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THE MAPPING CLASS GROUP ACTION ON THE HOMOLOGY OF THE CONFIGURATION SPACES OF SURFACES

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Abstract

In this paper, we study the natural action of the mapping class group $\mathcal{M}_{g,1}$ on the (co)homology groups of the configuration spaces of n-points on a surface Σ of genus g with the boundary $\partial\Sigma \cong S^1$. We present two main results in this paper. The first result is that the kernel of the action of $\mathcal{M}_{g,1}$ coincides with the kernel of the natural action on the nth lower central quotient group of the fundamental group of Σ. The second result is a new interpretation of the cohomology group $H^*(\mathcal{M}_{g,1}; T[H_1])$ of $\mathcal{M}_{g,1}$ with coefficients in the free tensor algebra $T[H_1]$ over $\mathbb{Z}$ generated by the first homology group H_1 of Σ, by using the configuration spaces. More precisely, we define a certain cochain complex C of $\mathcal{M}_{g,1}$-modules by using the configuration spaces and prove that $H^*(\mathcal{M}_{g,1}; C)$ is canonically isomorphic to $H^*(\mathcal{M}_{g,1}; T[H_1])$.

1. *Introduction*

Let Σ be a compact oriented surface of genus g with one boundary component, and with a base point $p_0 \in \partial\Sigma$, and let $\mathcal{M}_{g,1} = \pi_0(\mathrm{Diff}_+(\Sigma, \partial\Sigma))$ be the mapping class group of $(\Sigma, \partial\Sigma)$, where $\mathrm{Diff}_+(\Sigma, \partial\Sigma)$ is the group of orientation preserving diffeomorphisms of Σ restricting to the identity on $\partial\Sigma$. Let $H_1 = H_1(\Sigma; \mathbb{Z})$ be the first homology group of Σ; then $\mathcal{M}_{g,1}$ acts on H_1 naturally.

In this paper, we study the action of $\mathcal{M}_{g,1}$ on the (co)homology of the configuration space of points on Σ. There are two main results. The first result (Theorem A) is that the kernel of the natural action of $\mathcal{M}_{g,1}$ on the compact supported cohomology group (which will be denoted by H_n) of the configuration space of n-points on $\Sigma \setminus \{p_0\}$ coincides with the well-known subgroup $\mathcal{M}_{g,1}(n) \subset \mathcal{M}_{g,1}$, which is the kernel of the natural action of $\mathcal{M}_{g,1}$ on the nth lower central quotient group Γ_0/Γ_n of the fundamental group $\Gamma_0 = \pi_1(\Sigma, p_0)$ of Σ. The precise definition of $\mathcal{M}_{g,1}(n)$ will be given in Section 2. The second result (Theorem B) is a new interpretation of the cohomology group $H^*(\mathcal{M}_{g,1}; T[H_1])$ of $\mathcal{M}_{g,1}$ by using the configuration spaces, where $T[H_1]$ is the free tensor algebra over $\mathbb{Z}$ generated by H_1. More precisely, we will give an isomorphism

$$H^*(\mathcal{M}_{g,1}; T[H_1]) \cong H^*(\mathcal{M}_{g,1}; C),$$

where $C = \bigoplus_{p \geqslant 0} C^p$ is a certain cochain complex of $\mathcal{M}_{g,1}$-modules constructed from the homology groups H_n. The precise definition of C and its coboundary map $d_p : C^p \to C^{p+1}$ will be defined in Section 8. Roughly speaking, d_p represents how points on Σ make collisions with each other.

Johnson [4] introduced a homomorphism (the so-called Johnson homomorphism) $\tau_n : \mathcal{M}_{g,1}(n) \to H_1^{\otimes(n+2)}$ such that $\mathrm{Ker}\,\tau_n = \mathcal{M}_{g,1}(n+1)$. That the Johnson homomorphism is related to the Massey products of the mapping tori, associated to elements of $\mathcal{M}_{g,1}(n)$, was known to him [5], and later Kitano [8] proved that τ_n exactly measures the higher Massey products of length $n+1$ of the associated mapping tori. Morita [9] discovered a relation between the Johnson homomorphism and a secondary characteristic class of surface bundles, and the Casson invariant which is the simplest non-trivial finite type invariant of integral homology 3-spheres; see Morita [9, 10, 11] or Johnson's earlier results [4, 5] for more details.

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Additionally, [12] surveys recent works on the structure of the mapping class group of surfaces mainly from the topological viewpoint.

Now, we introduce some homology groups H_n $(n \geqslant 0)$ as follows. For each element $\varphi \in \mathrm{Diff}_+(\Sigma, \partial\Sigma)$, the diagonal action of φ on the nth Cartesian product Σ^n preserves the subsets Δ_n, A_n of Σ^n, which are defined by

$$\Delta_n = \{(x_1, x_2, \ldots, x_n) \in \Sigma^n \mid x_i = x_j \text{ for some } i \neq j\}\,,$$
$$A_n = \{(x_1, x_2, \ldots, x_n) \in \Sigma^n \mid x_i = p_0 \text{ for some } i\}\,.$$

Here, Σ^0 is understood to be the one-point set $\{\mathrm{pt}\}$ on which φ acts trivially, and Δ_0, A_0 are defined to be the empty set $\emptyset$. Define

$$H_n = H_n(\Sigma^n, \Delta_n \cup A_n; \mathbb{Z}),$$

which has an $\mathcal{M}_{g,1}$-module structure induced from the diagonal action. Let $C_n(\Sigma') = \Sigma'^n \setminus \Delta_n(\Sigma')$ be the configuration space of n-points on $\Sigma' = \Sigma \setminus \{p_0\}$. We remark that, as an $\mathcal{M}_{g,1}$-module, H_n is isomorphic to the compact supported cohomology group of $C_n(\Sigma')$:

$$H^n_c(C_n(\Sigma'); \mathbb{Z}) \cong H_n.$$

In this paper, we will use H_n instead of the left-hand side of the above equation.

Our first main theorem in this paper is the following.

THEOREM A. *For each integer $n \geqslant 0$, the kernel of the action of $\mathcal{M}_{g,1}$ on H_n is $\mathcal{M}_{g,1}(n)$. Namely, an element $\varphi \in \mathcal{M}_{g,1}$ acts on H_n trivially if and only if φ belongs to $\mathcal{M}_{g,1}(n)$.*

Therefore, for each $\varphi \in \mathcal{M}_{g,1}$, the following statements are equivalent.

(1) The element $\varphi \in \mathcal{M}_{g,1}(n)$; that is, φ acts on Γ_0/Γ_n trivially.

(2) For every integer $1 \leqslant i \leqslant n+1$, the Massey products of length i on the mapping torus, associated to φ, vanish.

(3) The element φ acts on H_n trivially.

The equivalence of (1) and (2) is due to Kitano [8], and Theorem A states that (1) and (3) are equivalent.

As a corollary of Theorem A, it turns out that the action of $\mathcal{M}_{g,1}$ on $\bigoplus_{n\geqslant 0} H_n$ is faithful (Corollary 7.2).

A related result was obtained by Groufalidis and Levine in [3]. They investigated relations among the Massey products of 3-manifolds, the Johnson homomorphism, and the Goussarov–Habiro theory of finite-type invariants of 3-manifolds.

In [7], Kawazumi determined $H^*(\mathcal{M}_{g,1}; T[H_1])$ explicitly modulo the stable cohomology group with trivial coefficients. As a corollary, the rational stable cohomology algebra of the extended mapping class group $\mathcal{M}_{g,1} \ltimes H_1$ was proved to be freely generated by the twisted Morita–Mumford classes (see [6]) over the rational stable cohomology algebra of $\mathcal{M}_{g,1}$. To describe a relationship between $H^*(\mathcal{M}_{g,1}; T[H_1])$ and the homology groups H_n, we introduce some notation. For integers n, l such that $n \geqslant 0$, let $S(n,l)$ be the Stirling number of the second kind, that is, the counting number of equivalence relations having l equivalence classes defined on a set with n elements. Here, $S(0,0)$ will be understood to be 1, and $S(n,l) = 0$ if $n < l$ or $n > 0 \geqslant l$. For example, for $n > 0$, $S(n,0) = 0$, $S(n,n) = S(n,1) = 1$ and $S(n,n-1) = n(n-1)/2$. In Section 8, we will give the definition of C:

$$0 \longrightarrow C^0 \xrightarrow{d_0} C^1 \xrightarrow{d_1} C^2 \xrightarrow{d_2} \cdots \xrightarrow{d_{l-1}} C^l \xrightarrow{d_l} \cdots\,,$$
$$C^p = \bigoplus_{q\geqslant 0} C^{p,q}, \quad C^{p,q} \cong H_{q-p}^{\oplus S(q,q-p)},$$

where $H_n = \{0\}$ if $n < 0$. Here is the table of $C^{p,q}$ for $0 \leqslant p \leqslant 4$ and $0 \leqslant q \leqslant 4$.

$$\begin{array}{r|ccccc} 4 & H_4 & H_3^{\oplus 6} & H_2^{\oplus 7} & H_1 & 0 \\ 3 & H_3 & H_2^{\oplus 3} & H_1 & 0 & 0 \\ 2 & H_2 & H_1 & 0 & 0 & 0 \\ 1 & H_1 & 0 & 0 & 0 & 0 \\ q=0 & \mathbb{Z} & 0 & 0 & 0 & 0 \\ \hline & p=0 & 1 & 2 & 3 & 4 \end{array} \tag{1.1}$$

Let $H^*(\mathcal{M}_{g,1}; C)$ be the cohomology group of $\mathcal{M}_{g,1}$ with coefficients in the cochain complex C (cf. [**1**, VII,5]).

