sum_n n^2 a_n$. Now Casson's theorem may be stated as follows.

THEOREM 3.1 (Casson). *There is an integer valued invariant λ for oriented homology 3-spheres such that*

(i) *λ mod 2 is equal to the Rohlin invariant.*
(ii) *If $\pi_1(M) = 1$, then $\lambda(M) = 0$.*
(iii) *$\lambda(-M) = -\lambda(M)$ and $\lambda(M_1 \# M_2) = \lambda(M_1) + \lambda(M_2)$.*
(iv) *Let K be a knot in an oriented homology 3-sphere M and let $M_n(K)$ be the homology 3-sphere obtained from M by performing the $1/n$ surgery on K. Then $\lambda(M_n(K)) = \lambda(M) + n\lambda'(K)$.*

Here observe that property (iv) with $n = \pm 1$ in the above theorem together with the fact $\lambda(S^3) = 0$ characterizes λ uniquely because any homology 3-sphere can be obtained from S^3 by successively applying ± 1 surgeries on knots. Now we show that the Casson invariant of a knot K can be expressed as a polynomial of degree *two* on the entries of any Seifert matrix of K. For that let F be an oriented Seifert surface of K and choose a symplectic basis $u_1, \ldots, u_{2h}$ (h = genus of F) of $H_1(F; \mathbb{Z})$ so that $u_i \cdot u_j = \delta_{i+hj}$ ($i < j$). Suppose that the homology class u_i is represented by an oriented simple closed curve υ_i on F and let υ_i^+ be the simple closed curve in the ambient homology 3-sphere which is obtained by pushing υ_i to the positive direction with respect to the orientations of F and the ambient manifold. Let l_{ij}

be the linking number of υ_i and υ_j^+ so that $L = (l_{ij})$ is the Seifert matrix of the knot K. With these notations we have

PROPOSITION 3.2. *Let K be a knot in an oriented homology 3-sphere and let $L = (l_{ij})$ be the Seifert matrix of K with respect to a symplectic basis of the homology of a Seifert surface of genus h. Then we have*

$$\lambda'(K) = \sum_{i=1}^{h} (l_{ii}l_{i+h,i+h} - l_{i,i+h}l_{i+h,i}) + 2\sum_{i<j\leq h} (l_{ij}l_{i+h,j+h} - l_{i,j+h}l_{j,i+h}).$$

Remark 3.3. Since $l_{ji} = l_{ij} + u_i \cdot u_j$, the number $l_{i,i+h}l_{i+h,i}$ is always an even integer. Hence $\lambda'(K) \equiv \sum_i l_{ii}l_{i+h,i+h}$ (mod 2) which is consistent with the fact that $\lambda'(K)$ mod 2 is equal to the Arf invariant of K.

Proof of Proposition 3.2. Let us write a_{ij}, b_{ij} and c_{ij} for l_{ij}, $l_{i,j+h}$ and $l_{i+h,j+h}$ $(i, j \leq h)$ respectively and define square matrices A, B and C of degree h by $A = (a_{ij})$, $B = (b_{ij})$ and $C = (c_{ij})$. Then we have

$$L = \begin{pmatrix} A & B \\ {}^tB + E & C \end{pmatrix}.$$

Hence the normalized Alexander polynomial $\Delta_K(t)$ is given by

$$\Delta_K(t) = \det\left(\frac{1}{\sqrt{t}}{}^tL - \sqrt{t}L\right)$$

$$= \det\left(\frac{1}{\sqrt{t}}P - \sqrt{t}Q\right), \quad \text{where}$$

$$P = \begin{pmatrix} A & B+E \\ {}^tB & C \end{pmatrix}, \quad Q = \begin{pmatrix} A & B \\ {}^tB + E & C \end{pmatrix} \quad \text{and so} \quad P - Q = J = \begin{pmatrix} O & E \\ -E & O \end{pmatrix}.$$

Let us write R for the matrix $(1/\sqrt{t})P - \sqrt{t}Q$. Also let p_i (resp. q_i) be the i-th column vector of the matrix P (resp. Q). For a $2h$-dimensional column vector v, we denote $R_i(v)$ (resp. $J_i(v)$) for the matrix obtained from R (resp. J) by replacing its i-th column by the vector v. Similarly for two vectors v_1, v_2, let $R_{ij}(v_1, v_2)$ (resp. $J_{ij}(v_1, v_2)$) be the matrix by replacing the i-th and the j-th columns of R (resp. J) by the vectors v_1 and v_2. With these notations we have

$$\Delta'_K(t) = \sum_{j=1}^{2h} \det R_j(-\tfrac{1}{2}t^{-3/2}p_j - \tfrac{1}{2}t^{-1/2}q_j)$$

and hence

$$\Delta''_K(t) = \sum_{j=1}^{2h} \Big\{ \det R_j(\tfrac{3}{4}t^{-5/2}p_j + \tfrac{1}{4}t^{-3/2}q_j)$$
$$+ \sum_{i\neq j} \det R_{ij}(-\tfrac{1}{2}t^{-3/2}p_i - \tfrac{1}{2}t^{-1/2}q_i, -\tfrac{1}{2}t^{-3/2}p_j - \tfrac{1}{2}t^{-1/2}q_j)\Big\}.$$

If we put $t = 1$, then the matrix R becomes J so that we have

$$\Delta''_K(1) = \sum_{j=1}^{2h} \Big\{ d_j + \sum_{i\neq j} d_{ij} \Big\}, \quad \text{where}$$

$$d_j = \det J_j(\tfrac{3}{4}p_j + \tfrac{1}{4}q_j) \quad \text{and}$$
$$d_{ij} = \det J_{ij}(-\tfrac{1}{2}p_i - \tfrac{1}{2}q_i, -\tfrac{1}{2}p_j - \tfrac{1}{2}q_j)\}.$$

Now direct computation shows that

$$d_j = -(b_{jj} + \tfrac{1}{4}) \qquad (j \le h)$$
$$d_j = b_{jj} + \tfrac{3}{4} \qquad (j > h)$$

and hence we have $\Sigma\, d_j = \frac{1}{2}h$. Similarly we have

$$d_{ij} = d_{i+h,\, j+h} = (b_{ii} + \tfrac{1}{2})(b_{jj} + \tfrac{1}{2}) - b_{ij}b_{ji} \quad (i < j \le h)$$
$$d_{i,\, j+h} = a_{ji}c_{ij} - (b_{ii} + \tfrac{1}{2})(b_{jj} + \tfrac{1}{2}) \qquad (i, j \le h, i \ne j)$$
$$d_{i,\, i+h} = a_{ii}c_{ii} - (b_{ii} + \tfrac{1}{2})(b_{ii} + \tfrac{1}{2}) \qquad (1 \le i \le h).$$

Hence we obtain

$$\tfrac{1}{2}\Delta''(1) = \tfrac{1}{2}\sum d_j + \sum_{i<j} d_{ij} = \sum_{i=1}^{h} \{a_{ii}c_{ii} - b_{ii}(b_{ii}+1)\} + 2\sum_{i<j\le h}(a_{ij}c_{ij} - b_{ij}b_{ji}).$$

Here we have used the fact that the matrix A is symmetric. This completes the proof.

