Math 8803: Characteristic Classes of Vector Bundles and Surface Bundles

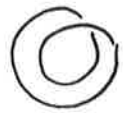
Fall 2013
Georgia Tech

Dan Margalit

Theory of Characteristic classes:

$$\boxed{\text{Bundles over } B} \longrightarrow \boxed{H^*(B)}$$

so as to distinguish bundles, e.g.

 vs 

This course: Vector bundles, surface bundles.

Vector Bundles

$$\begin{array}{c} E \\ p\big\downarrow \\ B \end{array}$$

B = base

$p^{-1}(B)$ = fiber $\leftarrow$ struct. of vector space V.

B covered by U s.t.

$p^{-1}(U) \longrightarrow U \times V$ homeo respecting v.s. structure of fibers

Important because smooth manifolds have tangent bundles, submanifolds have normal bundles.

e.g. can distinguish two smooth structures on a manifold if we can distinguish their tangent bundles using characteristic classes.

<u>Thm</u> (Milnor) $\exists$ exotic 7-spheres.

CHARACTERISTIC CLASSES

A char. class for vect. bundles is a function

$$\chi : \{V\text{-bundles over } B\} \longrightarrow H^k(B;G)$$

for fixed V, k, G (B allowed to vary!)
that is natural:

$$\chi(f^*(E)) = f^* \chi(E)$$

EULER CLASS

Take $V=\mathbb{R}^n$, $k=n$, $G=\mathbb{Z}$, restrict to oriented bundles.

$\rightsquigarrow$ *Euler class* e.

$$B=M \quad E=TM \rightsquigarrow \underset{\chi(M)}{\overset{\shortparallel}{e(TM)}} \in H^n(M;\mathbb{Z}) \cong \mathbb{Z}.$$

Euler char is a char. class. It has many interpretations, e.g.:

(1) Combinatorial: $\chi(M) = \Sigma(-1)^i(\#\ i\text{-cells})$
(2) Geometric: $\chi(M) = \frac{1}{\operatorname{vol} S^n}\int_M K(x)\, d\mathrm{vol}_M$
(3) Homological: $\chi(M) = \Sigma(-1)^i \operatorname{rank} H_i(M;\mathbb{Z})$
(4) Cohomological $\chi(M) =$ self-intersection of M in TM.

(4) implies $\chi(M)$ is obstruction to nonvanishing vector field (recall Thurston's proof).

Grassmann Manifolds

Euler class is so beautiful, we want to find all other char classes.

G_n = space of n-planes in $\mathbb{R}^\infty$.
E_n = canonical bundle over G_n:

$$(\text{n-plane in } \mathbb{R}^\infty, \text{ vector in that plane}) \subseteq G_n \times \mathbb{R}^\infty.$$

We will show:

$$\left\{ \begin{matrix} \mathbb{R}^n\text{-bundles} \\ \text{over } B \end{matrix} \right\} \Big/ \text{isomorp.} \quad \longleftrightarrow \quad \left\{ \begin{matrix} \text{maps} \\ B \to G_n \end{matrix} \right\} \Big/ \text{homotopy.}$$

$$f^*(E_n) \longleftarrow\!\shortmid f$$

This gives:

$$\left\{ \begin{matrix} \text{char. classes for } \mathbb{R}^n\text{-bundles} \\ G\text{-coeff} \end{matrix} \right\} \longleftrightarrow H^*(G_n; G).$$

Goal: compute the latter.

If we care about:

$$\text{complex bundles} \rightsquigarrow G_n(\mathbb{C})$$
$$\text{oriented real bundles} \rightsquigarrow \widetilde{G}_n$$

Stiefel-Whitney Classes

We will show: $H^*(G_n;\mathbb{Z}_2) \cong \mathbb{Z}_2[w_1,\dots,w_n]$

w_i called i^{th} SW class.

w_1 is very concrete $\in H^1(B;\mathbb{Z}_2) \cong \mathrm{Hom}(H_1(B;\mathbb{Z}_2);\mathbb{Z})$

It records whether the bundle is orientable over an element of H_1.

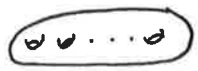
w_i = obstruction to finding $n-k+1$ indep. sections over the i-skeleton of B.

<u>Thm</u> (Thom). Two manifolds are cobordant iff their SW numbers of their tangent bundles are equal.

Other Characteristic Classes

vector bundle	coeff.	characteristic classes
real	$\mathbb{Z}_2$	SW
complex	$\mathbb{Z}$	Chern
real	$\mathbb{Z}$	Pontryagin, SW
oriented real	$\mathbb{Z}$	Pont., SW, Euler.

Surface Bundles

$S_g =$ 

S_g-bundle $\begin{array}{c} E \\ p \downarrow \\ B \end{array}$ $\qquad p^{-1}(U) \cong U \times S_g$

Important class of manifolds (also, they are the next-simplest bundles).