Theorem B. *There is a canonical isomorphism*

$$H^*(\mathcal{M}_{g,1}; T[H_1]) \cong H^*(\mathcal{M}_{g,1}; C).$$

In Section 2, we prepare the necessary notation and give a precise definition of the H_n. The kernel of the action of $\mathcal{M}_{g,1}$ on H_n will be denoted by $\mathcal{M}_{g,1}(n)'$.

In order to prove Theorem A, we propose several supporting statements, which we prove in Section 3. We shall see that $\hat{H} = \prod_{n \geqslant 0} H_n$ has some ring structure and study the relationships between H_n and H_{n-1} by using the product structure on $\hat{H}$. The nth symmetric group $\mathfrak{S}_n$ acts on H_n by the permutation of points on Σ. It is easy to check that H_n is an $(\mathfrak{S}_n \times \mathcal{M}_{g,1})$-module. In Section 4, we determine the $\mathfrak{S}_n$-module structure of H_n. Therefore, to compute the $\mathcal{M}_{g,1}$-action on H_n, it is enough to work with a generating set of H_n as an $\mathfrak{S}_n$-module. In Section 5, we define and study a $\mathcal{M}_{g,1}$-equivariant ring homomorphism $\Phi : \mathbb{Z}\pi_1(\Sigma, p_0) \to \hat{H}$ (from the group ring $\mathbb{Z}\pi_1(\Sigma, p_0)$ of $\pi_1(\Sigma, p_0)$ over $\mathbb{Z}$). By using Φ, we explore the relationship between $\mathcal{M}_{g,1}(n)$ and $\mathcal{M}_{g,1}(n)'$. Combining the results in Sections 4 and 5, we give a generating set R_n of H_n as an $\mathfrak{S}_n$-module. In particular, we can conclude that if a mapping class trivially acts on R_n then it trivially acts on H_n. In Section 7, by using the results obtained in previous sections, we give a proof of Theorem A, that is, $\mathcal{M}_{g,1}(n) = \mathcal{M}_{g,1}(n)'$ for all non-negative n. In Section 8, we give a precise construction of the cochain complex C, and give a proof of Theorem B.

2. *Notation*

Let $\pi_1(\Sigma, p_0) = \Gamma_0 \supset \Gamma_1 \supset \Gamma_2 \supset \dots$ be the lower central series of $\pi_1(\Sigma, p_0)$, namely, $\Gamma_n = [\Gamma_{n-1}, \Gamma_0]$, $n \geqslant 1$. Let

$$\rho_n : \mathcal{M}_{g,1} \longrightarrow \mathrm{Aut}\,(\Gamma_0/\Gamma_n)$$

be the action induced from the natural action $\rho : \mathcal{M}_{g,1} \to \mathrm{Aut}(\pi_1(\Sigma, p_0))$, where $\mathrm{Aut}(G)$ denotes the automorphism group of a group G. Let us write

$$\mathcal{M}_{g,1}(n) = \mathrm{Ker}\,\rho_n$$

for the kernel of ρ_n. By definition, $\mathcal{M}_{g,1}(0) = \mathcal{M}_{g,1}$, and $\mathcal{M}_{g,1}(1)$ is nothing but the Torelli group, which is the subgroup of $\mathcal{M}_{g,1}$ consisting of elements which act on $H_1(\Sigma; \mathbb{Z})$ trivially.

For an integer $n \geqslant 1$ and for a pair (X, Y) of topological spaces such that $Y \subset X$, we define

$$\begin{aligned} \Delta_n(X) &= \big\{(x_1, \dots, x_n) \in X^n \mid x_i = x_j \text{ for some } i \neq j\big\}, \\ A_n(X, Y) &= \big\{(x_1, \dots, x_n) \in X^n \mid x_i \in Y \text{ for some } i\ \big\}, \end{aligned}$$

and we write

$$(X,Y)^n = (X^n, A_n(X,Y)),$$
$$(X,Y)^{\overline{n}} = \big(X^n, \Delta_n(X) \cup A_n(X,Y)\big),$$

which are generalizations of the standard notation

$$(X,Y)^2 = (X \times X, (X \times Y) \cup (Y \times X)).$$

When $n = 0$, we define $X^0 = \{\mathrm{pt}\}$, and we will write $\Delta_0(X) = A_0(X,Y) = \emptyset$ so that $(X,Y)^0 = (X,Y)^{\overline{0}} = (\{\mathrm{pt}\}, \emptyset)$.

Since the diagonal action of $\mathrm{Diff}_+(\Sigma, \partial\Sigma)$ on Σ^n preserves the subset $\Delta_n(\Sigma) \cup A_n(\Sigma, p_0)$, the homology group $H_*((\Sigma,p_0)^{\overline{n}})$ is an $\mathcal{M}_{g,1}$-module. We will see later that $H_i((\Sigma,p_0)^{\overline{n}};\mathbb{Z})$ vanishes except when $i = n$ (Proposition 3.3(i)). For simplicity, we will write $\Delta_n = \Delta_n(\Sigma, p_0)$, $A_n = A_n(\Sigma, \partial\Sigma)$, and $H_n = H_n((\Sigma,p_0)^{\overline{n}};\mathbb{Z})$ as in Section 1. Denote by

$$\rho'_n : \mathcal{M}_{g,1} \longrightarrow \mathrm{Aut}(H_n)$$

the action of $\mathcal{M}_{g,1}$ on H_n and write $\mathcal{M}_{g,1}(n)' = \mathrm{Ker}\,\rho'_n$. Then Theorem A is equivalent to

$$\mathcal{M}_{g,1}(n)' = \mathcal{M}_{g,1}(n)$$

for all $n \geqslant 0$.

3. *Homology group of* $(\Sigma, p_0)^{\overline{n}}$

We introduce a formal series algebra $\hat{H} = \prod_{n\geqslant 0} H_n$, the elements of which are infinite formal sums $\sum_{n\geqslant 0} u_n$ $(u_n \in H_n)$. We will construct some algebraic structures on $\hat{H}$ as follows. The actions ρ'_n, $n \geqslant 0$, induce the action

$$\rho' = \prod_{n\geqslant 0} \rho'_n : \mathcal{M}_{g,1} \longrightarrow \mathrm{Aut}(\hat{H}).$$

The natural map $\iota_{m,n} : (\Sigma,p_0)^{\overline{m}} \times (\Sigma,p_0)^{\overline{n}} \to (\Sigma,p_0)^{\overline{m+n}}$ induces the multiplication $\mu_{m,n} = \iota_{m,n*} : H_m \otimes H_n \to H_{m+n}$, and we will write

$$uv = \mu(u,v) = \sum_{m,n\geqslant 0} \mu_{m,n}(u_m, v_n)$$

for $u = \sum_{n\geqslant 0} u_n,\ v = \sum_{n\geqslant 0} v_n \in \hat{H}$. By definition, μ is associative (that is, $(uv)w = u(vw)$ for any $u,v,w \in \hat{H}$), and it has the unit $1 = [\mathrm{pt}] \in H_0 \cong \mathbb{Z}$ (that is, $1u = u1 = u$ for any $u \in \hat{H}$). Note that ρ' commutes with μ, because $\iota_{m,n}$ commutes with the natural $\mathrm{Diff}_+(\Sigma,\partial\Sigma)$-action. Let $\mathfrak{S}_n$, $n \geqslant 0$, denote the nth symmetric group, where $\mathfrak{S}_0$ is the unit group. The $\mathfrak{S}_n$-action on H_n, given by permutation of n-points on Σ, commutes with the $\mathcal{M}_{g,1}$-action, and thus H_n is an $(\mathfrak{S}_n \times \mathcal{M}_{g,1})$-module. Let $\mathcal{F}$ be the descending filtration of $\hat{H}$ such that

$$\mathcal{F}_n\hat{H} = \prod_{i\geqslant n} H_i;$$

then $(\mathcal{F}_m\hat{H})(\mathcal{F}_n\hat{H}) \subset \mathcal{F}_{m+n}\hat{H}$. Clearly, the action ρ' preserves $\mathcal{F}$, namely, $\varphi_*(\mathcal{F}_n\hat{H}) \subset \mathcal{F}_n\hat{H}$ for all $\varphi \in \mathcal{M}_{g,1}$. The following lemma is easy to prove.

