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Winding Numbers on Surfaces, I.

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§ 0. Introduction

This paper describes a theory of winding numbers for homotopy classes of closed curves on surfaces, based on ideas introduced by Reinhart in [5]. In the sequel (to appear) we give some applications of the theory to the problem of determining when a given element of the fundamental group of a compact surface contains simple closed curves (see [2] for an announcement of the result), and to the study of the homeotopy group (the group of homeomorphisms modulo those isotopic to the identity) of a closed surface.

In § 1 we define the winding number $\omega_X(\gamma, x)$ of a regular closed curve γ with respect to a non-vanishing continuous vector field X on a smooth surface M with base-point $x \in \gamma$. This is a natural generalization of the well-known "rotation number" of a regular closed curve in the plane (see e.g. [10]), and is essentially the definition given in [5]. Here, however, we include the case when the surface is non-orientable. In § 2 we show how to define a "winding number" for any element of $\pi_1(M, x)$, using a preferred family of regular closed curves in each homotopy class: when the homotopy class contains simple closed curves this coincides with the definition in [5]. A theorem of Smale is used in § 3 to deduce a generalization to surfaces of the Whitney-Graustein Theorem [10] on regular closed curves in the plane, proving for example that two homotopic orientation-preserving regular closed curves on M which are not nullhomotopic are regularly homotopic if they contain no null-homotopic loops. § 4 relates winding numbers to cohomology and homology, § 5 is concerned with the removal of base-point restrictions, and § 6 shows how the theory can be modified to cope with closed surfaces which may not admit continuous non-vanishing vector fields. In § 7 we give formulae for calculating the winding number of any given element of $\pi_1(M, x)$ in terms of the winding numbers of a set of generators. Some possible directions in which the theory could be generalized are mentioned in § 8.

We assume a familiarity with some elementary homotopy theory and algebraic and differential topology. By a *surface* we mean a connected separable 2-manifold, with or without boundary. Any compact orientable surface may be regarded as a (punctured) 2-sphere with handles attached. The *genus* of the surface is the number of handles. Any compact non-orientable surface may be regarded as a (punctured) 2-sphere with Möbius strips attached, and the *genus* in this case is the number of Möbius strips. This classification of compact surfaces is standard, and can be found in many books, for example [4], and also in [12]. A *path* in a surface M is a continuous map of a closed interval into M; by abuse of terminology we shall usually think of the path as the image of the map. A path is *simple* (often called an *arc*) if the map is an embedding. A *closed curve* is (the image of) a continuous map of a circle S^1 into M; it is *simple* if the map is an embedding. We think of S^1 as the unit circle in the complex plane, and as being equipped with a fixed orientation. Closed curves in M are thus automatically oriented. The boundary of M is denoted by ∂M. If M is a smooth manifold then $TM \to M$ denotes the tangent bundle, although we do not always distinguish between a bundle and its total space. A *vector field* is a section of the tangent bundle: all vector fields will be assumed to be continuous. The symbols $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{Z}_m$ denote the real line, the additive group of integers, and the cyclic group of order m, respectively.

This paper and its sequel constitute part of a Ph. D. thesis submitted to the University of Cambridge in 1968.

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§ 1. The Winding Number of a Regular Closed Curve

Let M denote a smooth (say C^1) surface. Until stated otherwise M is assumed to be non-compact or to have non-empty boundary and thus to admit (continuous) non-vanishing vector-fields.

(1.1) **Definitions.** A closed curve on M given by

$$f : S^1 \to M$$

is *regular* if f is C^1 and the tangent map

$$Tf : TS^1 \to TM$$

is injective on each fibre. In other words, continuously-varying non-zero tangents exist at all points of the curve.

Let v_x denote a vector in the tangent plane T_xM to M at $x \in M$, i.e. in the fibre of TM over x. A regular closed curve as above is *based at* v_x if $f(1) = x$ and $Tf((1,1)) = v_x$, where $(1,1) \in TS^1 = S^1 \times \mathbb{R}$. Thus v_x is a tangent to the (oriented) curve at x.

Let X be a non-vanishing vector-field on M, and let $\gamma = f(S^1)$ be a regular closed curve based at v_x, $x \in M$. The *winding number* $\omega_X(\gamma, x)$ may be defined intuitively as the total number of rotations that the tangent vector to γ makes relative to the X-vector as the curve γ is traversed once in the positive sense. The sign of $\omega_X(\gamma, x)$ depends on a choice of orientation for T_xM, this orientation being transported to $T_{x'}M$ for each $x' \in \gamma$ by the appropriate path xx' in γ.

We shall now make this definition more precise.

Let M be given some Riemannian structure, automatically inducing a norm $\| \ \|_x$ on T_xM for each $x \in M$, and let

$$T_0M = \bigcup_{x \in M} \{v \in T_xM \mid \|v\|_x = 1\}.$$

Any continuous map $f: S^1 \to M$ induces a pull-back over S^1 from T_0M over M, i.e. a bundle $E^f \to S^1$ with fibre S^1 such that the diagram

$$\begin{array}{ccc} E^f & \xrightarrow{F} & T_0M \\ \downarrow{\scriptstyle p^f} & & \downarrow{\scriptstyle p_1} \\ S^1 & \xrightarrow{f} & M \end{array}$$

commutes, where p_1 and p^f are bundle projections and F is an isomorphism on each fibre. The total space E^f is a torus or a Klein bottle according to whether $\gamma = f(S^1)$ is an orientation-preserving curve or not.

A non-vanishing vector field X defines a section X_0 of T_0M by $X_0(x) = X(x)/\|X(x)\|_x$, $x \in M$, and the composition

$$X_0 f: S^1 \to T_0M$$

pulls back to a unique section

$$X^f: S^1 \to E^f$$

such that $FX^f = X_0 f$. If γ is regular the map $T_0 f: S^1 \to T_0M$ defined by

$$T_0 f(z) = Tf((z,1))/\|Tf((z,1))\|_{f(z)}$$

also pulls back to a unique section

$$Z^f: S^1 \to E^f$$

such that $FZ^f = T_0 f$. Furthermore, if γ is based at $X(x) \in T_xM$ then $X^f(1) = Z^f(1) = e_0$, say, and so X^f and Z^f represent elements $\{X^f\}, \{Z^f\}$

of $\pi_1(E^f, e_0)$. Since X^f and Z^f are sections of E^f the element $\{Z^f\}\{X^f\}^{-1} \in \pi_1(E^f, e_0)$ belongs to the kernel of the homomorphism

$$p_*^f : \pi_1(E^f, e_0) \to \pi_1(S^1, 1)$$

induced by p^f. From the bundle exact sequence

$$0 = \pi_2(S^1, 1) \to \pi_1(E_0, e_0) \xrightarrow{i_*^f} \pi_1(E^f, e_0) \xrightarrow{p_*^f} \pi_1(S^1, 1),$$

where $i^f : E_0 \to E^f$ is the inclusion of the fibre E_0 over $1 \in S^1$, it follows that

$$\{Z^f\}\{X^f\}^{-1} = i_*^f w^f$$

for some unique $w^f \in \pi_1(E_0, e_0)$. A choice of orientation of $T_x M$ induces an orientation of $E_0 \cong S^1$, and w^f may then be thought of as an integer, which is defined to be the *winding number* $\omega_X(\gamma, x)$ *of* γ *(based at* x*) with respect to* X. It is easy to check that this corresponds to the intuitive definition above, and to the definition given by Reinhart [5].

Remarks. 1. The definition is clearly independent of the choice of Riemannian metric on M.

2. The bundle language, used here both for precision and in order to indicate possible generalizations (see § 8), tends to obscure the simple underlying geometry. We have $\pi_1(E^f, e_0) = \{a, b \mid aba^{-1}b^{-1} = 1\}$ if γ is orientation-preserving (E^f is a torus) or $\{a, b \mid aba^{-1}b = 1\}$ if γ is orientation-reversing (E^f is a Klein bottle). We may choose $a = \{X^f\}$ and b to be represented by E_0: then $p_*^f(a)$ generates $\pi_1(S^1, 1) \cong \mathbb{Z}$ and $p_*^f(b) = 0$. Since $p_*^f(\{Z^f\}) = p_*^f(a)$ it follows from the defining relations that $\{Z^f\} = b^m a$ for some unique integer m; then $\omega_X(\gamma, x) = m$.

3. Recall that γ is based at $X(x)$ and we are assuming M is non-compact or $\partial M \neq \emptyset$. We later investigate the affects of removing these conditions.

4. The definition of winding number may easily be extended from regular closed curves to piecewise regular closed curves: see [6]. A piecewise regular closed curve is a closed curve which is regular except for a finite number of points at which the tangent may vanish or be discontinuous. It is often helpful to use piecewise regular curves to calculate winding numbers of homotopy classes ([6]). The calculations in § 7 below, for example, can be carried out more easily without the necessity of approximating piecewise regular curves by regular ones.

We now note two elementary properties of $\omega_X(\gamma, x)$. Recall that all vector fields are assumed continuous.

(1.2) **Definition.** Two non-vanishing vector fields X, Y on M are *homotopic* (write $X \simeq Y$) if the maps $X, Y : M \to TM$ are homotopic via $X_t : M \to TM$ $(0 \leqq t \leqq 1)$ which are non-vanishing vector fields. They are *homotopic relative to* x or $rel\, x$ ($x \in M$) if $X_t(x)$ remains fixed throughout the homotopy.

(1.3) **Lemma.** $X \simeq Y\ rel\, x \Rightarrow \omega_X(\gamma, x) = \omega_Y(\gamma, x)$ *for any regular closed curve* γ *based at* $X(x) = Y(x)$.

Proof. Since $\omega_{X_t}(\gamma, x)$ is integer-valued and varies continuously with t it must be constant. Alternatively $\{X^f\} = \{Y^f\}$ (where $\gamma = f(S^1)$) so $\{Z^f\}\{X^f\}^{-1} = \{Z^f\}\{Y^f\}^{-1}$.

(1.4) **Definition.** Two regular closed curves $\gamma = f(S^1)$, $\delta = g(S^1)$ on M are *regularly homotopic* (write $\gamma \underset{R}{\simeq} \delta$) if f and g are homotopic via C^1 maps $f_t: S^1 \to M$ $(0 \leqq t \leqq 1)$ such that $Tf_t: TS^1 \to TM$ is injective on each fibre (so $\gamma_t = f_t(S^1)$ is regular) and varies continuously with respect to t. They are *regularly homotopic relative to* v_x or $rel\, v_x$ ($v_x \in T_x M$, $x \in M$) if they are based at v_x and $Tf_t((1, 1))$ remains fixed ($= v_x$) throughout the homotopy.