Characteristic class

$$\chi : \left\{ \begin{array}{c} \text{oriented} \\ S_g\text{-bundles} \\ \text{over } B \end{array} \right\} \Big/ \text{isom.} \longrightarrow H^k(B; G)$$

naturality $\chi(f^*(E)) = f^*(\chi(E))$

Classifying space

$$\left\{ \begin{array}{c} \text{oriented} \\ S_g\text{-bundles} \\ \text{over } B \end{array} \right\} \Big/ \text{isom.} \longleftrightarrow \left\{ \begin{array}{c} \text{maps} \\ B \to B\mathrm{Homeo}^+(S_g) \end{array} \right\} \Big/ \text{hom.}$$

$$\begin{aligned} B\mathrm{Homeo}^+(S_g) &= \text{space of } S_g\text{-submanifolds of } \mathbb{R}^\infty \\ &= K(\mathrm{MCG}(S_g), 1) \end{aligned}$$

So: Char. classes for orient. S_g-bundles $\Longleftrightarrow H^*(\mathrm{MCG}(S_g); G)$.

We do not have a full list, but

$$e_i \in H^{2i}(\mathrm{MCG}(S_g); \mathbb{Z}) \qquad \text{Morita-Mumford-Miller classes}$$

generate $H^*(\mathrm{MCG}(S_g))$ stably (Madsen-Weiss).

Morita's Theorem

$\pi : \mathrm{Diff}^+(S_g) \longrightarrow MCG(S_g)$ has no section $g \gg 0$.

Proof: $e_3 \neq 0$, $\pi^*(e_3) = 0$.

Odd MMM classes are geometric.

$e_1 \in H^2(B; \mathbb{Z})$ WLOG: B = surface.
$\Rightarrow E$ = 4-manifold M

Hirzebruch: $e_1(E) = \sigma(M)$ signature.

But σ (hence e_1) ignores bundle structure even though e_1 defined via bunde structure.
Say e_1 is geometric.

<u>Thm</u> (Church-Farb-Thibault) e_{2i+1} is geometric.

e.g. $\exists$ S_4-bundle over $S_{17} \cong S_{49}$ bundle over S_2.

Pf that e_1 is geometric: $e_1(E) = p_1(M) \leftarrow$ 1st Pontryagin class.
$= \sigma(M)$ (Hirzebruch).

Vector Bundles

Fix a vector space V

$$\begin{array}{ccc} V & \to & E \\ & & \downarrow p \\ & & B \end{array}$$

① Fibers $p^{-1}(b)$ have structure of V.

② B covered by U s.t. $\exists$

$p^{-1}(U) \longrightarrow U \times V$ homeo resp. structure on fibers.

local trivialization

Examples

① Trivial bundle $E = B \times V$.

② Möbius bundle over S^1.

③ Tangent bundle to a smooth manifold M

$TM = \{(x, v) : v \in T_x M\}$

$p(x, v) = x$

V.S. structure: $k_1(x, v_1) + k_2(x, v_2) = (x, k_1 v_1 + k_2 v_2)$

By defn, M locally diffeo to $U \subseteq \mathbb{R}^n$ open.
So suffices to show TU locally trivial. <u>easy</u>

④ Normal bundle to $M \hookrightarrow N$

Locally: $\mathbb{R}^n \hookrightarrow \mathbb{R}^{n+k}$ (Tubular nbhd thm).

⑤ Canonical bundle over $\mathbb{RP}^n$

$\mathbb{RP}^n$ = space of lines in $\mathbb{R}^{n+1}$ $\cong S^n/\text{antipode}$
Canonical line bundle: $\{(\ell, v) : v \in \ell\}$
Local trivialization near ℓ: orthog. proj. to ℓ in $\mathbb{R}^{n+1}$.
e.g. $(\ell', v) \mapsto (\ell', \text{proj}_\ell(v)) \in U \times \ell$.

Allow $n = \infty$.

⑥ Orthogonal complement to ⑤

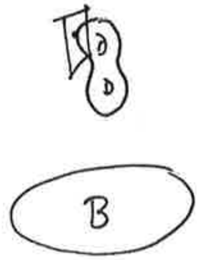
$E^\perp = \{(\ell, v) \in \mathbb{RP}^n \times \mathbb{R}^{n+1} : v \perp \ell\}$
Again, orthog proj gives local trivialization.

Q. $E^\perp \cong T\mathbb{RP}^n$?

⑦ Grassmann manifold

G_n = space of n-planes in $\mathbb{R}^\infty$ thru 0.
$E_n = \{(P, v) \in G_n \times \mathbb{R}^\infty : v \in P\}$
& $E_n^\perp = \{(P, v) \in G_n \times \mathbb{R}^\infty : v \perp P\}$

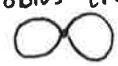
⑧ Vertical bundle of surface bundle



Char. classes for surface bundles defined in terms of char. classes for these vector bundles.

Isomorphism

$p_1 : E_1 \to B$ is isomorphic to $p_2 : E_2 \to B$
if $\exists$ homeo $h : E_1 \to E_2$ s.t. $h|_{p_1^{-1}(b)}$ is a v.s. $\cong$ to $p_2^{-1}(b)$.

N.B. 

$\not\cong$ 

& bundles over different spaces can't be isomorphic(!)

Examples

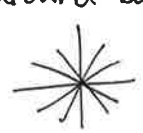
① $NS^n \cong S^n \times \mathbb{R}$
via $(x, tx) \mapsto (x, t)$

② $TS^1 \cong S^1 \times \mathbb{R}$
via $(z, izt) \mapsto (z, t)$

We say S^1 is <u>parallelizable</u> 

Q. Which manifolds are parallelizable? S^2?
(All 3-manifolds!)

③ Cancn. line bundle over $\mathbb{R}P^1$ $\cong$ Möbius bundle over $\mathbb{R}P^1$
after traveling around base, fibers get flipped:



Q. Is $T\mathbb{R}P^n \cong E^\perp$?

Sections

A section of $p: E \to B$ is $s: B \to E$ s.t. $p \circ s = id$.

e.g. 0-section

Some bundles have nonvanishing sections, some do not.