Lemma 3.1. *The algebra $\hat{H}$ is an associative, filtered algebra over $\mathbb{Z}$ with the multiplication μ and the filtration $\mathcal{F}$. The action ρ' preserves μ and $\mathcal{F}$. For each $n \geqslant 0$, H_n is an $(\mathfrak{S}_n \times \mathcal{M}_{g,1})$-module.*

Proof. The proof follows from the above discussion. □

In order to consider the multiplication $\mu_{n-1,1} : H_{n-1} \otimes H_1 \to H_n$, we introduce some notation as follows. Fix an integer $n \geqslant 2$ and write $Y_n = (\Delta_{n-1} \times \Sigma) \cup A_n$ so that

$$(\Sigma, p_0)^{\overline{n-1}} \times (\Sigma, p_0)^{\overline{1}} = (\Sigma^n, Y_n). \tag{3.1}$$

Set $I_{n-1} = \{1, 2, \ldots, n-1\}$ on which $\mathrm{Diff}_+(\Sigma, p_0)$ acts trivially, and let

$$f : I_{n-1} \times \Sigma^{n-1} \longrightarrow \Delta_n \cup A_n$$

be the $\mathrm{Diff}_+(\Sigma, p_0)$-equivariant map defined by

$$f(i, (x_1, x_2, \ldots, x_{n-1})) = (x_1, x_2, \ldots, x_{n-1}, x_i).$$

By definition, $f(I_{n-1} \times (\Delta_{n-1} \cup A_{n-1})) \subset Y_n$. There is a natural isomorphism

$$H_*(I_{n-1} \times \Sigma^{n-1}, I_{n-1} \times (\Delta_{n-1} \cup A_{n-1}); \mathbb{Z}) \cong H_*((\Sigma, p_0)^{\overline{n-1}}; \mathbb{Z})^{\oplus n-1},$$

and we identify these groups as $\mathcal{M}_{g,1}$-modules.

Lemma 3.2. *Let $n \geqslant 2$ be an integer. The homomorphism*

$$f_* : H_*\big((\Sigma, p_0)^{\overline{n-1}}; \mathbb{Z}\big)^{\oplus n-1} \longrightarrow H_*(\Delta_n \cup A_n, Y_n; \mathbb{Z})$$

induced from f is an $\mathcal{M}_{g,1}$-module isomorphism.

Proof. Let

$$f_1 : I_{n-1} \times (\Delta_{n-1} \cup A_{n-1}) \longrightarrow Y_n$$

be the restriction of f, and let $V = Y_n \bigcup_{f_1} \big(I_{n-1} \times \Sigma^{n-1}\big)$ be the attaching space. One can regard Y_n as a subset of V in the standard way. To distiniguish the notation Y_n $(\subset V)$ from the original Y_n $(\subset \Sigma^n)$, we rewrite $W = Y_n \subset V$. Note that there is a natural isomorphism

$$H_*\big((\Sigma, p_0)^{\overline{n-1}}; \mathbb{Z}\big)^{\oplus n-1} \cong H_*\big(V, W; \mathbb{Z}\big). \tag{3.2}$$

It is easy to check that the map f is injective on the complement of $I_{n-1} \times (\Delta_{n-1} \cup A_{n-1})$ in $I_{n-1} \times \Sigma^{n-1}$. Therefore, f induces the injective continuous map

$$f_2 : (V, W) \longrightarrow (\Delta_n \cup A_n, Y_n),$$

which restricts to the identity $W \to Y_n$.

For any $x = (x_1, x_2, \ldots, x_n) \in (\Delta_n \cup A_n) \setminus Y_n$, there exists i, $1 \leqslant i \leqslant n-1$, such that $x_i = x_n$, and so $f(i, (x_1, x_2, \ldots, x_{n-1})) = x$. This means that f_2 is surjective. Moreover, f_2 is a homeomorphism, because it is a continuous bijective map from a compact space to a Hausdorff space. Thus we obtain an isomorphism

$$f_{2*} : H_*(V, W; \mathbb{Z}) \longrightarrow H_*(\Delta_n \cup A_n, Y_n; \mathbb{Z}),$$

which coincides with f_* via the isomorphism (3.2). □

Let us write $\partial_* : H_*((\Sigma, p_0)^{\overline{n}}; \mathbb{Z}) \to H_{*-1}(\Delta_n \cup A_n, Y_n; \mathbb{Z})$ for the homology connecting homomorphism of the triple $(\Sigma^n, \Delta_n \cup A_n, Y_n)$. The composition

$$\partial'_* = f_*^{-1} \circ \partial_* : H_*((\Sigma, p_0)^{\overline{n}}; \mathbb{Z}) \longrightarrow H_{*-1}((\Sigma, p_0)^{\overline{n-1}}; \mathbb{Z})^{\oplus n-1}$$

is an $\mathcal{M}_{g,1}$-module homomorphism.

Proposition 3.3. *Let $n \geqslant 1$ be an integer.*

(i) *If $i \neq n$, then $H_i((\Sigma, p_0)^{\overline{n}}; \mathbb{Z}) = 0$.*

(ii) *The sequence*

$$0 \longrightarrow H_{n-1} \otimes H_1 \xrightarrow{\mu_{n-1,1}} H_n \xrightarrow{\partial'_*} H_{n-1}^{\oplus n-1} \longrightarrow 0$$

is exact.

(iii) *H_n is a free abelian group of rank $2g(2g+1)\dots(2g+(n-1))$.*

Proof. The proof proceeds by induction on n. The statements are obvious when $n=1$. Fix an integer $n \geqslant 2$. Let us consider the homology exact sequence of the triple $(\Sigma^n, \Delta_n \cup A_n, Y_n)$:

$$\cdots \longrightarrow H_i(\Sigma^n, Y_n; \mathbb{Z}) \longrightarrow H_i((\Sigma, p_0)^{\overline{n}}; \mathbb{Z}) \xrightarrow{\partial_*} H_{i-1}(\Delta_n \cup A_n, Y_n; \mathbb{Z}) \longrightarrow \cdots .$$

By Lemma 3.2, we can replace the last group with $H_{i-1}\big((\Sigma, p_0)^{\overline{n-1}}; \mathbb{Z}\big)^{\oplus n-1}$, and then ∂_* becomes ∂'_*. There is an $\mathcal{M}_{g,1}$-module isomorphism

$$H_i(\Sigma^n, Y_n; \mathbb{Z}) \cong H_{i-1}((\Sigma, p_0)^{\overline{n-1}}; \mathbb{Z}) \otimes H_1$$

induced from (3.1). If $i \neq n$, then by the induction hypothesis, the groups of both ends on the above sequence vanish, and hence we obtain (i). In the case $i = n$, the sequence is nothing but the short exact sequence in (ii). (iii) is immediate from (ii). □

Let $\operatorname{gr} \hat{H} = \bigoplus_{n \geqslant 0} \operatorname{gr}_n \hat{H}$, $\operatorname{gr}_n \hat{H} = \mathcal{F}_n \hat{H}/\mathcal{F}_{n+1}\hat{H}$ denote the associated graded algebra of $\hat{H}$, and we will identify $\operatorname{gr}_n \hat{H}$ with H_n. Let $T[H_1] = \bigoplus_{n \geqslant 0} H_1^{\otimes n}$ be the free tensor algebra generated by H_1 over $\mathbb{Z}$. By Proposition 3.3, we obtain the following corollary.

Corollary 3.4. *Let $n \geqslant 1$ be an integer.*
(i) *$\mathcal{M}_{g,1}(n-1)' \supset \mathcal{M}_{g,1}(n)'$.*
(ii) *The homomorphism $\psi_n : {H_1}^{\otimes n} \to H_n$, defined by*

$$u_1 \otimes u_2 \otimes \dots \otimes u_n \longmapsto u_1 u_2 \dots u_n,$$

where $u_i \in H_1$, is an injective $\mathcal{M}_{g,1}$-module homomorphism. Consequently, the induced homomorphism

$$\Psi = \bigoplus_{n \geqslant 0} \psi_n : T[H_1] \longrightarrow \operatorname{gr} \hat{H}$$

is an injective graded algebra homomorphism.

Proof. The group $\mathcal{M}_{g,1}(n)'$ acts trivially on $H_{n-1}^{\oplus n-1}$ because of the surjectivity of ∂'_* (Proposition 3.3(ii)). Thus, $\mathcal{M}_{g,1}(n-1)' \supset \mathcal{M}_{g,1}(n)'$. Next, we prove part (ii) as follows. Since ρ' and μ commute with each other, ψ_n is an $\mathcal{M}_{g,1}$-module homomorphism. The proof of the injectivity of ψ_n proceeds by induction on n. Obviously, $\psi_1 = \mathrm{id}$ is injective. If $n \geqslant 2$, then ψ_n coincides with the composition

$$H_1^{\otimes n} = H_1^{\otimes n-1} \otimes H_1 \xrightarrow{\psi_{n-1} \otimes \mathrm{id}} H_{n-1} \otimes H_1 \xrightarrow{\mu_{n-1,1}} H_n.$$

The homomorphism $\psi_{n-1} \otimes \mathrm{id}$ is injective by the induction hypothesis, and $\mu_{n-1,1}$ is injective by Proposition 3.3(ii). Thus, ψ_n is injective. □

4. *Cell decomposition of* $(\Sigma, p_0)^{\overline{n}}$

In this section, we use a cell decomposition of $(\Sigma, p_0)^{\overline{n}}$ to determine the $\mathfrak{S}_n$-module structure of H_n. Let $X = S_1^1 \vee S_2^1 \vee \dots \vee S_{2g}^1$ be the wedge of $2g$ copies of the oriented circle $S^1 = \mathbb{R}/\mathbb{Z}$ with the base point $0 \in S^1$. Let $\alpha_j \in \pi_1(X, p_0)$ denote the homotopy class of S_j^1; then $\alpha_1, \alpha_2, \dots, \alpha_{2g}$ generate $\pi_1(X, p_0)$ freely. We assume that X is a subspace of Σ such that $\bigcap_{1 \leqslant j \leqslant 2g} S_j^1 = p_0$ and that the inclusion is a homotopy equivalence. The symmetric group $\mathfrak{S}_n$ acts on $(X, p_0)^{\overline{n}}$ and $(\Sigma, p_0)^{\overline{n}}$ by permutation of n-points, and the map $(X, p_0)^{\overline{n}} \to (\Sigma, p_0)^{\overline{n}}$ induced from the inclusion is a homotopy equivalence of $\mathfrak{S}_n$-spaces. Hence H_n can be identified with $H_n((X, p_0)^{\overline{n}}; \mathbb{Z})$ as an $\mathfrak{S}_n$-module. For simplicity, we will write $\Delta'_n = \Delta_n(X)$ and $A'_n = A_n(X, p_0)$.