(1.5) **Lemma.** $\gamma \underset{R}{\simeq} \delta\ rel\, v_x \Rightarrow \omega_X(\gamma, x) = \omega_X(\delta, x)$ *for any non-vanishing vector field* X *on* M *such that* $X(x) = v_x$.

Proof. Since $\omega_X(\gamma_t, x)$ is integer-valued and varies continuously with t it must be constant. Alternatively one may construct a cumbersome algebraic argument using a bundle isomorphism $E^f \cong E^g$ induced by the homotopy between f and g.

§ 2. The Winding Number of an Element of $\pi_1(M, x)$

In this section we show how to pick out from each homotopy class of closed curves on M based at x a particular family of regular curves which are, roughly speaking, straightened out as much as possible. The main theorem then states that the winding number of such a regular curve (with respect to some given X) depends only on the original homotopy class. Using this we define winding numbers for elements of $\pi_1(M, x)$.

Let $\gamma = f(S^1)$ be any closed curve on M. A point $x \in \gamma$ (where x does not now necessarily denote the base-point) is a *self-intersection point* of γ if $f^{-1}(x)$ contains more than one point, or a *double point* if $f^{-1}(x)$ contains exactly two points. A *loop* in γ with *vertex* x is $f(\alpha)$ where α is an arc $\overline{zz'}$ ($z \neq z'$) in S^1 with $f(z) = f(z') = x$. The *complementary loop* is $\overline{f(S^1 - \alpha)}$.

(2.1) **Lemma.** *Let* γ *be a closed curve on* $S^1 \times \mathbb{R}$ *having at most a finite number of self-intersection points, each a double point. Suppose* γ *represents* ng *where* g *generates* $\pi_1(S^1 \times \mathbb{R}) \cong \mathbb{Z}$ *(with additive notation) and* $n > 0$. *Then* γ *contains a loop* λ *representing* g.

Proof. Let x be the vertex of a loop δ_x containing no other loop: such δ_x obviously exists. Since δ_x has no self-intersection points (except x) we must have $\{\delta_x\}=0$ or $\pm g$, where $\{\ \}$ as usual denotes homotopy class. If $\{\delta_x\}=+g$ take $\lambda=\delta_x$. If $\{\delta_x\}\neq +g$ we proceed by induction on the number $S(\gamma)$ of self-intersection points of γ. Clearly the lemma is true when $S(\gamma)=0$. Assume the lemma to be true for any γ with $S(\gamma)\leqq k$, and suppose now that γ has $k+1$ self-intersection points. Let ε_x be the complementary loop to δ_x in γ. Since $\{\varepsilon_x\}=ng$ or $(n+1)g$ and $S(\varepsilon_x)<S(\gamma)$ the induction hypothesis applies, and so there is a loop μ_y with vertex y in ε_x such that $\{\mu_y\}=g$. Now we consider two cases.

Case (i). $\{\delta_x\}=0$, $\{\varepsilon_x\}=ng$. If $x\notin\mu_y$ then μ_y is a loop in γ, so take $\lambda=\mu_y$. If $x\in\mu_y$ take λ to be that loop of γ with vertex y which contains x. Then $\{\lambda\}=\{\delta_x\}+\{\mu_y\}=0+g=g$.

Case (ii). $\{\delta_x\}=-g$, $\{\varepsilon_x\}=(n+1)g$. If $x\notin\mu_y$ take $\lambda=\mu_y$. If $x\in\mu_y$ let v_y be the loop of γ with vertex y containing x. Then $\{v_y\}=\{\delta_x\}+\{\mu_y\}=g-g=0$. Let α_y be the complementary loop to v_y in γ. Then by the induction hypothesis (since $\{\alpha_y\}=ng$) there is a loop β_t with vertex t in α_y such that $\{\beta_t\}=g$. If $y\notin\beta_t$ take $\lambda=\beta_t$. If $y\in\beta_t$ take λ to be that loop of γ with vertex t that contains y (and x). Then $\{\lambda\}=\{v_y\}+\{\beta_t\}=0+g=g$.

(2.2) **Corollary.** *If $n=1$ and γ is not simple then γ contains a nullhomotopic loop.*

Proof. The complementary loop to λ in γ is nullhomotopic since $\{\lambda\}=g=\{\gamma\}$.

Let $\gamma=f(S^1)$ be a closed curve in M with $f(1)=x\in M$. Let $p: M^*\to M$ be a covering of M corresponding to the cyclic subgroup of $\pi_1(M,x)$ generated by the homotopy class $\{\gamma\}$. Let γ^* be a closed curve covering γ so that $\{\gamma^*\}$ generates $\pi_1(M^*,x^*)$ where $x^*=\gamma^*\cap p^{-1}(x)$.

(2.3) **Definition.** An orientation-preserving closed curve γ on M is *direct* if the corresponding closed curve γ^* on M^* is simple. An orientation-reversing closed curve γ is *direct* if 2γ (the curve obtained by going twice round γ) is direct.

We now give a geometric characterization of direct curves which is more convenient than the definition for use in applications. It follows from the characterization that the definition does not depend on the choice of γ^*.

(2.4) **Lemma.** *An orientation-preserving closed curve γ with at most a finite number of self-intersection points, each a double point, is direct if and only if it contains no nullhomotopic loop.*

Proof. If γ contains a nullhomotopic loop so does γ^* by the homotopy lifting property.

If γ is itself nullhomotopic then the interior of M^* is homeomorphic to $\mathbb{R}^2$ and so if γ^* is not simple it contains a nullhomotopic loop. (We need not assume γ lies in the interior of M.) If γ is not nullhomotopic then the interior of M^* is homeomorphic to $S^1 \times \mathbb{R}$ and from Corollary (2.2) it follows that if γ^* is not simple it contains a nullhomotopic loop. In both cases projection by p gives a nullhomotopic loop in γ.

Note. The above lemma requires that γ be orientation-preserving. One may construct on a Möbius strip a curve as shown in Fig. 1 which generates the fundamental group and has two self-intersections yet contains no nullhomotopic loop.

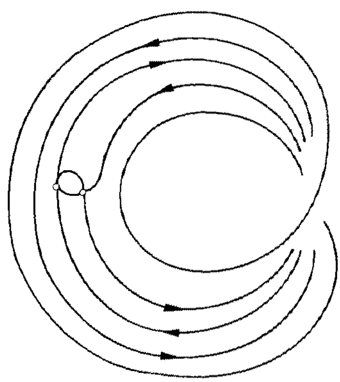

Fig. 1

(2.5) **Lemma.** *Every homotopy class* $c \in \pi_1(M, x)$ *contains a direct regular curve.*

Proof. The lemma clearly holds when c is the identity element $1 \in \pi_1(M, x)$, so assume $c \neq 1$. The interior of M^* is homeomorphic to $S^1 \times \mathbb{R}$ or a Möbius strip. In both cases a generator of $\pi_1(M^*, x^*)$ can be represented by a regular simple closed curve (where M^* is given the smooth structure induced from that of M by p) and $p(\gamma^*)$ represents c and is direct.

(2.6) **Lemma.** *Let* X *be any non-vanishing vector field on* M, *and* $x \in M$. *If* γ *is a nullhomotopic simple closed regular curve based at* $X(x)$ *then*

$$\omega_X(\gamma, x) = \pm 1 ,$$

the sign depending on the orientation of γ.

Proof. The proof is essentially that given by Reinhart in [5]. By attaching a collar smoothly to ∂M and extending X over the collar we may assume that γ lies in the interior M_0 of M. Let M_0' ($\cong \mathbb{R}^2$) be the universal cover of M_0 with smooth structure induced from M_0 by the projection $p_0 : M_0' \to M_0$. Let $x' \in p_0^{-1}(x)$ and let γ' be the connected component of $p_0^{-1}(\gamma)$ containing x'. Since γ is nullhomotopic, γ' is a (simple) closed curve. Now p_0 induces a non-vanishing vector field X' on M_0' from X on M_0, and $X' \simeq X_0' \, rel \, x'$ where X_0' is a field of parallel vectors, since $\mathbb{R}^2$ is contractible. Clearly

$$\omega_X(\gamma, x) = \omega_{X'}(\gamma' x')$$

and by Lemma (1.3)

$$\omega_{X'}(\gamma', x') = \omega_{X_0'}(\gamma', x').$$

The proof is now completed by using the classical result due to Whitney [10] which states that

$$\omega_{X_0'}(\gamma', x') = \pm 1,$$

the sign depending on the orientation of γ'.

We next come to the main result which is the key to the theory of winding numbers which follows.

(2.7) **Theorem.** *Let X be a non-vanishing vector field on M, and let γ_1, γ_2 be regular closed curves based at $X(x)$ ($x \in M$). Suppose $\{\gamma_1\} = \{\gamma_2\}$ but $\{\gamma_1\} \neq 1 \in \pi_1(M, x)$, and suppose that γ_i is direct ($i = 1, 2$). Then*

$$\omega_X(\gamma_1, x) = \omega_X(\gamma_2, x)$$

if the γ_i are orientation-preserving, or

$$\omega_X(\gamma_1, x) \equiv \omega_X(\gamma_2, x) \, mod 2$$

if the γ_i are orientation-reversing.

Using this theorem we are able to make the following definition:

(2.8) **Definition.** Let $c \neq 1 \in \pi_1(M, x)$ and let X be a non-vanishing vector field on M. The *winding number of c with respect to X*, denoted by $\omega_X(c)$, is defined to be

$$\omega_X(\gamma, x)$$

if M is orientable, or

$$\omega_X(\gamma, x) \quad \text{reduced} \quad mod 2$$

if M is non-orientable, where γ is any direct regular curve in M based at $X(x)$ and representing c.

Theorem (2.7) shows that $\omega_X(c)$ is thus well defined ($c \neq 1$) as an element of $\mathbb{Z}$ when M is orientable or of $\mathbb{Z}_2$ when M is non-orientable. By convention we define

$$\omega_X(1) = 1 \in \mathbb{Z} \quad \text{or} \quad \mathbb{Z}_2 .$$

Clearly we could define $\omega_X(c) \in \mathbb{Z}$ whenever c contains orientation-preserving curves, but it turns out to be more convenient to reduce everything *mod* 2 when M is non-orientable.

Proof of the Theorem. In view of Lemma (1.5) we may, by first altering γ_1 and γ_2 by regular homotopies if necessary, assume that $\gamma_1 \cap \gamma_2$ consists of a finite number of points none of which is a self-intersection point of γ_1 or γ_2, and that γ_1 and γ_2 do not cross (although they touch) at the base-point x. As in Lemma (2.6) we may also assume $\gamma_1 \cup \gamma_2 \subset M_0$.