Now let $\lambda: \mathscr{H}(3) \to \mathbb{Z}$ be the mapping defined by the Casson invariant. In view of Proposition 2.3, it defines a mapping $\lambda^*: \mathscr{K}_{g,1} \to \mathbb{Z}$ defined by $\lambda^*(\varphi) = \lambda(M_\varphi) \in \mathbb{Z}$ $(\varphi \in \mathscr{K}_{g,1})$ (see §2).

LEMMA 3.4. *Let $\varphi \in \mathscr{K}_{g,1}$ be the Dehn twist on a bounding simple closed curve K on Σ_g^0, which is considered to be a fixed Heegaard surface of S^3 so that K is a knot in S^3. Then the oriented homology 3-sphere M_φ is equal to the one obtained from S^3 by applying -1 Dehn surgery on K.*

Proof. Let N be a tubular neighborhood of K and let m (resp. l) be an oriented meridian (resp. longitude) curve on ∂N with $[m]\cdot[l] = 1$. Then to prove the Lemma it suffices to show that a simple closed curve on ∂N which represent the homology class $[m] - [l]$ bounds a disc in M_φ. But this can be checked easily.

PROPOSITION 3.5. *The mapping $\lambda^*: \mathscr{K}_{g,1} \to \mathbb{Z}$ is a homomorphism.*

Proof. Since $\mathscr{K}_{g,1}$ is the subgroup of $\mathscr{M}_{g,1}$ generated by all the Dehn twists on bounding simple closed curves on Σ_g^0, it is enough to prove the following. Namely if φ is an arbitrary map in $\mathscr{K}_{g,1}$ and if $\psi \in \mathscr{K}_{g,1}$ is a single twist on a bounding simple closed curve K on Σ_g^0, then the equality $\lambda^*(\varphi\psi) = \lambda^*(\varphi) + \lambda^*(\psi)$ holds. By the definition of the map λ^*, this is equivalent to $\lambda(M_{\varphi\psi}) = \lambda(M_\varphi) + \lambda(M_\psi)$. Now write $S^3 = H_+ \cup H_-$ for the Heegaard splitting we have fixed. The knot K is a bounding simple closed curve on $\Sigma_g^0 \subset \partial H_+$ $(= \partial H_-)$ and let F be the compact surface in S^3 which K bounds on Σ_g^0. It is a Seifert surface for the knot K in S^3. However observe that both of K and F remain to sit inside $M_\varphi = H_+ \cup_\varphi H_-$ (∂H_+ is attached to ∂H_- by the map φ). We write K^* and F^* for them. K^* is a knot in M_φ and F^* is a Seifert surface for it. Then a slightly modified argument as in the proof of Lemma 3.4 implies that the homology 3-sphere $M_{\varphi\psi}$ is nothing but the one which is obtained from M_φ by performing the -1 Dehn surgery on K^*. By Theorem 3.1 and Lemma 3.4, we have $\lambda(M_{\varphi\psi}) = \lambda(M_\varphi) - \lambda'(K^*)$ and also $\lambda(M_\psi) = -\lambda'(K)$. Hence we have only to prove $\lambda'(K^*) = \lambda'(K)$. Now since the element φ can be expressed as a composition of Dehn twists on bounding simple closed curves, say p_i, on Σ_g^0, the argument of Lickorish in [14] implies that M_φ is obtained from S^3 by performing Dehn surgeries on a link L with ± 1 framings, where L is defined as follows. Choose mutually disjoint compact embedded surfaces Σ_g^i in $H_- \subset S^3$ such that each Σ_g^i is parallel to $\Sigma_g^0 \subset \partial H_-$ and the "ordering" of Σ_g^i's, is determined by that of the Dehn twists on p_i's in the expression of the element φ. Let p_i' be the bounding simple closed curve on Σ_g^i corresponding to p_i on Σ_g^0 and we put $L = \bigcup_i p_i'$. Then the linking number of any component of L with any homology class of the surface F is clearly zero. It is easy to deduce from this fact that the Seifert matrices of the two knots K and K^* with respect to the Seifert surfaces F and F^* coincide each other. Hence we have $\lambda'(K) = \lambda'(K^*)$, completing the proof.

Remark 3.6. (i) The definition of the mapping λ^* makes sense also on the Torelli group so that we have indeed a mapping $\lambda^*: \mathscr{I}_{g,1} \to \mathbb{Z}$ defined by $\lambda^*(\varphi) = \lambda(M_\varphi)$ $(\varphi \in \mathscr{I}_{g,1})$. For $g \ge 3$ this mapping is no longer a homomorphism. However we can completely describe its deviation from the additivity (see [26]).

(ii) In the above definition of the mapping λ^* we have fixed a Heegaard splitting of the 3-sphere: $S^3 = H_g \cup_{\iota_g} (-H_g)$. If we change the pasting map ι_g, then the resulting map λ^* may also change. However we can describe the variation completely (see Remark 6.3).