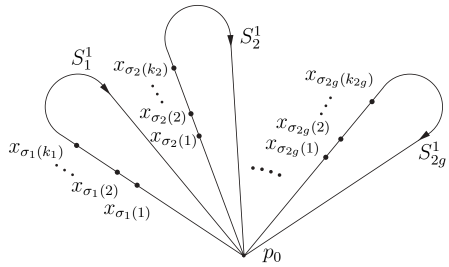

FIGURE 1. *A point* $x = (x_1, x_2, \ldots, x_n)$ *on* $X^n - (\Delta'_n \cup A'_n)$.

The subspace $X \setminus (\Delta'_1 \cup A'_1) \subset X$ is homeomorphic to the disjoint union of $2g$ copies of the open interval $(0,1) \cong S^1 \setminus \{0\}$. By induction on n, one can see that $X^n \setminus (\Delta'_n \cup A'_n)$ consists of $2g(2g+1)\ldots(2g+(n-1))$ connected components, each of which is homeomorphic to an open n-disc.

Fix a point $x = (x_1, x_2, \ldots, x_n) \in X^n \setminus (\Delta'_n \cup A'_n)$. For $1 \leqslant j \leqslant 2g$, we define

$$k_j(x) = \#\big(\{x_1, x_2, \ldots, x_n\} \cap S_j^1\big),$$

where $\#$ denotes the number of elements in a set. There exists a unique injective map $\sigma_j : I_{k_j(x)} \to I_n$, where $I_i = \{1, 2, \ldots, i\}$, such that

$$\big\{x_{\sigma_j(1)}, x_{\sigma_j(2)}, \ldots, x_{\sigma_j(k_j(x))}\big\} = \{x_1, x_2, \ldots, x_n\} \cap S_j^1, \tag{4.1}$$

$$x_{\sigma_j(1)} < x_{\sigma_j(2)} < \ldots < x_{\sigma_j(k_j(x))}; \tag{4.2}$$

see Figure 1. In (4.2), we regard $x_{\sigma_j(i)}$ as points in $(0,1) \cong S_j^1 \setminus \{0\}$ so that the inequalities make sense. When $n \neq 0$, we define

$$\sigma(x) = \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{pmatrix} \in \mathfrak{S}_n \tag{4.3}$$

to be the element such that

$$\sigma(k_0(x) + k_1(x) + \ldots + k_{j-1}(x) + i) = \sigma_j(i) \tag{4.4}$$

for $1 \leqslant j \leqslant 2g$ and $1 \leqslant i \leqslant k_j(x)$, where $k_0(x) = 0$. If $n = 0$, then we define $\sigma(x) \in \mathfrak{S}_0$ to be the unit. The $2g$-tuple

$$k(x) = (k_1(x), k_2(x), \ldots, k_{2g}(x))$$

belongs to the set

$$K_n = \left\{ (k_1, k_2, \ldots, k_{2g}) \in \mathbb{Z}^{2g} \;\middle|\; \begin{matrix} k_1 + k_2 + \ldots + k_{2g} = n, \\ k_j \geqslant 0 \text{ for } 1 \leqslant j \leqslant 2g \end{matrix} \right\}.$$

If two points $x, y \in X^n \setminus (\Delta'_n \cup A'_n)$ are in a same connected component, then $(\sigma(x), k(x)) = (\sigma(y), k(y))$. Therefore, we obtain a well-defined map

$$h_n : \pi_0(X^n \setminus (\Delta'_n \cup A'_n)) \longrightarrow \mathfrak{S}_n \times K_n, \quad [x] \longmapsto (\sigma(x), k(x)),$$

where $[x] \subset X^n \setminus (\Delta'_n \cup A'_n)$ is the connected component containing x. Note that $h_0 : \pi_0(X^0 \setminus (\Delta'_0 \cup A'_0)) \to \mathfrak{S}_0 \times K_0$ is given by $h_0([\mathrm{pt}]) = (1, (0, 0, \ldots, 0))$.

LEMMA 4.1. *For* $n \geqslant 0$, *the map* h_n *as defined above is a bijection.*

Proof. It is not difficult to see that h_n is injective. In fact, if $h_n([x]) = h_n([y])$ then one can easily construct a path from x to y in $X^n \setminus (\Delta'_n \cup A'_n)$. Since

$$\#(\mathfrak{S}_n \times K_n) = \frac{(2g+n-1)!}{(2g-1)!} = \#\pi_0(X^n \setminus (\Delta'_n \cup A'_n)),$$

h_n is a bijection. □

For $(\sigma, k) \in \mathfrak{S}_n \times K_n$, let us write

$$e_{(\sigma,k)} = h_n^{-1}(\sigma, k),$$

which is a connected component of $X^n \setminus (\Delta'_n \cup A'_n)$. By Lemma 4.1, X^n can be decomposed into the disjoint union

$$X^n = (\Delta'_n \cup A'_n) \amalg \left(\coprod_{(\sigma,k)\in \mathfrak{S}_n \times K_n} e_{(\sigma,k)} \right).$$

Let Δ^n denote the n-simplex:

$$\Delta^n = \{(t_1, \ldots, t_n) \mid 0 \leqslant t_1 \leqslant t_2 \leqslant \ldots \leqslant t_n \leqslant 1\},$$

where Δ^0 is understood to be the one point set $\{0\}$. For any element $k = (k_1, k_2, \ldots, k_{2g}) \in K_n$, we write $\Delta^k = \Delta^{k_1} \times \Delta^{k_2} \times \ldots \times \Delta^{k_{2g}}$, and its interior will be denoted by $\mathrm{Int}\,\Delta^k$. We define an $\mathfrak{S}_n$-action on $\mathfrak{S}_n \times K_n$ by $\tau(\sigma, k) = (\tau\sigma, k)$ for $(\sigma, k) \in \mathfrak{S}_n \times K_n$ and $\tau \in \mathfrak{S}_n$. Let $\mathbb{Z}(\mathfrak{S}_n \times K_n)$ be the $\mathfrak{S}_n$-module over $\mathbb{Z}$ generated by $\mathfrak{S}_n \times K_n$.

Proposition 4.2. *Let $n \geqslant 0$ be an integer. There exists a family*

$$\{\varphi_{(\sigma,k)} \mid (\sigma, k) \in \mathfrak{S}_n \times K_n\}$$

of continuous maps

$$\varphi_{(\sigma,k)} : (\Delta^k, \partial\Delta^k) \longrightarrow (X^n, \Delta'_n \cup A'_n)$$

such that:

(i) *$\varphi_{(\sigma,k)}$ maps $\mathrm{Int}\,\Delta^k$ onto $e_{(\sigma,k)}$ homeomorphically, and*
(ii) *$\tau\varphi_{(\sigma,k)}(\mathbf{t}) = \varphi_{(\tau\sigma,k)}(\mathbf{t})$ for $\tau \in \mathfrak{S}_n$, $\mathbf{t} \in \Delta^k$.*

Moreover, the homomorphism

$$\mathbb{Z}(\mathfrak{S}_n \times K_n) \longrightarrow H_n, \tag{4.5}$$

taking $(\sigma, k) \in \mathfrak{S}_n \times K_n$ to the homology class $[\varphi_{(\sigma,k)}] \in H_n$ of $\varphi_{(\sigma,k)}$, is an $\mathfrak{S}_n$-module isomorphism.