For example: A section of TM is a vector field on M.

We showed nonvan vect field $\Rightarrow \chi(M) = 0$.

So $\chi(M) \neq 0 \Rightarrow TM$ has no nonvan. sec.

e.g. $\chi(S^n) = 2$ n even.

Can show S^n has nonvan. vect field n odd.

FACT: An n-dim bundle is trivial $\Leftrightarrow$ it has n sections s_i that are lin. ind. over each point of B.

$\Rightarrow$ obvious

$\Leftarrow$ there is a contin. map

$$B \times \mathbb{R}^n \longrightarrow E$$
$$(b, t_1, \ldots, t_n) \longmapsto \sum t_i s_i(b)$$

clearly isom. on fibers

need to show inverse is continuous

follows from: inversion of matrices is continuous.

Spheres: TS^1 trivial by $s(z) = iz$

TS^3 trivial by $s_1(z) = iz$, $s_2(z) = jz$, $s_3(z) = kz$

TS^7 trivial by similar construction w/ octonians.

(all other TS^n nontrivial!)

Direct Sum

$p_1 : E_1 \to B,\ p_2 : E_2 \to B \rightsquigarrow$

$$E_1 \oplus E_2 = \{(v_1, v_2) \in E_1 \times E_2 : p_1(v_1) = p_2(v_2)\}$$

$$\begin{aligned} p : E_1 \oplus E_2 &\to B \\ (v_1, v_2) &\to p(v_1) \end{aligned}$$

$E_1 \oplus E_2$ a vector bundle because ① products of vb's are vb's
② restrictions of vb's are vb's.

$E_1 \oplus E_2$ is restriction of $E_1 \times E_2$ to diagonal $B \subseteq B \times B$.

Trivial $\oplus$ trivial = trivial but
Nontrivial $\oplus$ trivial can be trivial!

e.g. $TS^n \oplus \underset{\text{trivial}}{NS^n}$ trivial. Say TS^n stably trivial.

also: $E \oplus E^\perp \to \mathbb{R}P^n$ trivial via $(\ell, v, w) \mapsto (\ell, v+w)$
$n=1$ case: Möbius $\oplus$ Möbius = trivial

A useful exercise related to last example: Show there are exactly two $\mathbb{R}^n$ bundles over S^1. Similarly, exactly two S^1-bundles over S^1.

EXAMPLE. $T\mathbb{R}P^n$ stably isom. to $\bigoplus E$ ← canon. line bundle.

Start with $TS^n \oplus NS^n \cong S^n \times \mathbb{R}^{n+1}$

Quotient by $(x,v) \sim (-x,-v)$ on both sides.

$TS^n/\sim \;\cong T\mathbb{R}P^n$ since $(x,v) \mapsto (-x,-v)$ is map on TS^n induced by $x \mapsto -x$.

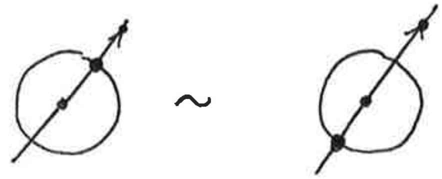
$NS^n/\sim \;\cong \mathbb{R}P^n \times \mathbb{R}$ via the section $x \mapsto (x,x)$

Claim: $(S^n \times \mathbb{R}^{n+1})/\sim \;\cong \bigoplus_{i=1}^{n+1} E$

First, $\sim$ preserves factors, so

$$(S^n \times \mathbb{R}^{n+1})/\sim \;\cong \bigoplus_{i=1}^{n+1} (S^n \times \mathbb{R})/\sim$$

But $(S^n \times \mathbb{R})/\sim \;\cong E$, as



Using quaternions, $T\mathbb{R}P^3 \cong \mathbb{R}P^3 \times \mathbb{R}^3$

~~As above~~ So $T\mathbb{R}P^3 \oplus$ trivial line bundle $\cong \mathbb{R}P^3 \times \mathbb{R}^4$

As above $T\mathbb{R}P^3 \oplus$ trivial line bundle $\cong \bigoplus_{i=1}^{4} E$

$$\Rightarrow \bigoplus_{i=1}^{4} E \cong \mathbb{R}P^3 \times \mathbb{R}^4.$$

NEXT GOAL

Prop. B = compact Hausdorff
$\forall\ E \to B\ \exists\ E' \to B$ s.t. $E \oplus E'$ trivial.

Step 1. Inner Products

Inner product on V: pos. def. symm. bilinear form.
Inner product on E: map $E \oplus E \to \mathbb{R}$ restricting to inner prod. on each fiber.

Paracompact: Hausdorff + every open cover admits a part. of unity.
Compact Hausdorff, CW complex, metric space $\Rightarrow$ paracompact

Prop. B paracompact $\Rightarrow$ $E \to B$ has an inner product.
Pf: Exercise.

Step 2. Orthogonal complements

Prop. B paracompact, $E_0 \to B$ subbundle of $E \to B$.
$\exists\ E_0^\perp$ s.t. $E_0 \oplus E_0^\perp \cong E$.