Choose $x' \in p_0^{-1}(x) \subset M_0'$ and let γ_i' be a path in M_0' starting at x' obtained by lifting γ_i, $i = 1, 2$. Each γ_i' is a simple arc since γ_i is direct, and γ_1' and γ_2' have the same end-point since $\{\gamma_1\} = \{\gamma_2\} \in \pi_1(M, x)$. Let t be the covering translation of M_0' corresponding to $\{\gamma_1\}$; we have $t^n \neq 0$ for all $n \neq 0$ since $\pi_1(M, x)$ has no non-trivial elements of finite order. (When M is compact ($\partial M \neq \emptyset$) this follows immediately from the fact that $\pi_1(M, x)$ is a free group since M deformation retracts onto a 1-complex; when M is non-compact note that if $n\gamma$ is nullhomotopic in M then it is nullhomotopic in a compact surface M_1 ($\partial M_1 \neq \emptyset$) contained in M, and so γ is nullhomotopic in M_1 and hence in M.) Define

$$T_m(\gamma_i) = \bigcup_{n=0}^{m} t^n(\gamma_i') \qquad (i = 1, 2;\ m = 0, 1, 2, \ldots).$$

Since M_0' is also the universal cover of the interior of M^* (see before Definition (2.3)) and γ_i is direct it follows that each $T_m(\gamma_i)$ is an arc.

We consider two cases separately.

Case (i): γ_i *orientation-preserving.* Let Γ_m denote the closed curve in M_0' consisting of $T_m(\gamma_1)$ followed by $T_m(\gamma_2)$ reversed, and let Γ_m' be a regular closed curve which differs from Γ_m only inside $U \cup t^{m+1} U$ where U is a neighbourhood of x' containing no point of

$$(\Gamma_m(\gamma_1) \cap \Gamma_m(\gamma_2)) \cup t^{-(m+1)}(\Gamma_m(\gamma_1) \cap \Gamma_m(\gamma_2))$$

except x'. Choose Γ_m' to have no self-intersections inside $U \cup t^{m+1} U$, and to be based at $X'(x')$, where X' is as before the vector field on M_0' induced from X by p_0. Thus Γ_m' is to be thought of as the result of "rounding off" the cusps of Γ_m at x' and $t^{m+1}(x')$. By suitably choosing Γ_m' the winding number $\omega_{X'}(\Gamma_m', x')$ can be made arbitrarily close to

$$(m+1)\,(\omega_X(\gamma_1, x) - \omega_X(\gamma_2, x)) \pm 1$$

since the contribution to $\omega_{X'}(\Gamma_m', x')$ from each $t^n(\gamma_1')$, $t^n(\gamma_2')$ is $\omega_X(\gamma_1, x)$, $-\omega_X(\gamma_2, x)$ respectively, and the effect of rounding off the cusps is to contribute ± 1 depending on the side from which γ_2 meets γ_1 at x. (We are assuming that M_0' is given the orientation induced by p_0 from the chosen orientation of M near x.) Since $\omega_{X'}(\Gamma_m', x') \in \mathbb{Z}$ we must in fact have

$$\omega_{X'}(\Gamma_m', x') = (m+1)\left(\omega_X(\gamma_1, x) - \omega_X(\gamma_2, x)\right) + \varepsilon \tag{2.9}$$

where $\varepsilon = \pm 1$.

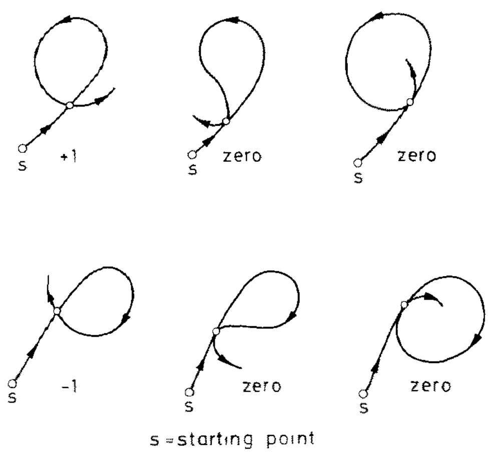

Fig. 2

We now apply Whitney's formula [10] which, after a trivial adjustment to take account of points of non-transverse self-intersection, states that the winding number with respect to a field of parallel vectors (and hence with respect to any non-vanishing vector field) of a regular closed curve in the plane having a finite number of self-intersections points, each a double point, is equal to

$$\pm 1 + (\text{algebraic number of self-intersections})$$

where the sign depends on the orientation of the curve, and the self-intersections are counted algebraically as indicated in Fig. 2 after an appropriate choice of starting point on the curve. The starting-point must be chosen to be an "outside" point, i.e. in the closure of the unbounded component of the complement of the curve[1]. Let $y' = (\gamma_1' \cup \gamma_2') \cap p_0^{-1}(y)$ where $y \in \gamma_1 \cup \gamma_2$ and lies in the closure of a component of

[1] In [10] the starting point must lie on a straight line having every point of the curve on it or on a single side of it. However, a regular homotopy of the curve introducing no further self-intersections can be chosen to take a starting point as we have defined it to a starting point of Whitney type, so the formula remains valid with our definition.

$M_0-(\gamma_1\cup\gamma_2)$ which has non-compact closure in M_0 (note that the hypotheses on M imply that M_0 is non-compact): then y' is an outside point of Γ'_m for all m. Assume without loss of generality that $y'\in\gamma'_1$. Every self-intersection point of Γ'_m belongs to $T_m(\gamma_1)\cap T_m(\gamma_2)$ since each $T_m(\gamma_i)$ is simple. Let the algebraic number of intersections of γ'_1 and $t^n\gamma'_2$ be denoted by k_n $(-m\leqq n\leqq m)$, the intersections being counted algebraically as in Fig. 3.

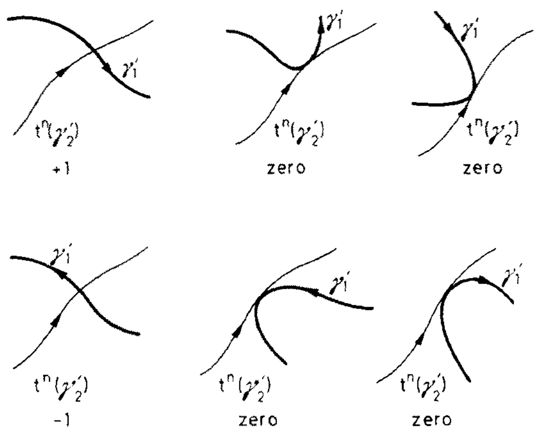

Fig. 3

Since the number of points of $\gamma_1\cap\gamma_2$ is finite there exists some integer $m_0>0$ such that $\gamma'_1\cap t^m\gamma'_2=\emptyset$ if $m>|m_0|$. The algebraic number of self-intersections of Γ'_m arising from $t^p(\gamma'_1)\cap t^q(\gamma'_2)$ $(1\leqq p\leqq m,\ 0\leqq q\leqq m)$ is k_{q-p} since

$$t^p(\gamma'_1)\cap t^q(\gamma'_2)=t^p(\gamma'_1\cap t^{q-p}(\gamma'_2))$$

and t^p preserves orientation, but the number l_q arising from $\gamma'_1\cap t^q(\gamma'_2)$ may not be equal to k_q since there may exist intersection points between x' and y' on γ'_1. Hence for each $m=0,1,2,\dots$ we have from (2.9)

$$\begin{aligned}(m+1)(\omega_X(\gamma_1,x)-\omega_X(\gamma_2,x))=\pm 1+\sum_{n=0}^{m} l_n+\sum_{n=0}^{m-1}(m-n)k_n\\ +\sum_{n=1}^{m}(m-n+1)k_{-n}-\varepsilon\,.\end{aligned}$$

Replacing m by $m+1$ and subtracting, we obtain

$$\omega_X(\gamma_1,x)-\omega_X(\gamma_2,x)=l_{m+1}+\sum_{n=-(m+1)}^{m} k_n\,,$$

the sign of ± 1 being the same in both cases since y' is an outside starting point for both Γ'_m and Γ'_{m+1}. However, if $m > m_0$ the right hand side is zero because $l_{m+1} = 0$ and

$$T(\gamma_2) = \bigcup_{n=-\infty}^{\infty} t^n(\gamma'_2)$$

is an embedded copy of $\mathbb{R}$ separating M'_0 (since γ_2^* separates M^*) and γ'_1 meets $T(\gamma_2)$ from the same side at x' and $t(x')$. This completes the proof in case (i).

Case (ii): γ_i *orientation-reversing.* The argument is similar to the above, although t now reverses orientation. Let $m \geqq 2$ be even, and let Γ_m, Γ'_m be constructed as before. Instead of (2.9) we have

$$\omega_{X'}(\Gamma'_m, x') = \omega_X(\gamma_1, x) - \omega_X(\gamma_2, x) \tag{2.10}$$

since the contribution to $\omega_{X'}(\Gamma'_m, x')$ from $t^n(\gamma'_i)$ cancels that from $t^{n+1}(\gamma'_i)$ $(0 \leqq n \leqq m-2,\ i = 1, 2)$, and the contributions from rounding off the cusps cancel each other out because t^{m+1} reverses orientation. Moreover,

$$\omega_{X'}(\Gamma'_{m+1}, x') = \pm 1 \tag{2.11}$$

since all the $t^n(\gamma'_i)$ contributions to $\omega_{X'}(\Gamma'_{m+1}, x')$ cancel out but the contributions from the cusps do not cancel out since t^{m+2} preserves orientation. However, using Whitney's formula as before to calculate $\omega_{X'}(\Gamma'_m, x')$ and $\omega_{X'}(\Gamma'_{m+1}, x')$ we now obtain

$$\omega_{X'}(\Gamma'_m, x') = \pm 1 + \sum_{n=0}^{m} l_n - (k_1 + k_3 + \cdots + k_{m-1}) + (k_{-2} + k_{-4} + \cdots + k_{-m})$$

and

$$\omega_{X'}(\Gamma'_{m+1}, x') = \pm 1 + \sum_{n=0}^{m+1} l_n - (k_0 + k_2 + \cdots + k_m) - (k_{-1} + k_{-3} + \cdots + k_{-(m+1)})$$

since t is orientation-reversing. Subtracting we finally obtain

$$\omega_X(\gamma_1, x) - \omega_X(\gamma_2, x) \equiv 1 + l_{m+1} + \sum_{n=-(m+1)}^{m} k_n \quad mod\, 2\,.$$

However, if $m > m_0$ the right hand side is zero $mod\, 2$ because $T(\gamma_2)$ separates M'_0 (since $(2\gamma_2)^*$ separates the double cover of M^*) and γ'_1 meets $T(\gamma_2)$ from opposite sides at x' and $t(x')$. This completes the proof in case (ii), and finishes the proof of Theorem (2.7).