4. THE NORMALIZING TERM

In this section we investigate how Johnson's homomorphism $\tau_3: \mathscr{K}_{g,1} \to \bar{T}$, which we studied in §1, is related to the Casson invariant or to the associated homomorphism λ^*: $\mathscr{K}_{g,1} \to \mathbb{Z}$ defined in the previous section. It will turn out that although the homomorphism τ_3 contains a good deal of informations about the homomorphism λ^* (see Remark 6.3), it is *impossible* to recover λ^* from τ_3 because the latter misses the most important factor (the constant term of the algebra $\mathscr{A}$ defined below) which is in some sense the essence of the Casson invariant (cf. the last sentence of [11], where Johnson already observed the corresponding fact for the Rohlin invariant). Now the aim of this section is to construct a homomorphism $q: \mathscr{K}_{g,1} \to \mathbb{Q}$ out of τ_3 in such a way that the difference $\lambda^* - q$ should be as simple as possible. In view of Remark 6.3 stated in the last section, we may call the homomorphism q the normalizing term of the homormorphism λ^* with respect to the fixed Heegaard embedding $\Sigma_g \subset S^3$.

Now let $\psi \in \mathscr{K}_{g,1}$ be a Dehn twist on a bounding simple closed curve p on Σ_g^0 and let M_ψ be the corresponding homology 3-sphere. Also let F be the compact surface which p bounds on Σ_g^0. Then in view of Theorem 3.1, (iv), Proposition 3.2 and Lemma 3.4, the Casson invariant $\lambda(M_\psi) = \lambda^*(\psi)$ can be read off from the linking pairing on the homology of F. Keeping in mind this fact we define a commutative algebra $\mathscr{A}$ over $\mathbb{Z}$ with unit 1 as follows. $\mathscr{A}$ has a generator $l(u, v)$ for any two elements $u, v \in H$ and we require the following relations to hold in $\mathscr{A}$:

(i) $l(v, u) = l(u, v) + u \cdot v$

(ii) $l(n_1u_1 + n_2u_2, v) = n_1 l(u_1, v) + n_2 l(u_2, v)$, $(n_1, n_2 \in \mathbb{Z})$.

$\mathscr{A}$ is the universal model for the linking pairing on H and is the lift of Johnson's $\mathbb{Z}/2$ algebra $\mathscr{B}$ in [9]. Moreover it is easy to see that $\mathscr{A}$ is nothing but the polynomial algebra over $\mathbb{Z}$ generated by the elements $l(x_i, x_i)$, $l(y_i, y_i)$, $l(x_i, x_j)$ $(i < j)$, $l(y_i, y_j)$ $(i < j)$ and $l(x_i, y_j)$. The group $\mathrm{Sp}(2g; \mathbb{Z})$ acts naturally on $\mathscr{A}$. Motivated by the formula in Proposition 3.2, we define a homomorphism $\theta: T \to \mathscr{A}$ as follows, where recall that T is the submodule of $\Lambda^2 H \otimes \Lambda^2 H$ generated by $t_i^{\otimes 2}$ and $t_i \leftrightarrow t_j$ (see §1).

PROPOSITION 4.1. *The following two types of correspondences*

(i) $(u \wedge v)^{\otimes 2} \to l(u, u)l(v, v) - l(u, v)l(v, u)$

(ii) $a \wedge b \leftrightarrow c \wedge d \to l(a, c)l(b, d) + l(c, a)l(d, b) - l(a, d)l(b, c) - l(d, a)l(c, b)$

define a well defined $\mathrm{Sp}(2g; \mathbb{Z})$*-equivariant homomorphism* $\theta\colon T \to \mathscr{A}$.

Proof. The well-definedness follows from the next two facts. One is that the above correspondences are linear with respect to any variables u, v, a, b, c, d and the other is the fact that type (i) correspondence is symmetric with respect to u, v, while type (ii) one is skew symmetric with respect to a, b and c, d respectively. That the resulting homomorphism θ is $\mathrm{Sp}(2g; \mathbb{Z})$-equivariant follows directly from the definition.

PROPOSITION 4.2. *Let p be a bounding simple closed curve of genus h on Σ_g^0 and let $u_1, \ldots, u_h, v_1, \ldots, v_h$ be a symplectic basis of the homology of the compact surface which p bounds. Then*

$$\theta((u_1 \wedge v_1 + \ldots + u_h \wedge v_h)^{\otimes 2}) = \sum_{i=1}^{h} \{l(u_i, u_i)l(v_i, v_i) - l(u_i, v_i)l(v_i, u_i)\}$$
$$+ 2 \sum_{i < j \leq h} \{l(u_i, u_j)l(v_i, v_j) - l(u_i, v_j)l(u_j, v_i)\}.$$

Remark 4.3. In view of Proposition 1.2, we can conclude that the homomorphism $\theta\colon T \to \mathscr{A}$ is characterized by the above property.

Proof of Proposition 4.2. We use the induction on h. If $h = 1$, then the assertion is clear. Next we assume that the assertion holds for h and prove it for $h + 1$. For simplicity we write $L(h)$ for the right hand side of the equality in our Proposition. Then we have

$$\theta\left(\left(\sum_{i=1}^{h+1} u_i \wedge v_i\right)^{\otimes 2}\right) = \theta\left(\left(\sum_{i=1}^{h} u_i \wedge v_i\right)^{\otimes 2} + \sum_{i=1}^{h} u_i \wedge v_i \leftrightarrow u_{h+1} \wedge v_{h+1} + (u_{h+1} \wedge v_{h+1})^{\otimes 2}\right)$$
$$= L(h) + \sum_{i=1}^{h} \{l(u_i, u_{h+1})l(v_i, v_{h+1}) + l(u_{h+1}, u_i)l(v_{h+1}, v_i)$$
$$- l(u_i, v_{h+1})l(v_i, u_{h+1}) - l(v_{h+1}, u_i)l(u_{h+1}, v_i)\}$$
$$+ l(u_{h+1}, u_{h+1})l(v_{h+1}, v_{h+1}) - l(u_{h+1}, v_{h+1})l(v_{h+1}, u_{h+1})$$
$$= L(h + 1).$$

This completes the proof.

Next let $\varepsilon_0\colon \mathscr{A} \to \mathbb{Z}$ be the ring homomorphism defined by $\varepsilon_0(l(u, v)) = lk(u, v^+)$. Here we consider $\Sigma_g = \partial H_g$ as a Heegaard surface in $S^3 = H_g \cup_{\iota_g} (-H_g)$ (see §2). Let $\theta_0\colon T \to \mathbb{Z}$ be the homomorphism defined as $\theta_0 = \varepsilon_0 \theta$.