Pf. Choose inner product, $E_0^\perp$ = orthog. comp. in each fiber.
Need to check local triviality
Over $U \subseteq B$ choose m sections s_i for E_0, $n-m$ for E.
Apply Gram-Schmidt — continuous.
New sections trivialize E_0 & $E_0^\perp$ simultaneously. ▨

Note: $E_0 \oplus E_0^\perp \cong E$ via FACT above.

To prove that any E has E' with $E \oplus E'$ trivial, it now suffices to show:

Prop. B = compact Hausdorff
Any $\mathbb{R}^n$-bundle $E \to B$ is a subbundle of $B \times \mathbb{R}^N$.

Pf. Choose: $U_1, \ldots, U_k$ s.t. $p^{-1}(U)$ trivial
$h_i : U_i \longrightarrow U_i \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$
φ_i = part of unity subord to U_i

Define: $g_i : E \to \mathbb{R}^n$ — linear inj. on each fiber
$v \mapsto \varphi_i(p(v)) h_i(v)$ — with $\varphi_i \neq 0$.

$g : E \to \mathbb{R}^{nk}$ — linear inj. on all fibers.
$v \mapsto (g_1(v), \ldots, g_k(v))$

$f : E \to B \times \mathbb{R}^{nk}$
$v \mapsto (p(v), g(v))$.

$\text{Im}(f)$ is a subbundle. Project in 2nd coord to get local triv. over U_i. ☒

The Grassmann Manifold.

We just showed

$$[B, G_n] \longrightarrow \{\mathbb{R}^n\text{-bundles over } B\}$$
$$f \longmapsto f^*(E_n)$$

is well defined.

Want to show it is a bijection. First, let's discuss the topology of G_n & E_n.

G_n = set of all n-dim subspaces of $\mathbb{R}^\infty$.
V_n = Stiefel manifold
= space of orthonormal n-frames in $\mathbb{R}^\infty$.

V_n has a natural topology as a subspace of S^∞, and there is a quotient

direct limit topology.

$$V_n \longrightarrow G_n.$$

Endow G_n with quotient topology.

Define $E_n = \{(\ell, v) \in G_n \times \mathbb{R}^\infty : v \in \ell\}$, $p(\ell, v) = \ell$.

<u>Lemma</u>. $E_n \xrightarrow{p} G_n$ is a vector bundle.

<u>Pf</u>. Let $\ell \in G_n$, $\pi_\ell : \mathbb{R}^\infty \to \ell$ orthog. proj.

$$U_\ell = \{\ell' \in G_n : \pi_\ell(\ell') \text{ has dim } n\}.$$

Steps: ① U_ℓ open (check preim in V_n open).

② $h: p^{-1}(U_\ell) \longrightarrow U_\ell \times \ell$ is a local triv.

$$(\ell', v) \longrightarrow (\ell', \pi_\ell(v))$$

h clearly a bij, lin. iso on each fiber.

Need: h, h^{-1} continuous (lin alg).

THEOREM. X paracompact. The map $[X, G_n] \longrightarrow \mathrm{Vect}^n(X)$, $f \mapsto f^*(E_n)$ is a bijection.

Example. $M \subseteq \mathbb{R}^N$ submanifold. Define $f: M \longrightarrow G_n$ by $x \mapsto T_xM$. Then $TM \cong f^*(E_n)$.

Pf. Key observation: For $E \to X$ an $\mathbb{R}^n$-bundle, an iso $E \cong f^*(E_n)$ is equivalent to a map $E \to \mathbb{R}^\infty$ that is a lin inj. on each fiber.

Indeed, given $f: X \to G_n$ and $E \xrightarrow{\cong} f^*(E_n)$ have:

$$\begin{tikzcd} E \arrow[r, "\cong"] \arrow[dr, "p"'] & f^*(E_n) \arrow[r] \arrow[d] & E_n \arrow[r] \arrow[d] & \mathbb{R}^\infty \\ & X \arrow[r, "f"] & G_n & \end{tikzcd}$$

Top row is the desired map.

Conversely, given $g: E \to \mathbb{R}^\infty$ lin inj. on each fiber, define $f: X \to G_n$ by $x \mapsto g(p^{-1}(x))$.
$\tilde{f}: E \to E_n$ by $v \mapsto g(v)$.

This gives diagram as above, by univ. prop. of pullbacks.

Surjectivity. Let $p: E \to X$ be an $\mathbb{R}^n$-bundle
(for simplicity, X = compact Hausdorff)

Choose cover $U_1, \dots, U_N$ s.t. E trivial over U_i.
& partition of unity $\varphi_1, \dots, \varphi_N$.
Define $g_i : p^{-1}(U_i) \longrightarrow U_i \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$
& $g : E \longrightarrow \mathbb{R}^n \times \dots \times \mathbb{R}^n \subseteq \mathbb{R}^\infty$

$$v \longmapsto (\varphi_1 g_1(v), \dots, \varphi_N g_N(v))$$

φ_i means $\varphi_i \circ p$ = scalar

Check g a lin. inj. on each fiber.