Note. A proof of this result when M is orientable and the γ_i simple can be found in [5], although more details in fact seem to be necessary than were given there.

We shall later make much use of the fact that the function

$$\omega_X : \pi_1(M, x) \to \mathbb{Z} \quad \text{or} \quad \mathbb{Z}_2$$

is *not* a homomorphism. We already have $\omega_X(1) = 1$ (instead of 0). Note also the following fact:

(2.12) **Proposition.** *$\omega_X(c) = -\omega_X(c^{-1})$ if and only if c contains orientation-preserving curves.*

Proof. If γ is a direct regular closed curve based at $X(x)$ with $\{\gamma\} = c$ then $\{-\gamma\} = c^{-1}$ (where $-\gamma$ means γ with reversed orientation) but $-\gamma$ is based at $-X(x)$. We may alter $-\gamma$ by a regular homotopy (not $rel - X(x)$) which turns the tangent at x back to $X(x)$: let the resulting curve, which may be taken to be direct, be called γ'. If γ is orientation-preserving then clearly $\omega_X(\gamma', x) = -\omega_X(\gamma, x)$ and so $\omega_X(c) = \omega_X(c^{-1})$: see also the discussion of winding numbers in a base-point free context in § 5 below. If γ is orientation-reversing then the homotopy turns the tangent to $-\gamma$ through an angle of $\pm\pi$ at the beginning of $-\gamma$ and the same angle again at the end. Hence $\omega_X(c^{-1}) = \omega_X(\gamma', x) \bmod 2 = -\omega_X(\gamma, x) \pm 1 \bmod 2$, so is not equal to $-\omega_X(c)$.

Consider also the following more instructive example. Let M be a punctured torus and let X be a standard "parallel" field on M. Choose $x \in M$ and generators $a = \{\alpha\}$, $b = \{\beta\}$ of $\pi_1(M, x)$ so that α is an orbit curve of X and β is perpendicular to all the orbits: thus $\omega_X(a) = \omega_X(b) = 0$. However, $aba^{-1}b^{-1}$ is represented by curves freely homotopic to ∂M, and it is easy to check that $\omega_X(aba^{-1}b^{-1}) = \pm 1$ (see also Corollary (5.8) below).

The way in which ω_X differs from a homomorphism will be used in the sequel to treat the problem of finding which elements of $\pi_1(M, x)$ contain simple closed curves.

It would be useful to have a clear picture of the algebraic structure of ω_X.

§ 3. Whitney's Theorems Generalized to 2-Manifolds

Combining results of the previous section with a theorem of Smale on regular homotopy [8] we can prove a generalization of the Whitney-Graustein Theorem [10] on regular closed curves in the plane. Recall that M is non-compact or with non-empty boundary, and is smooth.

(3.1) **Theorem.** *If γ_1 and γ_2 are orientation-preserving regular closed curves on M based at $X(x)$, where X is a non-vanishing vector field on M and $x \in M$, and if $\{\gamma_1\} = \{\gamma_2\} \in \pi_1(M, x)$, then*

$$\gamma_1 \underset{R}{\simeq} \gamma_2 \; rel\, X(x) \Leftrightarrow \omega_X(\gamma_1, x) = \omega_X(\gamma_2, x).$$

Proof. The implication one way is just that of Lemma (1.5). Note that if the γ_i are nullhomotopic then by lifting to the universal covering space of M we can apply the Whitney-Graustein Theorem, which is just Theorem (3.1) with $M = \mathbb{R}^2$, and obtain the result immediately. Compare the proof of Lemma (2.6). Therefore we now assume $\{\gamma_i\} \neq 1$ $(i = 1, 2)$.

We first state as a lemma a special case of Smale's general theorem in [8]. Let $\gamma_i = f_i(S^1)$, and recall the definition of $T_0 f$ from § 1.

(3.2) **Lemma.** *The maps $T_0 f_1$, $T_0 f_2 : S^1 \to T_0 M$ are homotopic relative to the base-point $y = X(x)/\|X(x)\|_x \in T_0 M$, if and only if $\gamma_1 \underset{R}{\simeq} \gamma_2$ rel $X(x)$.*

Thus it suffices for us to prove

$$\omega_X(\gamma_1, x) = \omega_X(\gamma_2, x) \Rightarrow T_0 f_1 \simeq T_0 f_2 \, rel \, y \, .$$

The group $\pi_1(T_0 M, y)$ is the free product of $\pi_1(M, x)$ and an infinite cyclic group corresponding to the fundamental group of the fibre S_0, say, over x of the S^1-bundle $T_0 M$, with amalgamation given by identifying the generator h of $\pi_1(S_0)$ with

$$g^{-1} h^{\varepsilon(g)} g$$

for every $g \in \pi_1(M, x)$, where $\varepsilon(g) = \pm 1$ according to whether g contains orientation-preserving or -reversing curves. This corresponds to the fact that each closed curve based at $x \in M$ gives rise to an automorphism of $\pi_1(S_0)$ which depends only on the homotopy class of the curve and is the identity if and only if the curve is orientation-preserving. (For an early study of $\pi_1(T_0 M, y)$ see [7].) Now recall from Remark 2 in § 1 that $\{Z^{f_i}\} = b^{m_i} a$ where $m_i = \omega_X(\gamma_i, x)$ and Z^{f_i} satisfies $F_i Z^{f_i} = T_0 f_i : S^1 \to T_0 M$ where $F_i : E^{f_i} \to T_0 M$ is the natural bundle map covering $f_i : S^1 \to M$, $i = 1, 2$. Thus

$$\{T_0 f_i\} = F_{i*}(b^{m_i} a)$$

where $F_{i*} : \pi_1(E^{f_i}, e_0) \to \pi_1(T_0 M, y)$ is the homomorphism induced by F_i. Clearly

$$F_{i*}(a) = \{\gamma_i\} = c$$

$$F_{i*}(b) = h$$

for $i = 1, 2$ and so $\{T_0 f_i\} = h^{m_i} c$. Hence

$$m_1 = m_2 \Rightarrow \{T_0 f_1\} = \{T_0 f_2\} \, ,$$

which completes the proof of Theorem (3.1).

In view of Theorem (2.7) we have the following

(3.3) **Theorem.** *If γ_1, γ_2 are direct orientation-preserving regular closed curves on M based at $v_x \in T_x M$, $x \in M$, and $\{\gamma_1\} = \{\gamma_2\} \neq 1 \in \pi_1(M, x)$, then $\gamma_1 \underset{R}{\simeq} \gamma_2$ rel v_x.*

From the methods of [10] and using Theorem (3.1) it is straightforward to derive the following formula for the winding number of a homotopically non-trivial regular closed curve γ with a finite number of self-intersection points, each a double point, on an orientable surface. The formula generalizes Whitney's formula [10] for the planar case, which was used to prove Theorem (2.7).

(3.4) **Formula.** *If M is orientable and X is a non-vanishing vector field on M such that γ is based at $X(x)$ then*

$$\omega_X(\gamma, x) = \omega_X(c) + N$$

where $c = \{\gamma\} \in \pi_1(M, x)$ and N is the algebraic number of self-intersections counting only those which are vertices of nullhomotopic loops, and each self-intersection is counted algebraically as indicated in Fig. 2 after a choice of starting point lying on no nullhomotopic loop.

§ 4. Winding Numbers, Cohomology and Homology

In this section we assume that M is compact and has non-empty boundary.

Let X_1, X_2 be two non-vanishing vector fields on M, and suppose that $\gamma = f(S^1)$ is any (not necessarily regular) oriented closed curve based at $x \in M$. Suppose also $X_1(x) = X_2(x)$. As in § 1 we may construct sections

$$X_i^f : S^1 \to E^f, \qquad i = 1, 2,$$

and since $p_*^f\{X_1^f\} = p_*^f\{X_2^f\}$ we obtain

$$\{X_1^f\}\{X_2^f\}^{-1} = i_*^f u^f$$

for some $u^f \in \pi_1(E_0, e_0) \cong \mathbb{Z}$. One can think of u^f as the total number of times that the X_1-vector rotates relative to the X_2-vector as γ is traversed once, the orientation of $T_x M$ being transported around γ.

If γ is a regular closed curve based at $X_1(x) = X_2(x)$ then from the definition of winding number we immediately have

(4.1) **Lemma.** $u^f = \omega_{X_2}(\gamma, x) - \omega_{X_1}(\gamma, x)$.

Let $v_1, v_2, \ldots, v_{2g+r-1}$ (M orientable of genus g with r boundary components) or $v_1, v_2, \ldots, v_{n+r-1}$ (M non-orientable of genus n with r boundary components) be a system of direct regular closed curves on M, all based at $v_x \in T_x M$, whose homotopy classes generate $\pi_1(M, x)$. Such a system is called a *regular generating system* (compare [5]). Since the homology classes of the v_i regarded as singular 1-cycles form a basis for $H_1(M; \mathbb{Z})$, any function which assigns an integer n_i to each v_i defines a homomorphism $H_1(M; \mathbb{Z}) \to \mathbb{Z}$ which can be identified with an element of $H^1(M;\mathbb{Z})$ since $H_1(M;\mathbb{Z})$ is free abelian (M has the homotopy type of a 1-complex). Let the element of $H^1(M;\mathbb{Z})$ defined by the function which assigns u^{f_i} to v_i (where $v_i=f_i(S^1)$) for each i be represented by a singular cocycle which we denote by $d(X_1, X_2)$: any two such representative cocycles differ by a coboundary and so take the same values on any given 1-cycle. (This construction is closely related to the standard obstruction theory for sections of bundles: see [9] for example.) If $\gamma=f(S^1)$ is any closed curve based at x then since γ is homotopic (and hence also homologous) to a sequence of the v_i and their inverses it is easy to check that in the case when M is orientable we have

$$u^f=\langle d(X_1, X_2), \gamma\rangle$$

where γ is regarded as a 1-cycle and $\langle s, \sigma\rangle$ denotes the value of the cochain s on the chain σ. When M is non-orientable the fact that if $f_{ij}: S^1 \to M$ represents the composition $v_i \circ v_j$ then $u^{f_{ij}}$ may not equal $u^{f_i}+u^{f_j}$ (it will do so if v_i is orientation-preserving) means that we at best have

$$u^f \equiv \langle d(X_1, X_2), \gamma\rangle \qquad mod 2 .$$

If X_1 is a given non-vanishing vector field and d is any singular 1-cocycle then when M is orientable we can construct a non-vanishing vector field X_2 on M such that $\langle d(X_1, X_2), v_i\rangle=\langle d, v_i\rangle$ for $1 \leqq i \leqq 2g+r-1$. This may be done by deformation retracting M onto a wedge W of circles in M, constructing X_2 on $W \subset M$ such that $u^{g_i}=\langle d, \mu_i\rangle$ for each $\mu_i=g_i(S^1)$ belonging to W, and extending X_2 to the whole of M. It then follows that $d(X_1, X_2)$ has the required property since $\langle d(X_1, X_2), \mu_i\rangle =\langle d, \mu_i\rangle$ for each μ_i and the homology classes of the μ_i generate $H_1(M;\mathbb{Z})$. When M is non-orientable the same will be true if we take coefficients all the time in $\mathbb{Z}_2$. Using Lemma (4.1) we thus easily obtain

(4.2) **Theorem.** *Let $c_1, c_2, \ldots, c_{2g+r-1}$ (M orientable) or $c_1, c_2, \ldots, c_{n+r-1}$ (M non-orientable) generate $\pi_1(M, x)$. A non-vanishing vector field X may be chosen to assign any given values (in $\mathbb{Z}$ or $\mathbb{Z}_2$ respectively) to the $\omega_X(c_i)$ $(1 \leqq i \leqq 2g+r-1$ or $n+r-1)$ and then for all other $c \in \pi_1(M, x)$ the value of $\omega_X(c)$ is independent of the choice of X.*

The following result relates winding numbers and homology.