LEMMA 4.4. *The value of the homomorphism θ_0 on each element in the basis of T is given by* $\theta_0(x_i \wedge x_j \leftrightarrow y_i \wedge y_j) = 1$ *and* θ_0 *(other element)* $= 0$.

Proof. The linking numbers are given by $lk(x_i, y_i^+) = -1$, $lk(x_i, x_j^+) = lk(y_i, y_j^+) = 0$ and $lk(x_i, y_j) = 0$ $(i \neq j)$. Then the result follows from a direct computation using the definition of the homomorphism θ together with the above facts.

Now we summarize the above results as

PROPOSITION 4.5. *Let $\psi \in \mathscr{K}_{g,1}$ be a Dehn twist on a bounding simple closed curve p on Σ_g^0 and let $u_1, \ldots, u_h, v_1, \ldots, v_h$ be a symplectic basis of the homology of the compact surface which p bounds. Define $t = (u_1 \wedge v_1 + \ldots + u_h \wedge v_h)^{\otimes 2} \in T$. Then we have $\lambda^*(\psi) = -\theta_0(t)$.*

Proof. By Theorem 3.1, (iv) and Lemma 3.4, we have

$$\lambda(\psi) = -\sum_{i=1}^{h} \{lk(u_i, u_i^+)lk(v_i, v_i^+) - lk(u_i, v_i^+)lk(v_i, u_i^+)\}$$
$$-2 \sum_{i < j \leq h} \{lk(u_i, u_j^+)lk(v_i, v_j^+) - lk(u_i, v_j^+)lk(v_i, u_j^+)\}.$$

Then Proposition 4.2 and the definition of the homomorphism $\theta_0\colon T \to \mathbb{Z}$ imply $\lambda^*(\psi) = -\theta_0(t)$. This completes the proof.

Now recall that $\bar{T}$ is the image of T in $\mathscr{L}_3 \otimes H$ under the projection $\Lambda^2 H \otimes \Lambda^2 H \to \mathscr{L}_3 \otimes H$ and we have Johnson's homomorphism $\tau_3\colon \mathscr{K}_{g,1} \to \bar{T}$ (see §1). In the situation of Proposition 4.5, let $\bar{t} \in \bar{T}$ be the image of the element t. Then Proposition 1.1 implies that $\tau_3(\psi) = -\bar{t}$. Hence we are naturally led to consider the value of the homomorphism θ on $\mathrm{Ker}\,(T \to \bar{T})$. First we prepare

PROPOSITION 4.6. $\mathrm{Ker}\,(\Lambda^2 H \otimes \Lambda^2 H \to \mathscr{L}_3 \otimes H)$ *is generated by the* $\mathrm{Sp}(2g; \mathbb{Z})$*-orbits of the following three elements.*

$$s_1 = x_1 \wedge y_1 \leftrightarrow x_2 \wedge y_2 - x_1 \wedge x_2 \leftrightarrow y_1 \wedge y_2 + x_1 \wedge y_2 \leftrightarrow y_1 \wedge x_2$$
$$s_2 = x_1 \wedge y_1 \leftrightarrow x_2 \wedge x_3 - x_1 \wedge x_2 \leftrightarrow y_1 \wedge x_3 + x_1 \wedge x_3 \leftrightarrow y_1 \wedge x_2$$
$$s_3 = x_1 \wedge x_2 \leftrightarrow x_3 \wedge x_4 - x_1 \wedge x_3 \leftrightarrow x_2 \wedge x_4 + x_1 \wedge x_4 \leftrightarrow x_2 \wedge x_3.$$

Proof. An easy argument in linear algebra shows that $\mathrm{Ker}\,(\Lambda^2 H \otimes \Lambda^2 H \to \mathscr{L}_3 \otimes H) = \Lambda^2 H \otimes \Lambda^2 H \cap \Lambda^3 H \otimes H$ is generated by the elements of the following form

$$a \wedge b \leftrightarrow c \wedge d - a \wedge c \leftrightarrow b \wedge d + a \wedge d \leftrightarrow b \wedge c \quad (a, b, c, d \in H).$$

The required assertion follows from this.

LEMMA 4.7. *The values $\theta(s_i) \in \mathscr{A}$ $(i = 1, 2, 3)$ are given by*

$$\theta(s_1) = -1, \theta(s_2) = \theta(s_3) = 0.$$

Proof. Direct computation.

Now observe that the homomorphism $\theta_0\colon T \to \mathbb{Z}$ is *not* $\mathrm{Sp}(2g; \mathbb{Z})$-equivariant. We seek for an $\mathrm{Sp}(2g; \mathbb{Z})$-equivariant homomorphism $T \to \mathbb{Z}$ such that its values on the elements s_i are proportional to those of θ_0. We find

PROPOSITION 4.8. *The following two types of correspondences*

(i) $\bar{d}((u \wedge v)^{\otimes 2}) = 0$

(ii) $\bar{d}(a \wedge b \leftrightarrow c \wedge d) = (a \cdot b)(c \cdot d) - (a \cdot c)(b \cdot d) + (a \cdot d)(b \cdot c)$

define a well defined $\mathrm{Sp}(2g; \mathbb{Z})$ *– equivariant homomorphism $\bar{d}\colon T \to \mathbb{Z}$ such that $\bar{d}(s_1) = 3$ and $\bar{d}(s_2) = \bar{d}(s_3) = 0$. Moreover we have $\bar{d}((u_1 \wedge v_1 + \cdots + u_h \wedge v_h)^{\otimes 2}) = \frac{1}{2}h(h-1)$ for any symplectic subbasis $u_1, \ldots, u_h, v_1, \ldots, v_h$ of H.*

Proof. The well-definedness of the homomorphism $\bar{d}$ follows from a similar argument as in the proof of Proposition 4.1. The values of $\bar{d}$ on s_i are obtained by a direct computation. The last assertion follows from an easy induction argument.

Remark 4.9. It can be shown that the above homomorphism $\bar{d}: T \to \mathbb{Z}$ can be characterized by the following two conditions: one is that it is $\mathrm{Sp}(2g; \mathbb{Z})$-equivariant and the other is their values on the elements s_i.

Next let us define a homomorphism $q_0: T \to \mathbb{Q}$ by setting $q_0 = \theta_0 + \frac{1}{3}\bar{d}$.