Injectivity. Say $E \cong f_0^*(E_n), f_1^*(E_n)$
for $f_0, f_1 : X \longrightarrow G_n$.
$\rightsquigarrow g_0, g_1 : E \longrightarrow \mathbb{R}^\infty$ lin inj on each fiber.
To show $g_0 \sim g_1$ via maps that are lin inj on each fiber.
$\Rightarrow f_0 \sim f_1$ via $f_t(x) = g_t(p^{-1}(x))$.
Use:

$$\begin{tikzcd}
g_0 \arrow[d, "①"] & & g_1 \arrow[d, "②"'] \\
\text{odd coords} \arrow[rr, "\text{straight line}", "③"'] & & \text{even coords}
\end{tikzcd}$$

N.B. ③ only makes sense b/c g_0, g_1 are both maps from a fixed space E to $\mathbb{R}^\infty$.

e.g. $g_0 \longrightarrow$ odd coords via $(x_1, x_2, \dots) \longmapsto (1-t)(x_1, x_2, \dots) + t(x_1, 0, x_2, \dots)$
At each stage, lin. inj. on fibers. ∎

The Thm has an immediate corollary: v.b.'s over paracompact bases have inner products. Pull back obvious one on $\mathbb{R}^\infty$.

We now know $\quad [B, G_n] \longleftrightarrow \{\text{vector bundles over } B\}$

so $\quad$ char. classes $\longleftrightarrow H^*(G_n)$

Cell Structure on G_n.

First recall cell structure on $G_1 = \mathbb{RP}^\infty$

one i-cell e_i $\forall i$.

e_i glued to e_{i-1} by degree 2 map

$e_i \longleftrightarrow \{\ell \in \mathbb{RP}^\infty : \ell \subseteq \mathbb{R}^{i+1}\}$

Will generalize this.

A Schubert symbol $\sigma = (\sigma_1, \dots, \sigma_n)$ is a seq of integers

s.t. $\quad 1 \leq \sigma_1 < \sigma_2 < \cdots < \sigma_n$

Let $e(\sigma) = \{\ell \in G_n : \dim(\ell \cap \mathbb{R}^{\sigma_i}) - \dim(\ell \cap \mathbb{R}^{\sigma_{i-1}}) = 1 \ \forall i\}$

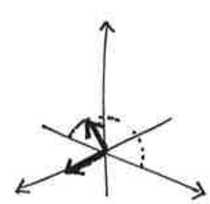
<u>Prop.</u> The $e(\sigma)$ are the cells of a CW structure on G_n.

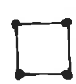
$$\dim e(\sigma) = \sum_{i=1}^{n} (\sigma_i - i)$$

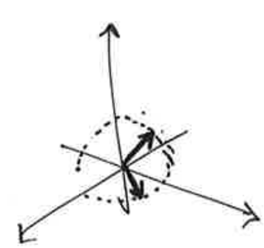
Examples. Consider in G_2:

$e(1,2) = \bullet$

$e(1,3) =$



$e(2,3) =$ 



Proof of Prop. Let H_i = hemisphere in $S^{\sigma_i - 1} \subseteq \mathbb{R}^{\sigma_i}$ s.t. σ_i-coord non-neg.

$$e(\sigma) \longleftrightarrow \{(b_1, \dots, b_n) \in V_n : b_i \in \operatorname{int} H_i\}$$

Let $E(\sigma) = \{(b_1, \dots, b_n) \in V_n : b_i \in H_i\}$

Main step: $E(\sigma)$ a closed ball of dim $\Sigma(\sigma_i - i)$

$n=1$ case: $E(\sigma) = H_1$ ✓

$n>1$ case: Define
$$\pi : E(\sigma) \longrightarrow H_1$$
$$(b_1, \dots, b_n) \mapsto b_1$$
$$p : E(\sigma) \longrightarrow \pi^{-1}(e_{\sigma_1})$$
rotate fiber over b_1 to $\pi^{-1}(e_{\sigma_1})$ by rotating b_1 to e_{σ_1}, fixing orthog. comp. of $\langle b_1, e_{\sigma_1} \rangle$

Then $\pi \times p : E(\sigma) \longrightarrow H_1 \times \pi^{-1}(e_{\sigma_1})$ is a contin. bij $\Longrightarrow$ homeo.
(exercise: Hausdorff)

Remains to check $\pi^{-1}(e_{\sigma_1})$ a ball.
Induct on n. $\pi^{-1}(e_{\sigma_1}) \longleftrightarrow E(\sigma_2 - 1, \dots, \sigma_n - 1)$

Span takes int $E(\sigma)$ to $e(\sigma)$ bijectively.
Since G_n has quotient top. from $V_n \rightsquigarrow$ homeo.

Need to check that the CW complex obtained from the $E(\sigma)$ give right topology. Induct on skeleta. ▨

Other versions: $\mathrm{Vect}^n_{\mathbb{C}}(X) \leftrightarrow [X, G_n(\mathbb{C})]$

$$\mathrm{Vect}^n_+(X) \leftrightarrow [X, \tilde{G}_n]$$

Note $\mathrm{Vect}^n_+(S^1)$ trivial $\Rightarrow [S^1, \tilde{G}_n]$ trivial
$\Rightarrow \pi_1(\tilde{G}_n) = 1$.
$\Rightarrow \tilde{G}_n =$ univ. cover of G_n.

For $f: X \to G_n$, $f^*(E)$ orientable iff f lifts to $\tilde{G}_n$ & in this case, orientations correspond to choices of lifts.

Prop. G_n is a manifold.

Pf. ~~First show~~ Clear for interior of a top-dim. cell. But G_n is homogeneous: $\exists$ homeo taking any pt to any other pt, ie the one induced by a linear map.