Proof. When $n = 0$, the unique map $\varphi_{(1,(0,0,\ldots,0))}$, such that $\varphi_{(1,(0,0,\ldots,0))}(0) = \mathrm{pt}$, satisfies the statement. Assume that $n \geqslant 1$. For $1 \leqslant j \leqslant 2g$, let $\tilde{\alpha}_j : ([0,1], \{0,1\}) \to (S^1_j, p_0)$ be the map induced from the identification $\mathbb{R}/\mathbb{Z} = S^1_j$. Fix any element $(\sigma, k) \in K_n \times \mathfrak{S}_n$, and let $\{\sigma_j\}_{j=1}^{2g}$ be the associated data determined by (4.4). We express the coordinates of points on Δ^k as follows:

$$\mathbf{t} = (\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_{2g}) \in \Delta^k,$$
$$\mathbf{t}_j = (t_{j,1}, t_{j,2}, \ldots, t_{j,k_j}) \in \Delta^{k_j} \quad (1 \leqslant j \leqslant 2g).$$

Define a map $\varphi_{(\sigma,k)} : \Delta^k \to X^n$ by

$$\varphi_{(\sigma,k)}(\mathbf{t}) = (x_1, x_2, \ldots, x_n),$$
$$x_{\sigma_j(i)} = \tilde{\alpha}_j(t_{j,i}) \quad (1 \leqslant j \leqslant 2g,\ 1 \leqslant i \leqslant k_j).$$

It is not difficult to check that $\varphi_{(\sigma,k)}(\partial\Delta^k) \subset \Delta'_n \cup A'_n$ and that $\varphi_{(\sigma,k)}(\operatorname{Int}\Delta^k) \subset e_{(\sigma,k)}$. Since $\tilde{\alpha}_j|_{(0,1)} : (0,1) \to S^1_j - \{p_0\}$ is a homeomorphism, so is

$$\varphi_{(\sigma,k)}|_{\operatorname{Int}\Delta^k} : \operatorname{Int}\Delta^k \longrightarrow e_{(\sigma,k)}.$$

This completes the proof of part (i).

For $\tau \in \mathfrak{S}_n$, let $\{(\tau\sigma)_j\}_{1\leqslant j\leqslant 2g}$ be the associated data determined by (4.4) with σ replaced by $\tau\sigma$. Then

$$(\tau\sigma)_j(i) = \tau(\sigma(k_0 + k_1 + \ldots + k_{j-1} + i)) = \tau(\sigma_j(i)).$$

This implies that

$$\tau(\varphi_{(\sigma,k)}(\mathbf{t})) = \varphi_{(\tau\sigma,k)}(\mathbf{t})$$

for $\mathbf{t} \in \Delta^k$, which proves part (ii).

By (i) and (ii), the map

$$\chi = \coprod_{(\sigma,k)\in\mathfrak{S}_n\times K_n} \varphi_{(\sigma,k)} : \coprod_{(\sigma,k)\in\mathfrak{S}_n\times K_n} (\Delta^k, \partial\Delta^k) \longrightarrow (X^n, \Delta'_n \cup A'_n)$$

induces an $\mathfrak{S}_n$-module homomorphism

$$\chi_* = \bigoplus_{(\sigma,k)\in\mathfrak{S}_n\times K_n} {\varphi_{(\sigma,k)}}_* : \bigoplus_{(\sigma,k)\in\mathfrak{S}_n\times K_n} H_n(\Delta^k, \partial\Delta^k; \mathbb{Z}) \longrightarrow H_n.$$

Since $H_n(\Delta^k, \partial\Delta^k; \mathbb{Z}) \cong \mathbb{Z}$, the group $\bigoplus_{(\sigma,k)\in\mathfrak{S}_n\times K_n} H_n(\Delta^k, \partial\Delta^k; \mathbb{Z})$ is identified with $\mathbb{Z}(\mathfrak{S}_n \times K_n)$, and so we can write

$$\chi_* : \mathbb{Z}(\mathfrak{S}_n \times K_n) \longrightarrow H_n.$$

By the definition of χ, we have $\chi_*(\sigma, k) = [\varphi_{\sigma,k}]$ for any $(\sigma, k) \in \mathfrak{S}_n \times K_n$. Since χ induces a homeomorphism

$$\bigvee_{(\sigma,k)\in\mathfrak{S}_n\times K_n} (\Delta^k/\partial\Delta^k) \longrightarrow X^n/(\Delta'_n \times A'_n),$$

χ_* is an isomorphism. □

Corollary 4.3. *The homology group H_n is a free $\mathfrak{S}_n$-module of rank $2g(2g+1)\ldots(2g+n-1)/n!$ with a basis $\{[\varphi_{(1_n,k)}] \mid k \in K_n\}$, where $1_n \in \mathfrak{S}_n$ is the unit.*

Proof. It is clear that $\mathbb{Z}(\mathfrak{S}_n \times K_n)$ is a free $\mathfrak{S}_n$-module with basis $\{(1_n, k) \mid k \in K_n\}$. The rank is equal to $\#K_n = 2g(2g+1)\ldots(2g+n-1)/n!$. □

Remark 1. Consequently, $H_n(\Sigma^n/\mathfrak{S}_n, (\Delta_n \cup A_n)/\mathfrak{S}_n; \mathbb{Z})$ is isomorphic to the nth symmetric tensor power $S^n H_1$ of H_1. The kernel of the action $\mathcal{M}_{g,1} \to \operatorname{Aut}(S^n H_1)$ is the Torelli group for any $n \geqslant 1$.

5. *A homomorphism* $\Phi : \mathbb{Z}\pi_1(\Sigma, p_0) \to \hat{H}$

Let $\mathbb{Z}\pi_1(\Sigma, p_0)$ be the group ring of $\pi_1(\Sigma, p_0)$ over $\mathbb{Z}$, which is an $\mathcal{M}_{g,1}$-module with the action induced from ρ. In this section we define some $\mathcal{M}_{g,1}$-equivariant ring homomorphism $\Phi : \mathbb{Z}\pi_1(\Sigma, p_0) \to \hat{H}$. Using this map, we will be able to compare the action of $\mathcal{M}_{g,1}$ on $\pi_1(\Sigma, p_0)$ with that on $\hat{H}$.

Let $\gamma \in \pi_1(\Sigma, p_0)$ be any element, and choose a representative $\tilde{\gamma} : [0,1] \to \Sigma$, $\tilde{\gamma}(0) = \tilde{\gamma}(1) = p_0$, of γ. For an integer $n \geqslant 1$, we define an n-chain $c^n_{\tilde{\gamma}} : \Delta^n \to \Sigma^n$ by

$$c^n_{\tilde{\gamma}}(t_1, t_2, \ldots, t_n) = (\tilde{\gamma}(t_1), \tilde{\gamma}(t_2), \ldots, \tilde{\gamma}(t_n)),$$

for $(t_1, t_2, \dots, t_n) \in \Delta^n$. Note that $c^n_{\tilde{\gamma}}(\partial\Delta^n) \subset \Delta_n \cup A_n$ and that the homology class $[c^n_{\tilde{\gamma}}] \in H_n$ of $c^n_{\tilde{\gamma}}$ depends only on γ and does not depend on the choice of $\tilde{\gamma}$.

Definition 1. Define an additive homomorphism $\phi_n : \mathbb{Z}\pi_1(\Sigma, p_0) \to H_n$ by

$$\phi_n(\gamma) = \begin{cases} [c^n_{\tilde{\gamma}}] & \text{if } n \geqslant 1, \\ 1 & \text{if } n = 0, \end{cases}$$

for any $\gamma \in \pi_1(\Sigma, p_0)$, and define the map $\Phi : \mathbb{Z}\pi_1(\Sigma, p_0) \to \hat{H}$ to be the formal series $\Phi = \sum_{n \geqslant 0} \phi_n$.

Clearly, every ϕ_n, $n \geqslant 0$, and Φ are $\mathcal{M}_{g,1}$-module homomorphisms. Let $I = \operatorname{Ker}\phi_0$ denote the augmentation ideal of $\mathbb{Z}\pi_1(\Sigma, p_0)$. Then $\mathbb{Z}\pi_1(\Sigma, p_0)$ is a filtered algebra with the descending filtration $\{I^n\}_{n \geqslant 0}$. The action of $\mathcal{M}_{g,1}$ on $\mathbb{Z}\pi_1(\Sigma, p_0)$ preserves this filtration.

Proposition 5.1. *The map Φ is a filtered algebra homomorphism.*

Proof. We need only to prove that Φ preserves the product and the filtration. The function Φ preserves the product if and only if

$$\phi_n(\gamma\delta) = \sum_{i=0}^{n} \phi_i(\gamma)\phi_{n-i}(\delta) \tag{5.1}$$

for any $\gamma, \delta \in \pi_1(\Sigma, p_0)$ and $n \geqslant 0$. To prove this, we consider the partition of Δ^n as follows:

$$\Delta^n = D_0 \cup D_1 \cup \dots \cup D_n,$$
$$D_i = \left\{(x_1, \dots, x_n) \in \Delta^n \;\middle|\; x_i \leqslant \tfrac{1}{2} \leqslant x_{i+1}\right\}, \quad (0 \leqslant i \leqslant n),$$

where $x_0 = 0$, $x_{n+1} = 1$. Let $\tilde{\gamma}, \tilde{\delta} : ([0,1], \{0,1\}) \to (\Sigma, p_0)$ be the paths representing γ and δ respectively. Let $\tilde{\gamma}\tilde{\delta}$ be the path defined by

$$\tilde{\gamma}\tilde{\delta}(t) = \begin{cases} \tilde{\gamma}(2t) & 0 \leqslant t \leqslant \frac{1}{2}, \\ \tilde{\delta}(2t-1) & \frac{1}{2} \leqslant t \leqslant 1, \end{cases}$$

which represents $\gamma\delta$. Since $\tilde{\gamma}\tilde{\delta}(1/2) = p_0$, the homology class $[c^n_{\tilde{\gamma}\tilde{\delta}}|_{D_i}] \in H_n$ of $c^n_{\tilde{\gamma}\tilde{\delta}}|_{D_i} : (D_i, \partial D_i) \to (\Sigma, p_0)^{\overline{n}}$ can be defined, and we have

$$\phi_n(\gamma\delta) = [c^n_{\tilde{\gamma}\tilde{\delta}}|_{D_0}] + [c^n_{\tilde{\gamma}\tilde{\delta}}|_{D_1}] + \dots + [c^n_{\tilde{\gamma}\tilde{\delta}}|_{D_n}].$$

The equation

$$[c^n_{\tilde{\gamma}\tilde{\delta}}|_{D_i}] = [c^i_{\tilde{\gamma}}][c^{n-i}_{\tilde{\delta}}] = \phi_i(\gamma)\phi_{n-i}(\delta)$$

is a consequence of the natural homeomorphism $D_i \cong \Delta^i \times \Delta^{n-i}$. Therefore, we obtain (5.1) as required.