PROPOSITION 4.10. *The homomorphism q_0 is trivial on* $\mathrm{Ker}(T \to \bar{T})$ *so that it defines a homomorphism* $\bar{q}_0: \bar{T} \to \mathbb{Q}$.

Proof. In view of the values of the homomorphisms $\theta: T \to \mathscr{A}$ and $\bar{d}: T \to \mathbb{Z}$ on the elements s_1, s_2 and s_3 given in Lemma 4.7 and Proposition 4.8, the result follows from the fact that the homomorphisms θ and $\bar{d}$ are both $\mathrm{Sp}(2g; \mathbb{Z})$-equivariant (see Propositions 4.1 and 4.8). Here the point is that the values $\theta(s_i)$ are *constants* as elements of the polynomial algebra $\mathscr{A}$. (As was already mentioned before, the homomorphism $\theta_0: T \to \mathbb{Z}$ is *not* $\mathrm{Sp}(2g; \mathbb{Z})$-equivariant.) This completes the proof.

Now we define a homomorphism $q: \mathscr{K}_{g,1} \to \mathbb{Q}$ to be the composition $\bar{q}_0\tau_3$: $\mathscr{K}_{g,1} \to \bar{T} \to \mathbb{Q}$.

5. A MAPPING $d: \mathcal{M}_{g,1} \to \mathbb{Z}$.

In the previous section we have modified the homomorphism $\theta_0: T \to \mathbb{Z}$, which is closely related to the homomorphism $\lambda^*: \mathscr{K}_{g,1} \to \mathbb{Z}$, so as to obtain a uniquely defined homomorphism $\bar{d}: T \to \mathbb{Z}$ which is $\mathrm{Sp}(2g; \mathbb{Z})$-equivariant and whose values on s_i are proportional to those of θ_0 (see Proposition 4.8 and Remark 4.9). In this section we prove the existence of a homomorphism $d: \mathscr{K}_{g,1} \to \mathbb{Z}$ which "realizes" the above homomorphism $\bar{d}$ geometrically. In fact we define a mapping $d: \mathcal{M}_{g,1} \to \mathbb{Z}$ whose domain is the whole mapping class group and not just its subgroup $\mathscr{K}_{g,1}$. It should be considered as a secondary invariant associated with the first characteristic class of surface bundles $e_1 \in H^2(\mathcal{M}_{g,1}; \mathbb{Z})$ introduced in [20]. Here recall that for each oriented Σ_g-bundle $\pi: E \to X$ it is defined as $e_1(\pi) = \pi_*(e^2) \in H^2(X; \mathbb{Z})$, where $e \in H^2(E; \mathbb{Z})$ is the Euler class of the tangent bundle along the fibres of π and $\pi_*: H^4(E; \mathbb{Z}) \to H^2(X; \mathbb{Z})$ is the Gysin homomorphism. Now we recall several results from our previous papers [21, 22]. First of all we have proved that $H^1(\mathcal{M}_{g,1}; H) \cong \mathbb{Z}$. Let us fix a crossed homomorphism $k: \mathcal{M}_{g,1} \to H$ which represents a generator of the above cohomology group as follows (see [21] for details). Recall that we have fixed a system of free generators $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$ of $\Gamma_1 = \pi_1(\Sigma_g^0)$ (see Fig. 1a). Define a mapping $\varepsilon: \Gamma_1 \to \mathbb{Z}$ by $\varepsilon(\gamma) = \Sigma_{i=1}^{r-1} [\gamma_1] \cdot [\gamma_{i+1} \cdots \gamma_r]$, where $\gamma = \gamma_1 \cdots \gamma_r$ is the expression of element $\gamma \in \Gamma_1$ in terms of the generators $\alpha_i^{\pm 1}$ and $\beta_i^{\pm 1}$. For each element $\varphi \in \mathcal{M}_{g,1}$, the correspondence $\Gamma_1 \ni \gamma \to \varepsilon(\varphi(\gamma)) - \varepsilon(\gamma) \in \mathbb{Z}$ defines a homomorphism $\varepsilon(\varphi): H \to \mathbb{Z}$. Then we define $k(\varphi) \in H$ by the equation $\varepsilon(\varphi)(u) = k(\varphi) \cdot u$ for all $u \in H$. It can be shown that the equality $k(\varphi\psi) = \psi_*^{-1}(k(\varphi)) + k(\psi)$ holds for any $\varphi, \psi \in \mathcal{M}_{g,1}$ so that the mapping $k: \mathcal{M}_{g,1} \to H$ is a crossed homomorphism.

Alternatively we can use the crossed homomorphism $k': \mathcal{M}_{g,1} \to H$ defined as follows (this crossed homomorphism is more suitable for computations by means of computers than the former one). We write γ_i $(i = 1, \ldots, 2g)$ for the free generators of Γ_1 we have fixed. For each map $\varphi \in \mathcal{M}_{g,1}$ consider the element $\varphi_{ij} = \partial/\partial\gamma_i(\varphi_*(\gamma_j)) \in \mathbb{Z}[\Gamma_1]$, where $\partial/\partial\gamma_i$ is the Fox free differential with respect to the element γ_i and $\mathbb{Z}[\Gamma_1]$ is the group ring of Γ_1. Let

$\bar{\varphi}_{ij} \in \mathbb{Z}[H]$ be the projection of φ_{ij} under the abelianization $\Gamma_1 \to H$. Then we define a matrix $\|\varphi\| \in \mathrm{GL}(2g; \mathbb{Z}[H])$ to be the one whose (i, j)-component is $\bar{\varphi}_{ij}$. It can be shown that $\|\varphi\psi\| = \|\varphi\|^{\varphi}\|\psi\|$, where ${}^{\varphi}\|\psi\|$ is the matrix obtained from $\|\psi\|$ by applying the automorphism $\varphi_*: \mathbb{Z}[H] \to \mathbb{Z}[H]$ on each component of $\|\psi\|$. (See Birman's book [2] for more detailed discussion of these constructions.) Now we define the crossed homomorphism $k': \mathcal{M}_{g,1} \to H$ by $k'(\varphi) = \det \|\varphi^{-1}\| \in H$.