(4.3) **Lemma.** *If γ, γ' are regular closed curves based at $v_x \in T_x M$, $x \in M$, and γ is homologous to γ' with coefficients in $\mathbb{Z}$ (M orientable) or $\mathbb{Z}_2$ (M non-orientable) then*

$$\omega_{X_1}(\gamma, x)=\omega_{X_2}(\gamma, x) \Leftrightarrow \omega_{X_1}(\gamma', x)=\omega_{X_2}(\gamma', x)$$

or

$$\omega_{X_1}(\gamma, x)=\omega_{X_2}(\gamma, x)\ mod 2 \Leftrightarrow \omega_{X_1}(\gamma', x)=\omega_{X_2}(\gamma', x)\ mod 2$$

respectively, where X_1 *and* X_2 *are any non-vanishing vector fields with* $X_1(x)=X_2(x)=v_x$.

Proof. By symmetry it is enough to prove the implications in one direction. By Lemma (4.1) the left hand sides imply

$$\langle d(X_1, X_2), \gamma\rangle = 0$$

taking coefficients in $\mathbb{Z}$ or $\mathbb{Z}_2$ as appropriate, and so

$$\begin{aligned}\langle d(X_1, X_2), \gamma'\rangle &= \langle d(X_1, X_2), \gamma'-\gamma\rangle \\ &= \langle d(X_1, X_2), \partial\beta\rangle\end{aligned}$$

for some 2-chain β, and

$$\begin{aligned}\langle d(X_1, X_2), \partial\beta\rangle &= \langle \delta d(X_1, X_2), \beta\rangle \\ &= 0\end{aligned}$$

since $d(X_1, X_2)$ is a cocycle. The right hand sides then follow, again using Lemma (4.1).

(4.4) **Corollary.** *If* γ *is homologous to zero with coefficients in* $\mathbb{Z}$ *(M orientable) or* $\mathbb{Z}_2$ *(M non-orientable) then* $\omega_X(\gamma, x)$ *is independent, or independent mod 2, respectively, of the vector field* X.

Proof. Take γ' to be a nullhomotopic simple closed curve. By Lemma (2.6) the winding number $\omega_X(\gamma', x)$ is ± 1, the sign depending on the orientation of γ and not on X. Hence $\omega_X(\gamma, x)$ is independent (or independent *mod* 2) of X by Lemma (4.3).

§ 5. Removal of Base-Point Restrictions

So far we have considered only regular curves with a given base-point and base-vector. In this section we see how the definition of winding number can be extended to situations in which this restriction is removed.

We continue to assume that M is smooth and admits non-vanishing vector fields, although we do not assume as in § 4 that M is compact.

Let $\gamma = f(S^1)$ be a regular closed curve on M, and let X be a non-vanishing vector field on M. Two sections X^f, Z^f of E^f can be defined just as in § 1, giving closed curves in E^f based at $e_1 = X^f(1)$, $e_2 = Z^f(1)$ respectively. The homotopy class of Z^f determines a conjugacy class of elements in $\pi_1(E^f, e_1)$.

If γ is orientation-preserving then $\pi_1(E^f, e_1)$ is abelian so we think of $\{Z^f\}$ as an element of $\pi_1(E^f, e_1)$ and then $\{Z^f\}\{X^f\}^{-1}$ defines a unique element $w^f \in \pi_1(E_0, e_1) \cong \mathbb{Z}$ as before. If M is orientable we call this the *winding number* of γ with respect to X, denoted by $\omega_X(\gamma)$. If M is non-orientable we reduce w^f *mod* 2 before calling it the winding number $\omega_X(\gamma)$: this turns out to be convenient for simplifying the statements of some results below. It is easy to check that $\omega_X(\gamma)$ does not depend on the choice of base-point $1 \in S^1$, provided the orientation of the fibre of E^f over the base-point is chosen appropriately. Intuitively, $\omega_X(\gamma)$ is as before the total number of times the tangent vector rotates relative to the X-vector on going once around γ in the positive sense, taken *mod* 2 if M is non-orientable.

If γ is orientation-reversing then E^f is a Klein bottle and $u_1, u_2 \in \pi_1(E^f, e_1)$ are conjugate if and only if $u_1 = b^{2r} u_2$ for some $r \in \mathbb{Z}$ (see Remark 2 in § 1). Hence $w^f \in \pi_1(E_0, e_1)$ is defined only up to multiplication by an even power of the generator, so we can only define $\omega_X(\gamma)$ in this case as an element of $\mathbb{Z}_2$. Again, any choice of base-point z in S^1 gives the same result as $z = 1$.

In both cases $\omega_X(\gamma)$ can be described as follows. Choose $x \in \gamma$ and deform γ by a regular homotopy to a curve γ' based at $X(x)$. Then

$$\omega_X(\gamma) = \omega_X(\gamma', x)$$

if M is orientable, or

$$\omega_X(\gamma) = \omega_X(\gamma', x) \text{ reduced } mod\, 2$$

if M is non-orientable.

Lemmas (1.3) and (1.5) generalize immediately to the base-point free case:

(5.1) **Lemma.** $X \simeq Y \Rightarrow \omega_X(\gamma) = \omega_Y(\gamma)$ *for all* γ.

(5.2) **Lemma.** $\gamma \underset{R}{\simeq} \delta \Rightarrow \omega_X(\gamma) = \omega_X(\delta)$ *for all* X.

In the new context Theorem (2.7) becomes

(5.3) **Theorem.** *Let X be a non-vanishing vector field on M, and let γ_1, γ_2 be direct regular closed curves which are (freely) homotopic but not nullhomotopic. Then $\omega_X(\gamma_1) = \omega_X(\gamma_2)$.*

Note. The original definition of *direct* (2.3) involved a base-point, but is clearly independent of the choice of base-point.

Proof of Theorem. Free homotopy classes correspond to conjugacy classes in the fundamental group. Deform γ_2 to a curve γ_2' by a regular homotopy such that γ_1 and γ_2' have the same base-vector $v_x \in T_x M$, $x \in M$: then $\{\gamma_2'\} = e\{\gamma_1\} e^{-1}$ for some $e \in \pi_1(M, x)$. Deform γ_2' by a further regular homotopy which takes x around a path corresponding to e to obtain γ_2'' with $\{\gamma_2''\} = \{\gamma_1\} \in \pi_1(M, x)$. We suppose that γ_2'' is still direct (since the regular homotopies may be performed using smooth isotopies of the whole of M) and based at v_x. Then if X is a non-vanishing vector field with $X(x) = v_x$ we have

$$\omega_X(\gamma_2) = \omega_X(\gamma_2'')$$

by Lemma (5.2), and

$$\omega_X(\gamma_1, x) = \omega_X(\gamma_2'', x)$$

(taken *mod* 2 if γ_1 is orientation reversing) by Theorem (2.7). Hence $\omega_X(\gamma_1) = \omega_X(\gamma_2)$.

(5.4) **Corollary.** *If c_1 and c_2 are conjugate elements of $\pi_1(M, x)$ then $\omega_X(c_1) = \omega_X(c_2)$.*

We can therefore define a winding number $\omega_X(C)$ for every conjugacy class C of elements of the fundamental group of M. Namely, if $1 \notin C$ let

$$\begin{aligned} \omega_X(C) &= \omega_X(c), \qquad c \in C \\ &= \omega_X(\gamma) \end{aligned}$$

where γ is a direct regular curve representing C. Thus $\omega_X(C) \in \mathbb{Z}$ or $\mathbb{Z}_2$ according to whether M is orientable or non-orientable. If $1 \in C$ define $\omega_X(C) = 1$.

Applying Theorem (3.3) to γ_1 and γ_2'' in the proof of Theorem (5.3) we immediately deduce the base-point free version of Theorem (3.3).

(5.5) **Theorem.** *If γ_1, γ_2 are direct, orientation-preserving regular closed curves on M, and neither curve is nullhomotopic, then*

$$\gamma_1 \simeq \gamma_2 \Rightarrow \gamma_1 \underset{R}{\simeq} \gamma_2 .$$

An *unbased regular generating system* for M is a collection of direct regular closed curves $v_i (i = 1, \ldots, 2g + r - 1$ or $n + r - 1)$ on M such that there is a (based) regular generating system consisting of curves v_i' with v_i' freely homotopic to v_i for each i. Using Theorem (5.3) and an unbased regular generating system it is easy to show

(5.6) **Theorem.** *Let $C_1, C_2, \ldots, C_{2g+r-1}$ (M orientable) or $C_1, C_2, \ldots, C_{n+r-1}$ (M non-orientable) be conjugacy classes of generators of the fundamental group of M. A non-vanishing vector field X may be chosen to assign any given values ($\in \mathbb{Z}$ or $\mathbb{Z}_2$ as appropriate) to the $\omega_X(C_i)$ ($1 \leqq i \leqq 2g + r - 1$ or $n + r - 1$) and then for all other conjugacy classes C the value of $\omega_X(C)$ is independent of the choice of X.*

It is obvious that, after suitable rewording, Lemma (4.3) and Corollary (4.4) continue to hold in the base-point free case. (The re-wording must take account of the fact that $\omega_X(\gamma) \in \mathbb{Z}_2$ for all γ on a non-orientable surface.)

We end this section with an elementary but useful geometric result.