By Lemma 5.2(i) below, the restriction $(\phi_0 + \phi_1 + \dots + \phi_{n-1})|_{I^n}$ is zero; that is, $\Phi(I^n) \subset \mathcal{F}_n(\hat{H})$. Hence Φ preserves the filtration. □

Lemma 5.2. *Let $n \geqslant 1$ be an integer. For any element of the form $(\gamma_1 - 1)(\gamma_2 - 1)\dots(\gamma_n - 1) \in I^n$, $\gamma_i \in \pi_1(\Sigma, p_0)$, we have*

$$\Phi\big((\gamma_1 - 1)(\gamma_2 - 1)\dots(\gamma_n - 1)\big) \equiv \phi_1(\gamma_1)\phi_1(\gamma_2)\dots\phi_1(\gamma_n)$$

modulo $\mathcal{F}_{n+1}\hat{H}$. In particular, we have:

(i) $\operatorname{Ker}\phi_{n-1} \supset I^n$,

(ii) $\phi_n\big((\gamma_1 - 1)(\gamma_2 - 1)\dots(\gamma_n - 1)\big) = \phi_1(\gamma_1)\phi_1(\gamma_2)\dots\phi_1(\gamma_n)$.

Proof. The statement is immediate from the facts that $\Phi(\gamma_i - 1) \equiv \phi_1(\gamma_i)$ modulo $\mathcal{F}_2\hat{H}$ and that Φ is a ring homomorphism. □

6. *Properties of* Φ

Let $q_n : \mathbb{Z}\pi_1(\Sigma, p_0) \to \mathbb{Z}\pi_1(\Sigma, p_0)/I^{n+1}$ be the quotient map. Since $\operatorname{Ker}\phi_n \supset I^{n+1}$ by Proposition 5.2(i), ϕ_n induces the homomorphism

$$\phi'_n : \mathbb{Z}\pi_1(\Sigma, p_0)/I^{n+1} \longrightarrow H_n$$

such that $\phi'_n \circ q_n = \phi_n$. The homomorphism Φ gives the associated graded homomorphism

$$\operatorname{gr}\Phi : \operatorname{gr}\mathbb{Z}\pi_1(\Sigma, p_0) \longrightarrow \operatorname{gr}\hat{H}$$

such that $\operatorname{gr}\mathbb{Z}\pi_1(\Sigma, p_0) = \bigoplus_{n\geqslant 0}\operatorname{gr}_n\mathbb{Z}\pi_1(\Sigma, p_0)$, $\operatorname{gr}_n\mathbb{Z}\pi_1(\Sigma, p_0) = I^n/I^{n+1}$ and $\operatorname{gr}_n\Phi = \phi'_n|_{I^n/I^{n+1}} : I^n/I^{n+1} \to H_n$.

Remember that there is an injective graded algebra homomorphism $\Psi : T[H_1] \to \operatorname{gr}\hat{H}$ (Corollary 3.4(ii)).

Lemma 6.1. *The graded homomorphism* $\operatorname{gr}\Phi$ *is an isomorphism onto the subalgebra* $\operatorname{Im}\Psi \subset \hat{H}$.

Proof. Clearly, $\operatorname{gr}_0\Phi$ is an isomorphism. Suppose that $n \geqslant 1$. By Lemma 5.2, for $\gamma_1, \gamma_2, \ldots, \gamma_n \in \mathbb{Z}\pi_1(\Sigma, p_0)$, we have

$$\operatorname{gr}_n\Phi((\gamma_1 - 1)(\gamma_2 - 1)\ldots(\gamma_n - 1)) = \psi_n([\gamma_1] \otimes [\gamma_2] \otimes \ldots \otimes [\gamma_n]),$$

where $[\gamma_i]$ is the homology class of γ_i. Since the homomorphism $I^n/I^{n+1} \to H_1^{\otimes n}$ defined by

$$(\gamma_1 - 1)(\gamma_2 - 1)\ldots(\gamma_n - 1) \longmapsto [\gamma_1] \otimes [\gamma_2] \otimes \ldots \otimes [\gamma_n]$$

is an isomorphism, $\operatorname{gr}_n\Phi$ is an isomorphism onto $\operatorname{Im}\psi_n$. Hence, $\operatorname{gr}\Phi$ is an isomorphism onto $\operatorname{Im}\Psi$. □

Let $\Phi_n : \mathbb{Z}\pi_1(\Sigma, p_0)/I^{n+1} \to \hat{H}/\mathcal{F}_{n+1}\hat{H}$ be the homomorphism induced from Φ.

Proposition 6.2. *The homomorphism* Φ_n *is injective.*

Proof. Write $Q_n = \mathbb{Z}\pi_1(\Sigma, p_0)/I^n$, $n \geqslant 0$. Since Φ preserves the filtrations (Proposition 5.1), there exists a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I^n/I^{n+1} & \longrightarrow & Q_{n+1} & \longrightarrow & Q_n & \longrightarrow & 0 \\ & & {\scriptstyle \operatorname{gr}_n\Phi}\big\downarrow & & {\scriptstyle \Phi_n}\big\downarrow & & {\scriptstyle \Phi_{n-1}}\big\downarrow & & \\ 0 & \longrightarrow & H_n & \longrightarrow & \hat{H}/\mathcal{F}_{n+1}\hat{H} & \longrightarrow & \hat{H}/\mathcal{F}_n\hat{H} & \longrightarrow & 0 \end{array}$$

where the two rows are short exact sequences. By Lemma 6.1, $\operatorname{gr}_n\Phi$ is injective for any $n \geqslant 0$, and therefore the injectivity of Φ_n is proved by induction on n. □

Lemma 6.3. *Let* $\alpha_1, \alpha_2, \ldots, \alpha_{2g}$ *be the free generators of* $\pi_1(\Sigma, p_0)$ $(\cong \pi_1(X, p_0))$ *introduced in Section* 4. *For any* $(\sigma, k) \in \mathfrak{S}_n \times K_n$ *such that* $k = (k_1, \ldots, k_{2g})$,

$$[\varphi_{(\sigma,k)}] = \sigma_*\left(\phi_{k_1}(\alpha_1)\phi_{k_2}(\alpha_2)\ldots\phi_{k_{2g}}(\alpha_{2g})\right).$$

Proof. Since $[\varphi_{(\sigma,k)}] = \sigma_*[\varphi_{(1_n,k)}]$ by Proposition 4.2(ii), we prove the statement only when $\sigma = 1_n$. Let $l_i = (0, \ldots, 0, k_i, 0, \ldots, 0) \in K_{k_i}$ be the $2g$-tuple of integers such that the only ith component is k_i and the others are zero. Referring to the construction of $\varphi_{(\sigma,k)}$ in the proof of Proposition 4.2, one can verify that

$$[\varphi_{1_{k_i},l_i}] = \phi_{k_i}(\alpha_i)$$

and

$$[\varphi_{(1_n,k)}] = [\varphi_{(1_{k_1},l_1)}][\varphi_{(1_{k_2},l_2)}]\dots[\varphi_{(1_{k_{2g}},l_{2g})}].$$

Therefore, we obtain

$$[\varphi_{(1_n,k)}] = \phi_{k_1}(\alpha_1)\phi_{k_2}(\alpha_2)\dots\phi_{k_{2g}}(\alpha_{2g}).$$ □

Let $R \subset \mathrm{gr}\,\hat{H}$ be the subalgebra, generated by $\bigcup_{n\geqslant 0}\operatorname{Im}\phi_n$, over $\mathbb{Z}$.