The crossed homomorphism k (or k') vanishes on $\mathrm{Ker}(\mathcal{M}_{g,1} \to \mathcal{M}_{g,*})$, where $\mathcal{M}_{g,*} = \pi_0(\mathrm{Diff}_+(\Sigma_g, *))$ is the mapping class group of Σ_g with respect to the base point $* \in D$. Hence it induces a crossed homomorphism $k: \mathcal{M}_{g,*} \to H$. If we restrict k to the Torelli group $\mathscr{I}_{g,*} \subset \mathcal{M}_{g,*}$, then we obtain a homomorphism $k: \mathscr{I}_{g,*} \to H$ (which is independent of the choice of k) and in [21, 22] we proved that the equality $k(\gamma) = (2 - 2g)[\gamma]$ holds for all $\gamma \in \pi_1(\Sigma_g) \subset \mathscr{I}_{g,*}$. Also it was proved in [22] that the homomorphism $k: \mathscr{I}_{g,1} \to H$ coincides with Johnson's homomorphism $t: \mathscr{I}_{g,1} \to H$ given in [10], §5 up to signs. It follows that the crossed homomorphism k is trivial on the subgroup $\mathscr{K}_{g,1}$. Alternatively this fact can be duduced directly from the definition of k along the lines of the proof of Proposition 1.1. Also it is easy to see that k is "stable with respect to g" in the sense that the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{M}_{g,1} & \xrightarrow{k} & H_1(\Sigma_g; \mathbb{Z}) \\ \downarrow & & \downarrow \\ \mathcal{M}_{g+1,1} & \xrightarrow[k]{} & H_1(\Sigma_{g+1}; \mathbb{Z}). \end{array}$$

Next in [22] we proved that the cohomology class $e_1 \in H^2(\mathcal{M}_{g,1}; \mathbb{Z})$ is represented by the 2-cocycle c of $\mathcal{M}_{g,1}$ given by

$$c(\varphi, \psi) = k(\varphi) \cdot k(\psi^{-1}) \qquad (\varphi, \psi \in \mathcal{M}_{g,1}).$$

Finally we have shown that $i^*(e_1) = 0$ in $H^2(\mathscr{I}_{g,1}; \mathbb{Z})$ where $i: \mathscr{I}_{g,1} \to \mathcal{M}_{g,1}$ is the inclusion. Therefore there exists a mapping $d: \mathscr{I}_{g,1} \to \mathbb{Z}$ such that $\delta d = i^*(c)$. At this stage it is well defined only up to elements of $H^1(\mathscr{I}_{g,1}; \mathbb{Z})$, however there exists a canonical choice for it as will be seen below. Let $\pi: E \to X$ be an oriented Σ_g-bundle over a closed oriented surface X so that E is an oriented closed 4-manifold. Then it is easy to deduce from the definition of e_1 that

$$\langle [X], e_1 \rangle = 3 \operatorname{sign} E.$$

Namely the cohomology class $e_1 \in H^2(\mathcal{M}_{g,1}; \mathbb{Z})$ "represents" thrice of the signature of oriented surface bundles over surfaces. On the other hand sign E depends only on the action of $\pi_1(X)$ on the *homology* of the fibre. In fact Meyer [16] defined a 2-cocyle τ of the symplectic group $\mathrm{Sp}(2g; \mathbb{Z})$ which represents $-$sign E as follows. For each element φ, $\psi \in \mathrm{Sp}(2g; \mathbb{Z})$ define

$$V_{\varphi,\psi} = \{(u, v) \in H \oplus H; (\varphi^{-1} - 1)u + (\psi - 1)v = 0\}.$$

Then the pairing $V_{\varphi,\psi} \times V_{\varphi,\psi} \to \mathbb{Z}$ defined by $((u_1, v_1), (u_2, v_2)) \to (u_1 + v_1) \cdot (1 - \psi)v_2$ ($(u_i, v_i) \in V_{\varphi,\psi}, i = 1, 2$) turns out to be symmetric and bilinear and he defines $\tau(\varphi, \psi)$ to be its signature. The cocycle τ is stable with respect to g in an obvious sense. If we consider τ as a 2-cocycle of $\mathcal{M}_{g,1}$ via the representation $\tau_1: \mathcal{M}_{g,1} \to \mathrm{Sp}(2g; \mathbb{Z})$, then we can conclude that there exists a mapping

$$d: \mathcal{M}_{g,1} \to \mathbb{Z}$$

such that $\delta d = c + 3\tau$. Moreover since the group $\mathcal{M}_{g,1}$ is perfect for $g \geq 3$ (see [7]), such a mapping is *unique*. Also d is stable with respect to g, namely the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{M}_{g,1} & \xrightarrow{d} & \mathbb{Z} \\ \downarrow & \nearrow_{d} & \\ \mathcal{M}_{g+1,1} & & \end{array}$$

If we represent any element $\varphi \in \mathcal{M}_{g,1}$ as a product of commutators, then we can evaluate the number $d(\varphi)$ explicitly. For example we have shown that for any of the Lickorish generators of $\mathcal{M}_{g,1}$ (see [13]), the value of d is equal to 3 (see [25]).

PROPOSITION 5.1. *For two elements $\varphi, \psi \in \mathcal{M}_{g,1}$, we have*

(i) $d(\varphi\psi) = d(\varphi) + d(\psi) + k(\varphi)\cdot\psi_* k(\psi) - 3\tau(\varphi, \psi)$
(ii) $d(\varphi^{-1}) = -d(\varphi)$
(iii) $d(\varphi\psi\varphi^{-1}) = d(\psi) + k(\varphi)\cdot(\psi_* k(\psi) + k(\varphi\psi))$
(iv) *If $\varphi, \psi \in \mathcal{I}_{g,1}$, then $d([\varphi, \psi]) = 2k(\varphi)\cdot k(\psi)$.*

Proof. Since $\partial(\varphi, \psi) = (\psi) - (\varphi\psi) + (\varphi)$, we have $d(\varphi\psi) = d(\varphi) + d(\psi) - (c + 3\tau)(\varphi, \psi)$. (i) follows from this. (ii) is an easy consequence of (i) together with the facts that $d(\mathrm{id}) = 0$ and $\tau(\varphi, \varphi^{-1}) = 0$ for any $\varphi \in \mathcal{M}_{g,1}$. To prove (iii), using (i) and (ii) we compute

$$\begin{aligned} d(\varphi\psi\varphi^{-1}) &= d(\varphi\psi) + d(\varphi^{-1}) + k(\varphi\psi)\cdot\varphi_*^{-1}k(\varphi^{-1}) - 3\tau(\varphi\psi, \varphi^{-1}) \\ &= d(\psi) + k(\varphi)\cdot\psi_* k(\psi) - 3\tau(\varphi, \psi) + k(\varphi\psi)\cdot\varphi_*^{-1}k(\varphi^{-1}) - 3\tau(\varphi\psi, \varphi^{-1}) \\ &= d(\psi) + k(\varphi)\cdot(\psi_* k(\psi) + k(\varphi\psi)). \end{aligned}$$