(5.7) **Lemma.** *Let $\gamma_1, \ldots, \gamma_r$ be disjoint regular simple closed curves on M which form the boundary components of a compact surface $N \subset M$, where N has genus denoted by g' or n' according to whether N is orientable or not. If N is orientable suppose that all the γ_i have orientations consistent with the orientation of ∂N induced by a chosen orientation of N. Then*

$$\sum_{i=1}^{r} \omega_X(\gamma_i) = \pm(r + 2g' - 2)$$

when N is orientable, the sign depending on the orientation of N, or

$$\sum_{i=1}^{r} \omega_X(\gamma_i) = r + n' \ \text{reduced} \ mod 2$$

when N is non-orientable, for any non-vanishing vector field X on M.

Proof. Span the γ_i by discs D_i to obtain a smooth closed surface N' from N. Let v_i belong to the interior of D_i, and extend $X|N$ to a non-vanishing vector field on $N' - \bigcup_{i=1}^{r} v_i$. Let X_i be a field of "parallel" vectors in D_i, defined using some specific diffeomorphism between D_i and the unit disc in $\mathbb{R}^2$. Now

$$\omega_{X_i}(\gamma_i) = \pm 1 \qquad (mod 2 \text{ if } N \text{ is non-orientable})$$

by Lemma (2.6), and $\omega_{X_i}(\gamma_i) - \omega_X(\gamma_i) = \pm \varrho_i$ (reduced $mod 2$ if N non-orientable) where ϱ_i is the *index* of X at v_i, that is the number of rotations the X-vector makes relative to the X_i-vector, with respect to a chosen orientation of a neighbourhood of v_i, on a (small) suitably oriented closed curve around v_i (see Lemma (4.1)). When N is orientable the signs are both + or − together, and depend only on the orientation of γ_i; since the γ_i are oriented consistently the signs are the same for all i. The sum of the ϱ_i is well known to be equal to the Euler characteristic $\chi_{N'}$ of N', and $\chi_{N'} = 2 - 2g'$ or $2 - n'$. This gives the result.

Taking $r = 1$ we have

(5.8) **Corollary.** *If γ is a regular simple closed curve homologous to zero, bounding a surface N of genus g' (n') on M, then when N is orientable*

$$\omega_{X|N}(\gamma) = \pm(2g' - 1)$$

or when N is non-orientable

$$\omega_{X|N}(\gamma) = n' - 1 \ \text{reduced} \ mod 2 .$$

Note. In fact it is easy to show that in the latter case $\omega_X(\gamma, x) = \pm(n' - 1)$, where x is any base-point on γ, and the sign depends on a choice of orientation of a neighbourhood of γ in M.

Taking $r=2$ gives

(5.9) **Corollary.** *If γ_1, γ_2 are disjoint regular simple closed curves which are the boundary components of an orientable surface $N \subset M$ then*

$$\omega_{X|N}(\gamma_1) \equiv \omega_{X|N}(\gamma_2) \bmod 2$$

for every non-vanishing vector field X on M.

Note. Knoppers has shown that if γ_1, γ_2 are homologous regular simple closed curves (not necessarily disjoint) on an orientable surface then $\omega_X(\gamma_1) \equiv \omega_X(\gamma_2) \bmod 2$. If "simple" is replaced by "direct", however, there exist easy examples (which may be constructed using the formulae in § 7, for example) which show that the result then fails.

§ 6. Winding Numbers on a Closed Surface

If M is a *closed* surface (i.e. compact and with empty boundary) then M admits continuous non-vanishing vector fields if and only if χ_M is zero, but if $v \in M$ then $M-v$ always admits such fields. We now show how this allows us to define a notion of *winding number* for curves or homotopy classes of curves on a closed surface, although no longer necessarily as an element of $\mathbb{Z}$ or $\mathbb{Z}_2$.

Let M for the remainder of this section denote a closed surface, with $x, v \in M$ and $x \neq v$. Let X be a (continuous) non-vanishing vector field on $M-v$, and let $\gamma \subset M-v$ be a regular closed curve based at $X(x)$. The index of X at v depends only on M and not on X: it is the Euler characteristic of M, which is $2-2g$ (M orientable of genus g) or $2-n$ (M non-orientable of genus n). Let N denote the complement in M of the interior D_0 of a closed disc D in $M-x$ such that $v \in D_0$. The restricted bundle $T_0 M | D \to D$ is trivial, having total space $D \times S^1$ with infinite cyclic fundamental group generated by H, say. Using van Kampen's Theorem it is easy to see that $\pi_1(T_0 M, y)$ (where $y = X(x)/\|X(x)\|_x$) is obtained from $\pi_1(T_0 N, y)$ by adjoining the generator H and the relation

$$H^{2g-2} d = 1$$

or

$$H^{n-2} d = 1$$

where d is an element of $\pi_1(T_0 N, y)$ corresponding to a closed curve in $T_0 N$ covering some simple closed curve in N based at x and freely homotopic to $\partial N = \partial D$. It follows from Smale's theorem (Lemma (3.2)) that two regular curves on N which are based at $X(x)$ and which are regularly homotopic $\mathit{rel}\, X(x)$ on M have winding numbers which may differ by a multiple of $2g-2$ (or $n-2$) (see [5]). Intuitively one may imagine the winding number changing by $2g-2$ $(n-2)$ as the curve passes through v during the homotopy.

Clearly, then, we cannot hope for Theorem (2.7) to hold for closed surfaces. Instead, we have the following.

(6.1) **Theorem.** *Let γ_1, γ_2 be direct regular curves based at $X(x)$ on M, where M is a closed surface and X is a vector field vanishing only at $v \notin \gamma_1 \cup \gamma_2$. Suppose $\{\gamma_1\} = \{\gamma_2\}$ but $\{\gamma_1\} \neq 1 \in \pi_1(M, x)$. Then*

$$\omega_X(\gamma_1, x) \equiv \omega_X(\gamma_2, x) \quad mod\, 2g-2$$

if M is orientable, and if M is non-orientable then

$$\omega_X(\gamma_1, x) \equiv \omega_X(\gamma_2, x) \quad mod\, n-2$$

if the γ_i are orientation-preserving, or

$$\omega_X(\gamma_1, x) \equiv \omega_X(\gamma_2, x) \quad mod\,(2, n-2)$$

if the γ are orientation-reversing, where g or n is the genus of M and $(2, n-2)$ denotes the highest common factor of 2 and $n-2$.

Proof. If M is the projective plane then $n=1$ and the result holds trivially. Hence assume M is not the projective plane.

The technique of proof is to construct direct regular curves γ_i' $(i=1,2)$ on N based at $X(x)$ such that γ_1' and γ_2' are homotopic to each other on N and such that

$$\gamma_i \underset{R}{\simeq} \gamma_i' \; rel\, X(x)$$

on M. The result then follows by applying Theorem (2.7) to γ_1', γ_2' on N, since we have already shown above that

$$\omega_X(\gamma_i, x) \equiv \omega_X(\gamma_i', x) \quad mod\, 2g-2 \quad \text{or} \quad n-2\,.$$

We construct γ_i' $(i=1,2)$ as follows. First assume that γ_1, γ_2 are orientation-preserving. Let $p: M^* \to M$ be the covering defined in § 2, corresponding to the cyclic subgroup of $\pi_1(M, x)$ generated by $\{\gamma_1\} = \{\gamma_2\}$, with differential structure on M^* induced from M by p. There exists a point $x^* \in p^{-1}(x)$ with two regular closed curves γ_1^*, γ_2^* based at $X^*(x^*)$, where X^* is the vector field on M^* induced from X on M, such that $p(\gamma_i^*) = \gamma_i$ $(i=1,2)$ and γ_1^*, γ_2^* both represent the same element of $\pi_1(M^*, x^*)$. Note that γ_1^*, γ_2^* are simple since γ_1, γ_2 are direct. Let $D^* = p^{-1}(D)$, which consists of the disjoint union of a countable number of discs in M^*. Let U be a deformation retract of M^*, under a deformation retraction which keeps x^* fixed, such that $U \cap D^* = \emptyset$: it is easy to construct U since M^* is an open cylinder. Take a smooth ambient isotopy $H_t (0 \leqq t \leqq 1)$ of M^*, such that H_0 is the identity and $H_1(\gamma_1^* \cup \gamma_2^*) \subset U$, and such that H_t is the identity sufficiently close to x^*. This is easy to do since $\gamma_1^* \cup \gamma_2^*$ is compact. Define $\gamma_i' = pH_1(\gamma_i^*)$ for $i=1,2$. Each γ_i' is regular and direct (since $H_1(\gamma_i^*)$ is simple), and

$$H_1(\gamma_i^*) \simeq H_0(\gamma_i^*) = \gamma_i^*$$

so $H_1(\gamma_1^*)$ is homotopic to $H_1(\gamma_2^*)$ (relative to x^*) on M^*. Therefore $H_1(\gamma_1^*) \simeq H_1(\gamma_2^*)$ $(rel\, x^*)$ in U since U is a deformation retract of M^*. However, $p(U) \subset N$ and so $\gamma_1' \simeq \gamma_2'$ $rel\, x$ in N as required.

When γ_1, γ_2 are orientation-reversing the argument is similar, although we lift to the double cover M^{**} of M^* and use isotopies which commute with the covering translation of M^{**}. Here M^{**} is an open cylinder since M is not a projective plane.

Applying Theorem (3.3) to γ_1' and γ_2' on N also gives:

(6.2) **Theorem.** *Theorem* (3.3) *holds also when* M *is a closed surface.*

In view of Theorem (6.1) it is now clear that if M is a closed orientable surface of genus g we can define winding numbers for elements of $\pi_1(M, x)$ as integers $mod\, 2g-2$. We merely let

$$\omega_X(c) = \omega_{X|N}(\gamma, x) \quad mod\, 2g-2$$

for each vector field on M with one singularity v, where N is the complement of the interior of a closed disc in M containing v and γ is a direct regular closed curve on N based at $X(x)$ with $\{\gamma\} = c \neq 1 \in \pi_1(M, x)$. We also let $\omega_X(1) = 1$ $mod\, 2g-2$. Similarly, if M is non-orientable of even genus we can define such winding numbers as integers $mod\, 2$. If M is non-orientable of odd genus n then winding numbers cannot be defined meaningfully as above since $(2, n-2) = 1$. However the more interesting applications of winding numbers are to orientable surfaces, and in any case it is often quite convenient to use winding numbers of curves restricted to compact surfaces with non-empty boundary.

Note that Theorem (4.2) continues to hold when $\mathbb{Z}$ is replaced by $\mathbb{Z}_{2g-2}$, and vector fields are defined on M minus one point.

The removal of base-point restrictions makes little further difference in the case of a closed surface. We can define $\omega_X(C) \in \mathbb{Z}_{2g-2}$ or $\mathbb{Z}_2$ for a conjugacy class C of elements of the fundamental group of M, except in the case when M is non-orientable and of odd genus. Theorem (5.5) continues to hold (combine Theorem (6.2) and the proof of Theorem (5.3)), as does Theorem (5.6) with $\mathbb{Z}_{2g-2}$ in place of $\mathbb{Z}$, and vector fields defined on M minus one point.