Stiefel-Whitney and Chern Classes

First, we will show that characteristic classes exist by defining specific ones, the SW classes w_i and the Chern classes c_i. Then we will show these are all char. classes (in the $\mathbb{R}, \mathbb{Z}_2$ & $\mathbb{C}, \mathbb{Z}$ cases, resp.) by computing $H^*(G_n; \mathbb{Z}_2)$ and $H^*(G_n(\mathbb{C}); \mathbb{Z})$.

<u>Thm.</u> $\exists!$ seq. of fns $w_1, w_2, \ldots$ assigning to each real v.b. $E \to B$ a class $w_i(E) \in H^i(B; \mathbb{Z}_2)$ s.t.

(i) $w_i(f^*(E)) = f^*(w_i(E))$

(ii) $w(E_1 \oplus E_2) = w(E_1) \cup w(E_2)$ $\qquad w = 1 + w_1 + w_2 + \cdots$

(iii) $w_i(E) = 0 \quad i > \dim E$

(iv) $w_1(\text{canon. bundle} \to \mathbb{RP}^\infty)$ is gen. of $H^1(\mathbb{RP}^\infty; \mathbb{Z}_2)$.

w = total SW class. $\qquad$ (iii) $\Rightarrow$ it is a finite sum.

(ii) is Whitney sum formula.

(iv) $\Rightarrow$ the w_i are not all zero!

(i) $\Rightarrow w_i(B \times \mathbb{R}^n) = 0 \quad i > 0$. (ii) $\Rightarrow w_i$ stable $\qquad$ <u>Cor</u>: $w_i(TS^n) = 0$ $i > 0$. or: $w(TS^n) = 1$.

For complex bundles, have $c_i \in H^{2i}(B; \mathbb{Z})$. Thm is same except:

(iv) $c_1(\text{canon} \to \mathbb{CP}^\infty)$ gen. $H^2(\mathbb{CP}^\infty; \mathbb{Z})$.

Proof requires one tool from alg. top. ...

THE LERAY-HIRSCH THEOREM

When does $H^*(E)$ look like $H^*(F\times B)$? First, recall:

<u>KÜNNETH FORMULA</u>. $$H^*(X;R)\otimes_R H^*(Y;R) \xrightarrow{\cong} H^*(X\times Y;R)$$
$$a\otimes b \longmapsto p_1^*(a)\cup p_2^*(b)$$

For a fiber bundle, $H^*(E)\to H^*(F)$ not. nec. surj, so don't always have a map the other way. To get a Künneth-like formula, must add this to the assumptions.

General theme in bundle theory: try to extend an object related to the fiber (inner prod, cohom. class) to whole bundle.

<u>L-H</u> Theorem. Let $F\to E\to B$ be a fiber bundle, R a ring s.t.

(i) $H^n(F;R)$ is a free f.g. R-module $\forall n$.

(ii) $\exists c_j \in H^{k_j}(E;R)$ s.t. the $i^*(c_j)$ form a basis for $H^*(F;R)$

Then: $$H^*(B;R)\otimes_R H^*(F;R) \xrightarrow{\cong} H^*(E,R)$$
$$\sum b_i\otimes i^*(c_j) \longmapsto p^*(b_i)\cup c_j$$

In other words: $H^*(E;R)$ a free $H^*(B;R)$ module w/ basis c_j. Module structure given by $\cup$.

- The c_i do exist for product bundles: pull back via projection.
- The c_i do not exist for $S^1\to S^3\to S^2$ as $H^1(S^3)=1$.

Pf. of LH (a few words) Using long ex. seq. for a pair, plus excision, you reduce to understanding

$p^{-1}(B^{n-1}) \longrightarrow B^{n-1}$ (n-skeleton)

$p^{-1}(n\text{-cell}) \longrightarrow n\text{-cell}$

Former works by induction, latter by local triviality. ∎

Pf of SW Thm. $\pi : E \longrightarrow B$

$\rightsquigarrow P(\pi) : P(E) \longrightarrow B$ $P(E)$ = space of lines, fibers $\mathbb{R}P^{n-1}$

To use L-H, need $x_i \in H^i(P(E); \mathbb{Z}_2)$ restricting to gens for $H^i(\mathbb{R}P^{n-1}; \mathbb{Z}_2)$.

$(E \longrightarrow B) \rightsquigarrow g : E \longrightarrow \mathbb{R}^\infty$ lin. inj on fibers.

$\rightsquigarrow P(g) : P(E) \longrightarrow \mathbb{R}P^\infty$

Let α = gen for $H^1(\mathbb{R}P^\infty; \mathbb{Z}_2)$

$x = P(g)^*(\alpha)$ ← easy to see this generates $H^1(\text{fiber})$. also indep. of g

$x_i = x^i$.

i.e. $x \in \mathrm{Hom}(H_1(P(E)), \mathbb{Z}_2)$ records whether a line comes back w/ same or. after the loop.

L-H ⇒ $H^*(P(E))$ a free $H^*(B)$-module with basis $1, x, \ldots, x^{n-1}$

⇒ x^n = unique linear combo:

$$x^n + w_1(E)x^{n-1} + \cdots + w_n(E) \cdot 1 = 0.$$

for some $w_i(E) \in H^*(B; \mathbb{Z}_2)$.

Also set $w_i(E) = 0$ for $i > n$

$w_0(E) = 1$.

These are the SW classes. Need to check properties (i)–(iv), uniqueness.