Proposition 6.4. *The module $R_n = R \cap H_n$ generates H_n as an $\mathfrak{S}_n$-module.*

Proof. By Corollary 4.3, H_n is generated by the set $\{[\varphi_{(1_n,k)}] \mid k \in K_n\}$ as an $\mathfrak{S}_n$-module. This generating set is contained in R_n by Lemma 6.3. Therefore, R_n generates H_n as an $\mathfrak{S}_n$-module. □

7. *Proof of Theorem* A

In the following, we will work with the three $\mathcal{M}_{g,1}$-modules $\mathbb{Z}\pi_1(\Sigma,p_0)/I^{n+1}$, $\hat{H}/\mathcal{F}_{n+1}\hat{H}$, and R_n (instead of $\pi_1(\Sigma,p_0)$ and H_n) to compare the subgroups $\mathcal{M}_{g,1}(n)$ and $\mathcal{M}_{g,1}(n)'$. As we will see, the kernels of the actions on these modules coincide with $\mathcal{M}_{g,1}(n)$, $\mathcal{M}_{g,1}(n)'$, and $\mathcal{M}_{g,1}(n)'$, respectively. The last two kernels are determined with no difficulty and will be described in the proof of Theorem A below. For the first group $\mathbb{Z}\pi_1(\Sigma,p_0)/I^{n+1}$, the kernel can be determined as a consequence of a classical result given by Fox [2].

Lemma 7.1. *For any integer $n \geqslant 0$, the kernel of the action*

$$\mathcal{M}_{g,1} \longrightarrow \operatorname{Aut}(\mathbb{Z}\pi_1(\Sigma,p_0)/I^{n+1})$$

is $\mathcal{M}_{g,1}(n)$.

Proof. This statement is proved easily by using the fact that $\gamma \in \pi_1(\Sigma,p_0)$ belongs to Γ_{n+1} if and only if $\gamma - 1 \in I^{n+1}$ (see [2]). □

We are now ready to prove Theorem A. For this, let us recall that we have an $\mathcal{M}_{g,1}$-module homomorphism $\Phi_n : \mathbb{Z}\pi_1(\Sigma,p_0)/I^{n+1} \to \hat{H}/\mathcal{F}_{n+1}\hat{H}$. Also recall that, by definition, any element in R_n is the sum of elements of the form $a_1a_2\dots a_l$, such that $a_j \in \operatorname{Im}\phi_{k_j}$, $k_j \geqslant 0$, and $\sum_{j=1}^{l} k_j = n$.

Proof of Theorem A. First, we will prove that $\mathcal{M}_{g,1}(n)' \subset \mathcal{M}_{g,1}(n)$. Let S denote the kernel of the action of $\mathcal{M}_{g,1}$ on $\hat{H}/\mathcal{F}_{n+1}\hat{H}$. Since Φ_n is injective (Proposition 6.2), we have $S \subset \mathcal{M}_{g,1}(n)$ by Lemma 7.1. On the other hand, $\hat{H}/\mathcal{F}_{n+1}\hat{H}$ is isomorphic to $H_0 \oplus H_1 \oplus \dots \oplus H_n$ as an $\mathcal{M}_{g,1}$-module, and thus $S = \bigcap_{i=0}^{n}\mathcal{M}_{g,1}(n)' = \mathcal{M}_{g,1}(n)'$ by Corollary 3.4(i). Hence, we have $\mathcal{M}_{g,1}(n)' \subset \mathcal{M}_{g,1}(n)$.

Next, we will prove the converse result that $\mathcal{M}_{g,1}(n)' \supset \mathcal{M}_{g,1}(n)$. Since R_n generates H_n as an $\mathfrak{S}_n$-module (Proposition 6.4), it is enough to prove that $\mathcal{M}_{g,1}(n)$ acts on $\operatorname{Im}\phi_i$ trivially for all $i = 1,2,\dots,n$. Let $\varphi \in \mathcal{M}_{g,1}(n)$ and $\gamma \in \pi_1(\Sigma,p_0)$ be any two elements; then

$$\varphi_*(\phi_i(\gamma)) = \phi_i'(\varphi_*(q_i(\gamma))).$$

Here, the notation φ_* denotes the action of φ on H_i and $\mathbb{Z}\pi_1(\Sigma, p_0)/I^{i+1}$. It follows from Lemma 7.1 and $\mathcal{M}_{g,1}(i) \supset \mathcal{M}_{g,1}(n)$ that $\varphi_*(q_i(\gamma)) = q_i(\gamma)$, and therefore $\varphi_*(\phi_i(\gamma)) = \phi_i(\gamma)$. Hence, $\mathcal{M}_{g,1}(n)$ acts on $\operatorname{Im}\phi_i$ trivially.

This completes the proof of Theorem A. □

As a corollary of Theorem A, it turns out that $\rho' : \mathcal{M}_{g,1} \to \operatorname{Aut}(\hat{H})$ is faithful.

COROLLARY 7.2. *The action* $\rho' : \mathcal{M}_{g,1} \to \operatorname{Aut}(\hat{H})$ *is faithful.*

Proof. By Theorem A, $\operatorname{Ker}\rho' \subset \bigcap_{n\geqslant 0} \mathcal{M}_{g,1}(n) = \{1\}$. □

8. *Some cohomology of the mapping class group*

For integers n, l such that $n \geqslant 0$, let $\mathbf{S}(n,l)$ be the set of 'ordered partitions' of the set $I_n = \{1, 2, \ldots, n\}$ with l blocks. More precisely, $\mathbf{S}(n,l)$ is defined as follows. In the case $l > 0$, we define

$$\mathbf{S}(n,l) = \left\{ (J_1, J_2, \ldots, J_l) \;\middle|\; \begin{array}{c} \emptyset \subsetneq J_i \subset I_n \\ J_1 \cup J_2 \cup \ldots \cup J_l = I_n \\ J_i \cap J_j = \emptyset \quad (i \neq j) \end{array} \right\},$$

and in the case $l \leqslant 0$, we define

$$\mathbf{S}(n,l) = \begin{cases} \{()\} & \text{if } (n,l) = (0,0), \\ \emptyset & \text{otherwise.} \end{cases}$$

Here, $\mathbf{S}(0,0)$ is understood as the set consisting of the 'empty partition' $()$ of $I_0 = \emptyset$. By definition, $\mathbf{S}(n,l) = \emptyset$ if $n < l$ or $n > l = 0$. The set $\mathbf{S}(n,l)$ has a free right $\mathfrak{S}_l$-action defined by $J\sigma = (J_{\sigma^{-1}(1)}, J_{\sigma^{-1}(2)}, \ldots, J_{\sigma^{-1}(l)})$ for $J = (J_1, J_2, \ldots, J_l) \in \mathbf{S}(n,l)$ and $\sigma \in \mathfrak{S}_l$, and it also has a left $\mathfrak{S}_n$-action as a permutation of I_n. Consequently, the free abelian group $\mathbb{Z}\mathbf{S}(n,l)$ generated by $\mathbf{S}(n,l)$ is a free right $\mathfrak{S}_l$-module and a left $\mathfrak{S}_n$-module. We remark that the number $S(n,l) = \#(\mathbf{S}(n,l)/\mathfrak{S}_l)$ of elements in the quotient set $\mathbf{S}(n,l)/\mathfrak{S}_l$ is nothing but the Stirling number of the second kind.

Recall that the $(\mathfrak{S}_n \times \mathcal{M}_{g,1})$-module H_n is defined only when n is a non-negative integer. For simplicity of notation, if $n < 0$ then we denote by $H_n = \{0\}$ the trivial $(\mathfrak{S}_n \times \mathcal{M}_{g,1})$-module, where $\mathfrak{S}_n = \{1\}$ is the trivial group. Similarly, if $n < 0$ then we define $\Sigma^n = A_n = \emptyset$, on which $\mathfrak{S}_n$ acts trivially.

For integers p, q such that $q \geqslant 0$, we define an $(\mathfrak{S}_q \times \mathcal{M}_{g,1})$-module $C^{p,q}$ by

$$C^{p,q} = \mathbb{Z}\mathbf{S}(q, q-p) \bigotimes_{\mathbb{Z}\mathfrak{S}_{q-p}} H_{q-p}.$$

Note that $C^{p,q}$ is isomorphic to $\mathbb{Z}(\mathbf{S}(q,q-p)/\mathfrak{S}_{q-p}) \bigotimes_{\mathbb{Z}} H_{q-p} \cong H_{q-p}^{\oplus S(q,q-p)}$ with the $\mathfrak{S}_q$-action induced from the left action on $\mathbf{S}(q,q-p)/\mathfrak{S}_{q-p}$. For example,

$$\mathbf{S}(3,2) = \begin{Bmatrix} (\{1\},\{2,3\}), & (\{2\},\{3,1\}), & (\{3\},\{1,2\}) \\ (\{2,3\},\{1\}), & (\{3,1\},\{2\}), & (\{1,2\},\{3\}) \end{Bmatrix},$$

and so $C^{1,3}$ is isomorphic to $H_2^{\oplus S(3,2)} = H_2^{\oplus 3}$ (with the permutation action). See also equation (1.1) of $C^{p,q}$ given in Section 1. By definition, $C^{p,q} = \{0\}$ if $0 = p \leqslant q$ or $p > q$.