Since we shall not in fact need to use winding numbers on closed surfaces in the applications which follow, we leave to the reader the straightforward task of supplying proofs of the above statements and constructing any other analogues of results in § 5 that may be of interest.

§ 7. The Calculation of $\omega_X(c)$

In this section we suppose that M is compact.

Let x be a point in the interior of M. If $\omega_X(c_i)$ is known for each element c_i of a set of generators of $\pi_1(M, x)$ then $\omega_X(c)$ is determined for all other $c \in \pi_1(M, x)$ (Theorem (4.2) and § 6), although we recall that ω_X is not a homomorphism. Since c is expressible in terms of the c_i it is useful to have an explicit formula for calculating $\omega_X(c)$ in terms of the $\omega_X(c_i)$. In this section we derive such a formula in the case when the c_i are suitable "standard" generators for $\pi_1(M, x)$. The method generalizes that of [5, 6] where a formula was given for the case when M is orientable and c contains simple curves.

Suppose first that M is orientable of genus g with r ($r \geqq 1$) boundary components $\varrho_1, \ldots, \varrho_r$. We can choose in the interior of M a system Σ of oriented simple closed curves α_i $(1 \leqq i \leqq g)$, β_j $(1 \leqq j \leqq g)$ and σ_k $(1 \leqq k \leqq r)$ with the following properties:

(i) the curves all meet at x, and are otherwise disjoint

(ii) σ_k together with ϱ_k bounds an annulus R_k whose interior is disjoint from all the other curves of Σ

(iii) cutting along the curves of Σ dissects M into the disjoint union of the R_k together with a disc whose boundary runs

$$\alpha_1 \beta_1 \alpha_1^{-1} \beta_1^{-1} \alpha_2 \beta_2 - \cdots - \alpha_g \beta_g \alpha_g^{-1} \beta_g^{-1} \sigma_1 \sigma_2 \ldots \sigma_r$$

where "α_i^{-1}" means "α_i backwards" etc.

We call Σ a *canonical curve-system* for M, as in [11], or [12, p. 114]. It is easily verified that around x the curves of Σ read as follows:

α_1 *leaving*
β_1 *arriving*
α_1 *arriving*
β_1 *leaving*
α_2 *leaving*
$\vdots$
β_g *leaving*
σ_1 *leaving*
σ_1 *arriving*
$\vdots$
σ_r *arriving*

with respect to some particular direction of rotation about x. We now assume the orientation of M to be chosen such that this direction is regarded as a negative rotation in a neighbourhood of x.

If a_i, b_j, s_k denote the homotopy classes of $\alpha_i, \beta_j, \sigma_k$ respectively, then $\pi_1(M, x)$ is generated by these with the one defining relation $d=1$ where d denotes the element

$$a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 - \cdots - a_g^{-1} b_g^{-1} s_1 \ldots s_r .$$

If M is closed and orientable the situation is the same, except that we dispense with the ϱ_k, σ_k, R_k and s_k.

In the case when M is non-orientable of genus n, the system Σ consists of oriented simple closed curves η_i $(1 \leqq i \leqq n)$ and σ_k $(1 \leqq k \leqq r)$ satisfying (i), (ii) above together with

(iii)′ cutting along the curves of Σ dissects M into the disjoint union of the R_k together with a disc whose boundary runs

$$\eta_1 \eta_1 \eta_2 \eta_2 \ldots \eta_n \eta_n \sigma_1 \ldots \sigma_r .$$

We can choose an orientation of a neighbourhood of x (and thus an orientation of the fibre of $T_0 M$ over x) such that the curves read

η_1 *leaving*
η_1 *arriving*
η_2 *leaving*
η_2 *arriving*
⋮
η_n *arriving*
σ_1 *leaving*
σ_1 *arriving*
⋮
σ_r *arriving*

with respect to a negative rotation about x. If u_i, s_k denote the homotopy classes of η_i, σ_k respectively, then $\pi_1(M, x)$ is generated by these with the one defining relation $d'=1$ where d' denotes the element

$$u_1^2 u_2^2 \ldots u_n^2 s_1 \ldots s_r .$$

As before, if M is closed there are no ϱ_k, σ_k, R_k or s_k.

(i) *Calculation when M is orientable.*

Let X be a given non-vanishing vector field on M, or on $M - v(v \neq x \in M)$ if $\partial M = \emptyset$, and let c be an element of $\pi_1(M, x)$. Let Σ

be a canonical curve-system for M chosen to have the following additional properties:

(iv) each curve of Σ is regular except at x, and if $\partial M = \emptyset$ then no curve passes through v

(v) if each curve is parametrized by $t (0 \leqq t \leqq 1)$ such that $\alpha_i(0) = \beta_j(0) = \sigma_k(0) = x = \alpha_i(1) = \beta_j(1) = \sigma_k(1)$ then the tangent to each curve approaches $X(x)$ as $t \to 0$ from above, and approaches $-X(x)$ as $t \to 1$ from below. See Fig. 4.

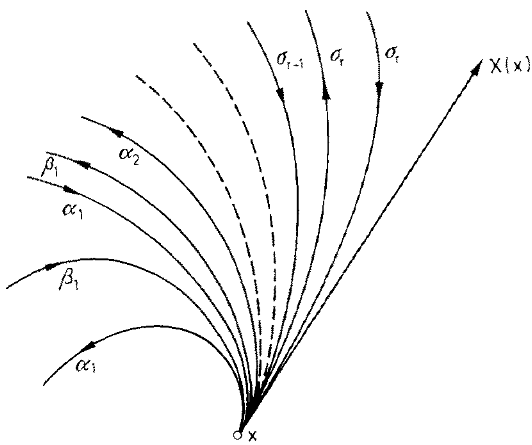

Fig. 4

Thus all the curves are bunched together to form a cusp at x, although they are still disjoint except at x.

Now any word in the "letters" $a_i^{\pm 1}$, $b_j^{\pm 1}$, $s_k^{\pm 1}$ $(1 \leqq i, j \leqq g;\ 1 \leqq k \leqq r$ if $r \geqq 1)$ represents an element of $\pi_1(M, x)$, and any element may be represented (not uniquely) by such a word. (The element $1 \in \pi_1(M, x)$ is represented by the "empty" word.) Write $c = [w]$ if the element c is represented by the word w.

If $c = 1$ then $\omega_X(c) = 1$ by definition, so there is nothing to calculate. For the rest of this section we assume $c \neq 1$.

Given $c \in \pi_1(M, x)$ choose w such that $c = [w]$ and w is reduced (i.e. no two adjacent letters in w are mutual inverses) and contains no subword which is more than half of a cyclic permutation of $d^{\pm 1}$. It follows from [3] if $r = 0$ or from [12, p. 83] in general[2] that w then contains no *null*

[2] It must here be assumed that there are at least 3 generators and that d (or d' in the non-orientable case) contains at least 5 letters. However, in the few exceptional cases it is easy to check by other direct means whether or not w contains any null subword.

subword, i.e. subword representing $1 \in \pi_1(M, x)$. Suppose $w = zvy$, where z, v and y are subwords such that $[yz] = 1$. Then $[w]$ is conjugate to $[v]$, and so by Corollary (5.4) we have $\omega_X([w]) = \omega_X([v])$. Therefore it suffices to be able to calculate $\omega_X(c)$ for elements $c = [w]$ such that no cyclic permutation of w contains a null subword.

Let γ with $\{\gamma\} = c$ be the closed curve which is the composition of curves in Σ and their inverses in the order in which their homotopy classes appear in w. By (iv) above, γ is regular except for the cusps at x. To calculate $\omega_X(c)$ it is necessary to find a direct regular curve γ' with $\{\gamma'\} = c$. This is now easily done as follows.

Let γ^* be a closed curve on M^* covering γ as in the paragraph preceding Definition (2.3). Since w contains (cyclically) no null subword it is impossible for γ to contain a nullhomotopic loop. Therefore as in Lemma (2.4) the curve γ^* is simple. Each of the cusps of γ^* may be "rounded off" (keeping all points of $p^{-1}(x)$ fixed) to give a regular simple curve γ^{**} on M^* such that $\gamma' = p\gamma^{**}$ is a regular curve on M whose tangents at x (which may be a self-intersection point) are all in the same direction as $X(x)$, and such that $\{\gamma'\} = c$. If $\partial M = \emptyset$ we may take γ' to lie in a neighbourhood of Σ not containing v. The curve γ' is direct since γ^{**} is simple.

Now by definition $\omega_X(c) = \omega_X(\gamma', x)$ $(mod\, 2g-2$ if $\partial M = \emptyset)$. We interpret $\omega_X(\gamma', x)$ as the total increase in the angle Θ that the tangent vector to γ makes with the X-vector on traversing γ once in the positive sense, and we consider the two contributions (1) from the "branches" of γ corresponding to $\alpha_i^{\pm 1}$ etc., (2) from the rounding off from one branch to another near the cusp at x to obtain γ'. If we denote the contributions (1) by $\Theta_X(\alpha_i^{\pm 1})$ etc., then it is not difficult to verify that when $\partial M \neq \emptyset$

$$\Theta_X(\alpha_i^{\pm 1}) = \pm 2\pi\omega_X(a_i) \pm \pi \quad (1 \leqq i \leqq g)$$
$$\Theta_X(\beta_j^{\pm 1}) = \pm 2\pi\omega_X(b_j) \mp \pi \quad (1 \leqq j \leqq g)$$
and
$$\Theta_X(\sigma_k^{\pm 1}) = \pm 2\pi\omega_X(s_k) \pm \pi \quad (1 \leqq k \leqq r),$$

with similar expressions, taking account of the fact that $\omega_X(a_i) \in \mathbb{Z}_{2g-2}$ etc., when $\partial M = \emptyset$. For (2), we let $\mathscr{R}$ denote the following ordering of the letters $a_i^{\pm 1}$ etc.:

$$a_1, b_1^{-1}, a_1^{-1}, b_1, a_2, b_2^{-1}, \ldots, a_g^{-1}, b_g, s_1, s_1^{-1}, s_2, \ldots, s_r^{-1}.$$

This describes the way the curves of Σ arrive at and leave x, rotating round x in the negative sense. If $\partial M = \emptyset$ then $\mathscr{R}$ contains no s_k terms. Let x_m and x_{m+1} be two consecutive letters in the word w taken cyclically, and let χ_m, χ_{m+1} be the curves of Σ (or their inverses) whose homotopy classes are x_m, x_{m+1} respectively. If x_m^{-1} appears *before* x_{m+1} in the ordering $\mathscr{R}$ then to round off the cusp between the end of χ_m and the beginning of χ_{m+1} we have to turn the tangent through an angle of $\pi + \eta_m$, where η_m may be made arbitrarily small. On the other hand, if x_m^{-1} appears *after* x_{m+1} in $\mathscr{R}$ the tangent turns through an angle of $-\pi + \eta_m$. Notice that x_m^{-1} and x_{m+1} cannot be identical, since w contains no null subword. Let $w = x_1 \dots x_p$ with $p \geqq 1$. If $p = 1$ there is nothing to compute, since we are assuming the winding numbers of the generators of $\pi_1(M, x)$ to be known. Therefore suppose $p > 1$. The winding number of γ' is the sum of all the increases in Θ on going round the χ_m $(1 \leqq m \leqq p)$ outside a small neighbourhood of x, together with the sum of the changes in going from χ_m to χ_{m+1} $(1 \leqq m \leqq p$, where χ_{p+1} means $\chi_1)$. From the above this total is

$$\frac{1}{2\pi}\left(\sum_{m=1}^{p} \Theta_X(\chi_m) + (P-N)\pi + \sum_{m=1}^{p} \eta_m\right)$$

where each η_m may be made arbitrarily small, and

P = number of values of m for which x_m^{-1} appears *before* x_{m+1} in the ordering $\mathscr{R}$,

N = number of values of m for which x_m^{-1} appears *after* x_{m+1} in the ordering $\mathscr{R}$.