Here we have used the fact that $\tau(\varphi, \psi) + \tau(\varphi\psi, \varphi^{-1}) = 0$ (see [16], p. 245). Finally we prove (iv). If $\varphi, \psi \in \mathcal{I}_{g,1}$, then (iii) implies $d(\varphi\psi\varphi^{-1}) = d(\psi) + 2k(\varphi)\cdot k(\psi)$. Hence $d([\varphi, \psi]) = d(\varphi\psi\varphi^{-1}) + d(\psi^{-1}) + k(\varphi\psi\varphi^{-1})\cdot k(\psi^{-1}) = 2k(\varphi)\cdot k(\psi)$. This completes the proof.

Remark 5.2. By virtue of the above Proposition, if we represent any element $\varphi \in \mathcal{M}_{g,1}$ in terms of the Lickorish generators, then we can compute $d(\varphi)$ explicitly.

THEOREM 5.3. *Let $\mathcal{K}_{g,1}$ be the subgroup of the mapping class group $\mathcal{M}_{g,1}$ generated by all the Dehn twists on bounding simple closed curves. Then the mapping $d: \mathcal{K}_{g,1} \to \mathbb{Z}$ is a homomorphism. Moreover if $\psi \in \mathcal{K}_{g,1}$ is a Dehn twist on a bounding simple closed curve of genus h, then*

$$d(\psi) = 4h(h-1).$$

Proof. The mapping $d: \mathcal{K}_{g,1} \to \mathbb{Z}$ is a homomorphism because the 2-cocycles c and τ are zero on $\mathcal{K}_{g,1}$. Next according to Proposition 5.1, (iii), we have $d(\varphi\psi\varphi^{-1}) = d(\psi)$ for any $\psi \in \mathcal{K}_{g,1}$ and $\varphi \in \mathcal{M}_{g,1}$. Also the mapping d is stable with respect to g. Hence to prove the required assertion it is enough to prove the following. Namely if ψ is the Dehn twist on a simple closed curve on Σ_g^0 which is parallel to the boundary, then $d(\psi) = 4g(g-1)$. Now we assume first that $g \geq 2$ and let $\bar{\alpha}_i, \bar{\beta}_i$ be the elements of $\pi_1(\Sigma_g)$ which are the images of α_i, β_i under the projection $\pi_1(\Sigma_g^0) \to \pi_1(\Sigma_g)$. We can consider $\bar{\alpha}_i, \bar{\beta}_i$ as elements of $\mathcal{I}_{g,*}$ because $\pi_1(\Sigma_g)$ is naturally a subgroup of $\mathcal{I}_{g,*}$. Now choose elements $\tilde{\alpha}_i, \tilde{\beta}_i$ of $\mathcal{I}_{g,1}$ such that they project to $\bar{\alpha}_i, \bar{\beta}_i$ under the projection $\mathcal{I}_{g,1} \to \mathcal{I}_{g,*}$. Then a direct computation shows that

$$[\tilde{\alpha}_1, \tilde{\beta}_1] \cdots [\tilde{\alpha}_g, \tilde{\beta}_g] = \psi^{2g-2}$$

(Up to signs this also follows from [17].) If we apply Proposition 5.1 to the above equation,

we find

$$\begin{aligned} (2g-2)d(\psi) &= \sum_{i=1}^{g} 2k(\tilde{\alpha}_i)\cdot k(\tilde{\beta}_i) \\ &= \sum_{i=1}^{g} 2(2-2g)^2 x_i \cdot y_i \\ &= 2g(2-2g)^2. \end{aligned}$$

Hence $d(\psi) = 4g(g-1)$ as required. Next if $g = 1$ then ψ can be expressed as $(\lambda_1\mu_1\lambda_1)^4$ where as in §2 λ_1 (resp. μ_1) is the right handed Dehn twist on the longitude curve l_1 (resp. meridian curve m_1) on Σ_g^0 (see Fig. 2). Then a direct computation using Proposition 5.1 together with the fact that $d(\lambda_1) = d(\mu_1) = 3$ (see [25]) implies $d(\psi) = 0$. Alternatively this fact can be proved by applying our mapping d on a certain relation in $\mathcal{K}_{3,1}$, proved by Johnson ([8], §§IV, V), between Dehn twists on bounding simple closed curves of genus 1, 2 and 3 which is derived from the well known relation, due originally to Dehn, of the mapping class group of the 2-sphere with four open discs removed. This completes the proof.

In view of Proposition 1.1, Proposition 4.8 and the above theorem, we can say that the homomorphism $d: \mathcal{K}_{g,1} \to \mathbb{Z}$ is the geometric realization of the homomorphism $\bar{d}: T \to \mathbb{Z}$ defined in §4 (up to a non zero scalar).

6. MAIN THEOREM

Using the two homomorphisms $d: \mathcal{K}_{g,1} \to \mathbb{Z}$ defined in §5 and $q: \mathcal{K}_{g,1} \to \mathbb{Q}$ defined in §4, let us define a homomorphism $\delta: \mathcal{K}_{g,1} \to \mathbb{Q}$ by setting $\delta = \frac{1}{24}d + q$. The following is our main theorem.

THEOREM 6.1. (i) *Im δ is contained in $\mathbb{Z}$ so that we have a homomorphism $\delta: \mathcal{K}_{g,1} \to \mathbb{Z}$.*
(ii) *If two elements $\varphi, \psi \in \mathcal{K}_{g,1}$ are equivalent each other in the sense of Definition 2.1, then we have $\delta(\varphi) = \delta(\psi)$ so that we have a mapping $\delta: \mathcal{H}(3) \to \mathbb{Z}$. Namely δ gives rise to an integer valued invariant for oriented homology 3-spheres.*
(iii) *The mapping $\delta: \mathcal{H}(3) \to \mathbb{Z}$ coincides with the Casson invariant λ.*

Remark 6.2. Once we assume the existence of the Casson invariant λ, then the statement (ii) in the above theorem is an immediate consequence of (iii). The meaning of the above theorem is that we can prove the well-definedness of the invariant δ for oriented homology 3-spheres *without* assuming the existence of λ.