In order to explain a geometric meaning of $C^{p,q}$ (Lemma 8.1), we introduce a filtration of (Σ^n, A_n):

$$\ldots = A_{n,-1} \subset A_{n,0} \subset A_{n,1} \subset \ldots \subset A_{n,n-1} \subset A_{n,n} = \ldots \tag{8.1}$$

such that $A_{n,-1} = A_n$ and $A_{n,n} = \Sigma^n$ as follows. In the case $n > 0$, we define

$$\Delta_{n,l} = \{(x_1, x_2, \dots, x_n) \in \Sigma^n \mid \#\{x_1, x_2, \dots, x_n\} \leqslant l\},$$
$$A_{n,l} = \Delta_{n,l} \cup A_n,$$

and in the case $n = 0$, we define

$$A_{0,l} = \begin{cases} \Sigma^0 \ (= \{\mathrm{pt}\}) & \text{if } l \geqslant 0, \\ \emptyset & \text{if } l < 0. \end{cases}$$

Let

$$F_{n,l} : (\mathbf{S}(n,l) \times_{\mathfrak{S}_l} \Sigma^l, \mathbf{S}(n,l) \times_{\mathfrak{S}_l} A_l) \longrightarrow (A_{n,l}, A_{n,l-1})$$

be the map defined by

$$F_{n,l}([J,x]) = (y_1, y_2, \dots, y_n), \quad y_i = x_j \text{ if } i \in J_j$$

for $[J,x] = [(J_1, J_2, \dots, J_l), (x_1, x_2, \dots, x_l)] \in \mathbf{S}(n,l) \times_{\mathfrak{S}_l} \Sigma^l$, where $\mathbf{S}(n,l) \times_{\mathfrak{S}_l} \Sigma^l$ is the quotient space of $\mathbf{S}(n,l) \times \Sigma^l$ by the equivalence relation $(J\sigma, x) \sim (J, \sigma x)$, $(J,x) \in \mathbf{S}(n,l) \times \Sigma^l$, $\sigma \in \mathfrak{S}_l$, and $[J,x]$ is the quotient image of (J,x). Here, in the extreme case $(n,l) = (0,0)$, the set $\mathbf{S}(0,0) \times_{\mathfrak{S}_0} \Sigma^0$ consists of exactly one element $[(), \mathrm{pt}]$, and $F_{0,0}$ is defined to be the unique map, namely, $F_{0,0}([(), \mathrm{pt}]) = \mathrm{pt}$.

Immediately, there is a canonical isomorphism

$$H_l(\mathbf{S}(n,l) \times_{\mathfrak{S}_l} \Sigma^l, \mathbf{S}(n,l) \times_{\mathfrak{S}_l} A_l; \mathbb{Z}) \cong \mathbb{Z}\mathbf{S}(n,l) \bigotimes_{\mathbb{Z}\mathfrak{S}_l} H_l = C^{n-l,n},$$

and hence $F_{n,l}$ gives an $(\mathfrak{S}_n \times \mathcal{M}_{g,1})$-module homomorphism

$$F_{n,l*} : C^{n-l,n} \longrightarrow H_l(A_{n,l}, A_{n,l-1}; \mathbb{Z}). \tag{8.2}$$

Lemma 8.1. *For integers n, l such that $n \geqslant 0$, the homology group $H_m(A_{n,l}, A_{n,l-1}; \mathbb{Z})$ vanishes if $m \neq l$, and $F_{n,l*}$ is an isomorphism.*

Proof. The proof is very similar to that of Lemma 3.2. Let $f_{n,l} : \mathbf{S}(n,l) \times_{\mathfrak{S}_l} A_l \to A_{n,l-1}$ denote the restriction of $F_{n,l}$, and let

$$V = A_{n,l-1} \bigcup_{f_{n,l}} (\mathbf{S}(n,l) \times_{\mathfrak{S}_l} \Sigma^l)$$

be the attaching space. It follows from Proposition 3.3(i) that

$$H_m(V, A_{n,l-1}; \mathbb{Z}) \cong \mathbb{Z}\mathbf{S}(n,l) \bigotimes_{\mathbb{Z}\mathfrak{S}_l} H_m(\Sigma^l, \Delta_l \cup A_l)$$
$$= \begin{cases} C^{n-l,n} & \text{if } m = l, \\ 0 & \text{if } m \neq n. \end{cases}$$

There $F_{n,l}$ induces a homeomorphism

$$(V, A_{n,l-1}) \cong (A_{n,l}, A_{n,l-1}),$$

which is proved in the same way as the proof that the map f_2 (defined in the proof of Lemma 3.2) is a homeomorphism. Therefore, $H_l(V, A_{n,l-1}; \mathbb{Z}) \cong H_l(A_{n,l}, A_{n,l-1}; \mathbb{Z})$. □

From now on, we will identify $C^{p,q}$ with $H_{q-p}(A_{q,q-p}, A_{q,q-p-1}; \mathbb{Z})$ via the isomorphism (8.2). Let

$$d_{p,q} : C^{p,q} \longrightarrow C^{p+1,q}$$

denote the homology connecting homomorphism for the triple

$$(A_{q,q-p}, A_{q,q-p-1}, A_{q,q-p-2}),$$

and write

$$d_p = \bigoplus_{q\geqslant 0} d_{p,q} : C^p \longrightarrow C^{p+1}, \quad C^p = \bigoplus_{q\geqslant 0} C^{p,q}.$$

Since $d_{p+1}d_p = 0$, the sequence

$$0 \longrightarrow C^0 \xrightarrow{d_0} C^1 \xrightarrow{d_1} C^2 \xrightarrow{d_2} \dots \xrightarrow{d_{p-1}} C^p \xrightarrow{d_p} \dots$$

is a cochain complex of $\mathcal{M}_{g,1}$-modules, and we denote this cochain complex by C. Since the set $\mathbf{S}(q,q)$ consists of exactly one element $(\{1\},\{2\},\dots,\{q\})$ for $q \geqslant 0$, we shall identify the $(\mathfrak{S}_q \times \mathcal{M}_{g,1})$-modules $C^{0,q}$ with H_q. Consequently, $C^0 = \operatorname{gr} \hat{H}$. Let $H^*(\mathcal{M}_{g,1}; C)$ denote the cohomology group of the group $\mathcal{M}_{g,1}$ with coefficients in C (cf. [**1**, VII,5]).

Remember that there is an injective $\mathcal{M}_{g,1}$-module homomorphism (Corollary 3.4(ii)):

$$\Psi = \bigoplus_{q\geqslant 0} \psi_q : T[H_1] \longrightarrow \operatorname{gr} \hat{H} = C^0.$$

Proposition 8.2. *The sequence of $\mathcal{M}_{g,1}$-module homomorphisms*

$$0 \longrightarrow T[H_1] \xrightarrow{\Psi} C^0 \xrightarrow{d_0} C^1 \xrightarrow{d_1} \dots \xrightarrow{d_{p-1}} C^p \xrightarrow{d_p} \dots \tag{8.3}$$

is exact.

Proof. Fix an integer $n \geqslant 0$, and consider the homology spectral sequence associated to the filtration (8.1) of (Σ^n, A_n):

$$E^1_{r,s} = H_{r+s}(A_{n,r}, A_{n,r-1}; \mathbb{Z}) \quad \Longrightarrow \quad H_{r+s}(\Sigma^n, A_n; \mathbb{Z})$$

(cf. [**13**, pp. 472–473]). By Lemma 8.1,

$$E^1_{r,s} = \begin{cases} C^{n-r,n} & \text{if } s = 0, \\ 0 & \text{otherwise,} \end{cases}$$

and the differential $d^1_{r,0} : E^1_{r,0} \to E^1_{r-1,0}$ coincides with $d_{n-r,n}$. Since $E^1_{r,s} = 0$ unless $s = 0$, the spectral sequence converges at the E^2-term so that $E^2_{r,0} \cong E^\infty_{r,0}$. On the other hand,

$$E^\infty_{r,0} = H_r(\Sigma^n, A_n; \mathbb{Z}) = \begin{cases} {H_1}^{\otimes n} & \text{if } r = n, \\ 0 & \text{otherwise.} \end{cases}$$

This means that the sequence

$$0 \longrightarrow {H_1}^{\otimes n} \xrightarrow{\psi_n} C^{0,n} \xrightarrow{d_{0,n}} C^{1,n} \xrightarrow{d_{1,n}} \dots \xrightarrow{d_{n-1,n}} C^{n,n} \xrightarrow{d_{n,n}} 0 \longrightarrow \dots$$

is exact, and this is the degree-n part of (8.3). □

Let $H^*(\mathcal{M}_{g,1}; T[H_1])$ be the cohomology of $\mathcal{M}_{g,1}$ with coefficients in $T[H_1]$. Note that $H^*(\mathcal{M}_{g,1}; T[H_1])$ coincides with the cohomology group of $\mathcal{M}_{g,1}$ with coefficients in the trivial cochain complex $0 \to T[H_1] \to 0 \to 0 \to \dots$. Let

$$\Psi_* : H^*(\mathcal{M}_{g,1}; T[H_1]) \longrightarrow H^*(\mathcal{M}_{g,1}; C) \tag{8.4}$$

be the homomorphism obtained by regarding Ψ as a cochain homomorphism.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T[H_1] & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \dots \\ \downarrow & & \Psi\downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C^0 & \xrightarrow{d_0} & C^1 & \xrightarrow{d_1} & C^2 & \xrightarrow{d_2} & \dots \end{array} \tag{8.5}$$

We are now ready to prove Theorem B.

Proof of Theorem B. Proposition 8.2 implies that the cochain homomorphism (8.5) is a quasi-isomorphism. Consequently, Ψ_* is an isomorphism. □

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