Since $\Theta_X(\chi_m)$ is an integral multiple of π for each m, the term $\sum_{m=1}^{p} \eta_m$ must in fact be zero.

For each i $(1 \leqq i \leqq g)$ let A_i denote the number of letters x_m in w such that $x_m = a_i$, and let $\bar{A}_i$ be the total number for which $x_m = a_i^{-1}$. Similarly define B_j, $\bar{B}_j$, S_k, $\bar{S}_k$ and let $A = \sum_{i=1}^{g} A_i$, $\bar{A} = \sum_{i=1}^{g} \bar{A}_i$, etc.

Then when $\partial M \neq \emptyset$ we have

$$\begin{aligned} 2\pi\omega_X(\gamma', x) = &\sum_{i=1}^{g} (2\pi\omega_X(a_i) + \pi)(A_i - \bar{A}_i) \\ &+ \sum_{j=1}^{g} (2\pi\omega_X(b_j) - \pi)(B_j - \bar{B}_j) \\ &+ \sum_{k=1}^{r} (2\pi\omega_X(s_k) + \pi)(S_k - \bar{S}_k) \\ &+ (P-N)\pi\,. \end{aligned}$$

If $\partial M = \emptyset$ there are as usual no s_k terms, and integers are reduced $mod\, 2g-2$.

Let F be the free group generated by a_i $(1 \leqq i \leqq g)$, b_j $(1 \leqq j \leqq g)$ and (if $\partial M \neq \emptyset$) $s_k (1 \leqq k \leqq r)$, and let $\varphi_X : F \to \mathbb{Z}$ (or $\mathbb{Z}_{2g-2}$) be the homomorphism defined by $\varphi_X(a_i)=\omega_X(a_i)$, etc. Then regarding w as an element of F, and using the fact that

$$A-\bar{A}-B+\bar{B}+S-\bar{S}+P-N=2(A+\bar{B}+S-N)$$

(since $A+\bar{A}+B+\bar{B}+S+\bar{S}=p=P+N$), we obtain

Formula 1.

$$\omega_X(c)-\varphi_X(w)=(A+\bar{B}+S)-N \qquad (\partial M \neq \emptyset)$$

or

$$\omega_X(c)-\varphi_X(w)=(A+\bar{B})-N \quad mod\, 2g-2 \quad (\partial M=\emptyset).$$

Since $P+N=p$ etc., we may also write the above as

Formula 1′.

$$\omega_X(c)-\varphi_X(w)=P-(\bar{A}+B+\bar{S}) \qquad (\partial M \neq \emptyset)$$

or

$$\omega_X(c)-\varphi_X(w)=P-(\bar{A}+B) \quad mod\, 2g-2 \quad (\partial M=\emptyset).$$

(ii) *Calculation when M is non-orientable.*

Assume that M has even genus in order that $\omega_X(c)$ be defined.

Let c be a non-trivial element of $\pi_1(M,x)$ represented by a word w in the letters $u_i^{\pm 1}$, $s_k^{\pm 1}$ ($1 \leqq i \leqq n$, $1 \leqq k \leqq r$ if $r \geqq 1$). As in (i) we consider only those elements $c=[w]$ for which every cyclic permutation of w is reduced and contains no subword which is more than half a cyclic permutation of $d'^{\pm 1}$; then no cyclic permutation of w contains a null subword [12] [2]. It is easy to check that except in the case $r=0$, $n=1$ (M = projective plane) this condition implies that no cyclic permutation of ww contains a null subword. The projective plane case can, however, be dismissed since $\pi_1(M,x)\cong\mathbb{Z}_2$ and so the only non-trivial element is a generator.

Write $w=x_1 \ldots x_p$ and suppose as in (i) that $p>1$. Let γ be a closed curve with $\{\gamma\}=c$ constructed from curves of Σ as in (i); let $\delta=2\gamma$, and let δ' be a direct regular curve approximating to δ also as in (i). Clearly δ' can be chosen such that $\delta'=2\gamma'$ for some orientation-reversing regular curve γ' which is thus by definition direct, and $\{\gamma'\}=c$. We now proceed as in the orientable case, but must beware of the following differences:

(a) the sign of the change in Θ in rounding off the cusp when going from χ_m to χ_{m+1} depends on whether the number of χ_j with $j \leqq m$ such that $\chi_j=$ some $\eta_i^{\pm 1}$ is odd or even; this is because each η_i is orientation-reversing.

(b) $\omega_X(u_i^{-1})=\omega_X(u_i)+1$ for all i ($1 \leqq i \leqq n$) but $\omega_X(s_k^{-1})=\omega_X(s_k)$ ($1 \leqq k \leqq r$) (see (2.12)).

Remember that all winding numbers of homotopy classes are elements of $\mathbb{Z}_2$.

Corresponding to equations in (i) we have the following:

$$\omega_X(u_i^{\pm 1}) = \frac{1}{2\pi}(\Theta_X(\eta_i^{\pm 1}) \pm \pi) \quad mod 2 \qquad (1 \leqq i \leqq n)$$

and if $\partial M \neq \emptyset$

$$\omega_X(s_k^{\pm 1}) = \frac{1}{2\pi}(\Theta_X(\sigma_k^{\pm 1}) \mp \pi) \quad mod 2 \qquad (1 \leqq k \leqq r)$$

where for example $\omega_X(\eta_i)$ is defined using the orientations at points $x' \in \eta_i$ induced from the orientation at x by the path xx' of η_i. Let $\mathscr{R}'$ denote the following ordering of the letters $u_i^{\pm 1}$ etc.:

$$u_1, u_1^{-1}, u_2, u_2^{-1}, \ldots, u_n, u_n^{-1}, s_1, s_1^{-1}, s_2, \ldots, s_r^{-1}.$$

If $\partial M = \emptyset$ then $\mathscr{R}'$ contains no s_k terms. Let F' be the free group generated by the u_i $(1 \leqq i \leqq n)$ and (if $\partial M \neq \emptyset$) the s_k $(1 \leqq k \leqq r)$, and let $\varphi'_X : F' \to \mathbb{Z}_2$ be the homomorphism defined by $\varphi'_X(u_i) = \omega_X(u_i)$, etc. A tedious but elementary computation gives

Formula 2.

$$\omega_X(c) - \varphi'_X(w) = P' - \overline{S} \quad mod 2 \qquad (\partial M \neq \emptyset)$$

or

$$\omega_X(c) - \varphi'_X(w) = P' \quad mod 2 \qquad (\partial M = \emptyset)$$

where P' is defined from $\mathscr{R}'$ in the same way that P was defined from $\mathscr{R}$.

Full details of all the above calculations can be found in [1, pp. 104—115].

Remark. We observe from the above formulae that the expression $\omega_X([w]) - \varphi_X(w)$ (or $\omega_X([w]) - \varphi'_X(w)$) is independent of X. It must thus have some algebraic significance in the structure of $\pi_1(M, x)$, which it would be necessary to elucidate in order to understand ω_X.

§ 8. Generalizations

Winding numbers can be directly generalized in at least two different directions. The most natural is to consider an immersion

$$f : M, m_0 \to Q, q_0$$

of an m-manifold M in a q-manifold Q, where M is orientable and Q is equipped with a distribution σ of oriented tangent m-planes – i.e. a section of the Grassmanian bundle, denoted by $G^+_{m,q}(Q)$. The pull-back $f^* G^+_{m,q}(Q)$ is a bundle over M with fibre $G^+_{m,q}$ (the space of oriented m-planes in q-space), and it has two sections induced (i) from the section σ and (ii) from the tangent map $Tf : TM \to TQ$ which are analogous to the sections X^f, Z^f in § 1. If the set of homotopy classes $[M, G^+_{m,q}]$ has a group structure (e.g. if M is an associative H'-space with inverse) then there is an exact homotopy sequence

$$[SM, M] \to [M, G^+_{m,q}] \to [M, f^* G^+_{m,q}(Q)] \to [M, M]$$

(where SM is the suspension of M) of the fibration

$$G^+_{m,q} \to f^* G^+_{m,q}(Q) \to M$$

and as in § 1 the two sections of $f^* G^+_{m,q}(Q)$ define an element of $[M, G^+_{m,q}]/\mathrm{im}([SM, M])$. This may be thought of as a generalized winding number.

(In § 1 we had $m=1$, $q=2$, $M=S^1$ and $G^+_{m,q}=S^1$; $[SM, M]=\pi_2(S^1)$ which is trivial.)

Alternatively, we can take M to be the m-sphere S^m ($m>1$), and assume that M and Q both admit continuous non-vanishing vector-fields (thus m is necessarily odd). Defining E^f as in § 1 we obtain as before an exact sequence

$$\pi_{m+1}(S^m) \to \pi_m(S^{q-1}) \to \pi_m(E^f) \to \pi_m(S^m)$$

which yields a "winding number" as an element of $\pi_m(S^{q-1})/\mathrm{im}(\pi_{m+1}(S^m))$, or of

$$\pi_m(S^{q-1})/(\text{elements of order } 2)$$

when $m>2$ since $\pi_{m+1}(S^m) \cong \mathbb{Z}_2$.

It would be interesting to know whether any of the results in §§ 2–6 can be generalized meaningfully in either of the above two contexts.

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