(i) Naturality

Say
$$\begin{tikzcd} E' \arrow[r, "\tilde{f}"] \arrow[d] & E \arrow[r, "g"] \arrow[d] & \mathbb{R}^\infty \\ B' \arrow[r, "f"] & B & \end{tikzcd}$$

$\rightsquigarrow \quad P(\tilde{f})^* \, x(E) = x(E')$

$\Rightarrow P(\tilde{f})^* \, x_i(E) = x_i(E')$

Commutativity $\Rightarrow$ module structure pulls back

i.e. $\quad x^n + w_1(E)x^{n-1} + \cdots + w_n(E)\cdot 1 = 0$

$\rightsquigarrow \quad x^n + f^*(w_1(E))\, x^{n-1} + \cdots + f^*(w_n(E))\cdot 1 = 0$

But this defines $w_i(E')$ so $w_i(E') = f^*(w_i(E)) \ \forall i$.

(ii) Whitney sum - similar flavor

(iii) $w_i(E) = 0 \quad i > n$ by definition.

(iv) $w_1(CB \to \mathbb{RP}^\infty) \neq 0$.

Almost by definition: x(loop in $P(E)$) measures whether or not a line comes back to where it started with same or different orientation.

$$x + w_1(CB)\cdot 1 = 0.$$
$$\Rightarrow w_1(CB) = x.$$

For uniqueness of w_i, need a tool.

Splitting Principle. Given $E \to B$ $\exists\, f: A \to B$ s.t.

(i) $f^*(E)$ splits as a sum of line bundles

(ii) $f^*: H^*(B) \to H^*(A)$ injective

Now, the w_i are unique because:

(iv) determines $w_1(CB \to \mathbb{RP}^\infty)$

(iii) determines $w_i(CB \to \mathbb{RP}^\infty)$ $\quad i > 1$.

(i) determines w_i (line bundles)

(ii) determines w_i (sum of line bundles)

SP + (i) determines w_i (any bundle).

Pf of SP. $\quad A = F(E) =$ flag bundle of E

$=$ space of orthog. splittings $l_1 \oplus \cdots \oplus l_n$ of E into lines

$f: A \to B$ projection

$f^*(E) = \{$(splitting of fiber over b, vector in fiber over b)$\}$

This has n obvious linear subbundles, which give the splitting.

For (ii) use Leray-Hirsch $\Rightarrow$ $H^*(B) \cdot 1$ a summand of $H^*(A)$.

IMPORTANT EXAMPLE.

$(E_1)^n \to (G_1)^n$ $\quad E_1 =$ Canon. line bundle

$(E_1)^n \cong \oplus \pi_i^*(E_1)$ $\quad \pi_i : (G_1)^n \to G_1$ $\quad$ true for any $E^n \to B^n$

$\Rightarrow w((E_1)^n) = \prod (1+\alpha_i) \in \mathbb{Z}_2[\alpha_1, \dots, \alpha_n] \cong H^*((\mathbb{R}P^\infty)^n; \mathbb{Z}_2)$

$\to w_i((E_1)^n) =$ i^{th} symm. poly σ_i in the α_j

e.g. for $n=3$:
$$\sigma_1 = \alpha_1 + \alpha_2 + \alpha_3$$
$$\sigma_2 = \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3$$
$$\sigma_3 = \alpha_1\alpha_2\alpha_3$$

So all w_i nonzero $i \leq n$.

Next: We'll use this to show
$$\mathbb{Z}_2[w_1, \dots, w_n] \hookrightarrow H^*(G_n; \mathbb{Z}_2)$$

Cohomology of Grassmannians

We showed $w_i\left((E_1)^n \to (G_1)^n\right) \neq 0 \quad 0 \le i \le n.$

Naturality $\Rightarrow$ $w_i(E_n) \neq 0 \quad 0 \le i \le n.$

Let $f: (\mathbb{RP}^\infty)^n \longrightarrow G_n$ be classifying map for $(E_1)^n$.

& $w_i = w_i(E_n)$.

Then:

$$\mathbb{Z}_2[w_1,\ldots,w_n] \longrightarrow H^*(G_n;\mathbb{Z}_2) \xrightarrow{f^*} H^*((\mathbb{RP}^\infty)^n;\mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha_1,\ldots,\alpha_n]$$

sends w_i to i^{th} symm. poly. σ_i in the α_j.

Fact. The σ_i are alg. indep.

$\Rightarrow$ above map is inj

$\Rightarrow$ $\mathbb{Z}_2[w_1,\ldots,w_n] \hookrightarrow H^*(G_n;\mathbb{Z}_2)$.

<u>Thm</u> $H^*(G_n;\mathbb{Z}_2) = \mathbb{Z}_2[w_1,\ldots,w_n]$

also: $H^*(G_n(\mathbb{C});\mathbb{Z}) = \mathbb{Z}[c_1,\ldots,c_n]$

<u>Pf.</u> We showed $\operatorname{im} f^*$ contains $\mathbb{Z}_2[\sigma_1,\ldots,\sigma_n]$

Also $\operatorname{im} f^*$ contained in $\mathbb{Z}_2[\sigma_1,\ldots,\sigma_n]$ since permuting the $\mathbb{RP}^\infty$ factors gives same bundle with α_i's permuted.