Proof of Proposition 6.1. Here we only prove (i) and (iii). The proof of (ii) will be given in the second paper of this series [25] mainly because of its length. Since both of the mappings δ and λ^* are homomorphisms from the group $\mathcal{K}_{g,1}$ to $\mathbb{Z}$ (or $\mathbb{Q}$) (see Proposition 3.5), to prove (i) and (iii), we have only to prove the following. Namely for each Dehn twist $\psi \in \mathcal{K}_{g,1}$ on a bounding simple closed curve p on Σ_g^0, we have $\delta(\psi) = \lambda^*(\psi)$. Now choose a symplectic basis $u_1, \ldots, u_h, v_1, \ldots, v_h$ of the homology of the compact surface which p bounds. Let $t = (u_1 \wedge v_1 + \ldots + u_h \wedge v_h)^{\otimes 2} \in T$ and let $\bar{t} \in \bar{T}$ be its image. Then we have $\tau_3(\psi) = -\bar{t}$ by Proposition 1.1 and $\lambda^*(\psi) = -\theta_0(t)$ by Proposition 4.5. On the other hand according to Proposition 4.10 we have

$$\begin{aligned} \theta_0(t) &= (\theta_0 + \tfrac{1}{3}\bar{d})(t) - \tfrac{1}{3}\bar{d}(t) \\ &= \bar{q}_0(\bar{t}) - \tfrac{1}{3}\bar{d}(t). \end{aligned}$$

But Proposition 4.8 and Theorem 5.3 imply $d(t) = \frac{1}{2}h(h-1) = \frac{1}{8}d(\psi)$. Hence we have

$$\theta_0(t) = -q(\psi) - \tfrac{1}{24}d(\psi).$$

Therefore $\lambda^* = \frac{1}{24}d + q = \delta$ as required. This completes the proof.

Remark 6.3. For each element $\varphi \in \mathcal{M}_{g,1}$, let $\lambda_\varphi^*\colon \mathcal{K}_{g,1} \to \mathbb{Z}$ be the homomorphism defined by $\lambda_\varphi^*(\psi) = \lambda^*(\varphi^{-1}\psi\varphi)(\psi \in \mathcal{K}_{g,1})$. Then by Theorem 6.1 together with the results of §§4, 5, we can write

$$\lambda_\varphi^* = \tfrac{1}{24}d + q_\varphi$$

where $q_\varphi\colon \mathcal{K}_{g,1} \to \mathbb{Q}$ is the homomorphism defined similarly as the homomorphism q: $\mathcal{K}_{g,1} \to \mathbb{Q}$. More precisely in the definition of q we just replace the ring homomorphism ε_0: $\mathcal{A} \to \mathbb{Z}$ by another one $\varepsilon_\varphi\colon \mathcal{A} \to \mathbb{Z}$ which is defined by $\varepsilon_\varphi(l(u, v)) = lk(\varphi_*(u), \varphi_*(v)^+)$ $(u, v \in H)$. The homomorphism q_φ should be considered as the normalizing term, in our expression of the Casson invariant, with respect to the Heegaard embedding $\Sigma_g \xrightarrow{\varphi} \Sigma_g \subset S^3$.

From the above formula we can conclude that the homomorphism $\tau_3\colon \mathcal{K}_{g,1} \to \bar{T}$ contains all the informations about the difference $\lambda^* - \lambda_\varphi^*$ for all $\varphi \in \mathcal{M}_{g,1}$.

Let us write $\mathcal{L}_{g,1}$ for the subgroup $\mathcal{M}(4) = \operatorname{Ker}\tau_3$ of $\mathcal{K}_{g,1}$ (see §1). Then we have

PROPOSITION 6.4. *$\lambda^* = \frac{1}{24}d$ on $\mathcal{L}_{g,1}$ and it is a nontrivial (in fact surjective) homomorphism $\mathcal{L}_{g,1} \to \mathbb{Z}$.*

Proof. We have only to prove the nontriviality. For that we consider the element $s_1 \in T$ which goes to zero in $\bar{T}$ (see Proposition 4.6). A similar argument as in the proof of Proposition 1.2, we can show that there are elements u_i, v_i with $u_i \cdot v_i = 1$ $(i = 1, \ldots, r$ for some $r)$ such that

$$\sum_{i=1}^{r} \pm (u_i \wedge v_i)^{\otimes 2} + 3(x_1 \wedge y_1 + x_2 \wedge y_2)^{\otimes 2} = s_1.$$

Hence there exists an element $\psi \in \mathcal{L}_{2,1}$ such that $d(\psi) = 24$, i.e. $\lambda^*(\psi) = 1$. This completes the proof.

Remark 6.5. The above element $\psi \in \mathcal{L}_{2,1}$ has the following interesting property. Namely for *any* embedding $i\colon \Sigma_2 \to S^3$ the homology 3-sphere obtained by cutting S^3 along $i(\Sigma_q)$ and regluing the resulting two pieces by the map ψ has Casson invariant 1 independent of the embedding i.

Remark 6.6. According to the computations the author has made so far, it seems to be very likely that $\mathcal{H}(3) = \lim_{g \to \infty} \mathcal{L}_{g,1}/\sim$. If this were true, in view of the above Proposition we can say that $\frac{1}{24}d$ is exactly equal to the Casson invariant.

As another application of the main theorem, we can answer a question of Johnson ([11], p. 172, Problem B) negatively.

COROLLARY 6.7. *The three-fold commutator subgroup $[[\mathcal{I}_{g,1}, \mathcal{I}_{g,1}], \mathcal{I}_{g,1}]$ of the Torelli group, which is a normal subgroup of $\mathcal{L}_{g,1}$, has an infinite index in $\mathcal{L}_{g,1}$.*

Proof. We have shown that the homomorphism $d\colon \mathcal{L}_{g,1} \to \mathbb{Z}$ is non-trivial. But it is easy to deduce from Proposition 5.1 that d is trivial on the three-fold commutator subgroup of the Torelli group. The result follows.

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