So:

$$\mathbb{Z}_2[w_1,\ldots,w_n] \longrightarrow H^*(G_n;\mathbb{Z}_2) \xrightarrow{f^*} \mathbb{Z}_2[\sigma_1,\ldots,\sigma_n] \cong \mathbb{Z}_2[w_1,\ldots,w_n]$$

f^* surjective. To show f^* injective.

Focus on r-grading:

$$(\mathbb{Z}_2[w_1,\ldots,w_n])_r \longrightarrow H^r(G_n;\mathbb{Z}_2) \longrightarrow (\mathbb{Z}_2[w_1,\ldots,w_n])_r$$

Since composition surj, suffices to show $\dim H^r(G_n;\mathbb{Z}_2) \leq \dim(\mathbb{Z}_2[w_1,\ldots,w_n])_r$.

Let $p(r,n)$ = #partitions of r into n nonneg integers.

Step 1. $\dim(\mathbb{Z}_2[w_1,\ldots,w_n])_r = p(r,n)$.

$w_1^{r_1}w_2^{r_2}\cdots w_n^{r_n} \in (\mathbb{Z}_2[w_1,\ldots,w_n])_r$ means

$r_1 + 2r_2 + \cdots + nr_n = r$ (since $w_i \in H^i$)

$\rightsquigarrow$ partition of r: $r_n \leq r_n + r_{n-1} \leq \cdots \leq r_n + \cdots + r_1$

Step 2. $\dim H^r(G_n;\mathbb{Z}_2) \leq$ # Schubert cells of dim r.

General fact about cell complexes

Step 3. # Schubert cells in G_n of dim r $= p(r,n)$.

A partition $a_1 \leq a_2 \leq \cdots \leq a_n$

$\rightsquigarrow$ Schubert symbol $(a_1+1, a_2+2, \ldots, a_n+n)$.

Example. $r=10$, $n=6$.

partition: $0,0,1,1,3,5$

Schubert cell: $(1,2,4,5,8,11)$

monomial: $w_1^2 w_2^2 w_4$

$\square$

The Group of Line Bundles

We'll first show: $\mathrm{Vect}^1(X)$ is a group under $\otimes$.

and then: $\mathrm{Vect}^1(X) \cong H^1(X;\mathbb{Z}_2)$. The isom. is w_1!

Gluing construction of vector bundles. Given $p: E \to B$, $\{U_\alpha\}$,

$h_\alpha : p^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{R}^n$, can recover

$$E = \left(\coprod U_\alpha \times \mathbb{R}^n\right)/\sim$$

where $(x,v) \in U_\alpha \times \mathbb{R}^n \sim h_\beta h_\alpha^{-1}(x,v) \in U_\beta \times \mathbb{R}^n \quad x \in U_\alpha \cap U_\beta$.

Write $g_{\beta\alpha}$ for the gluing func. $h_\beta h_\alpha^{-1} : U_\alpha \cap U_\beta \longrightarrow GL_n(\mathbb{R})$.

$\rightsquigarrow$ cocycle condition: $g_{\gamma\beta} g_{\beta\alpha} = g_{\gamma\alpha}$ on $U_\alpha \cap U_\beta \cap U_\gamma$

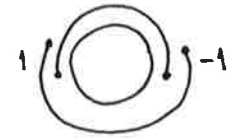
Conversely: any collection of gluing functions satisfying cocycle cond gives rise to a vector bundle.

The gluing functions for $E_1 \otimes E_2$ are the tensor products of the gluing functions for E_1, E_2.

In general, $\otimes$ on $\mathrm{Vect}^n(X)$ is comm, assoc, and has identity = trivial line bundle.

For $n=1$, also have inverses. In fact, each elt is its own inverse.

Example. Möbius $\to S^1$ has gluing fns $1, -1$



$1 \otimes 1 = 1 \quad -1 \otimes -1 = 1$

$\Rightarrow$ Möbius $\otimes$ Möbius $\longrightarrow S^1$ is trivial.

For general line bundles, we obtain inverse by replacing gluing matrices by their inverses, as $t \otimes t^{-1} = 1$.

Cocycle condition still works since 1×1 matrices commute.

Endow E w/ inner product $\rightsquigarrow$ rescale all h_α with isometries

$\Rightarrow$ all gluing fns ± 1. $\Rightarrow$ gluing fns for $E \otimes E$ all 1.

$\Rightarrow$ $E \otimes E$ trivial.

We have: $\underbrace{\mathrm{Vect}^1(X) = [X, G_1]}_{\text{isom. of sets}} \cong H^1(X; \mathbb{Z}_2)$ since $G_1 = \mathbb{RP}^\infty$ is $K(\mathbb{Z}_2, 1)$.

Prop. $w_1 : \mathrm{Vect}^1(X) \xrightarrow{\cong} H^1(X; \mathbb{Z}_2)$ $\qquad X$ = CW-complex.

Pf. First show w_1 a homomorphism.

Step 1. $w_1(L_1 \otimes L_2) = w_1(L_1) + w_1(L_2)$

for $L_i \to G_1 \times G_1$ the pullback of $E_1 \to G_1$ via $\pi_i : G_1 \times G_1 \to G_1$.

Have $H^*(G_1 \times G_1) \cong \mathbb{Z}_2[\alpha_1] \otimes \mathbb{Z}_2[\alpha_2] \cong \mathbb{Z}_2[\alpha_1, \alpha_2]$

$H^*(G_1 \vee G_2) \cong \mathbb{Z}_2[\alpha_1] \oplus \mathbb{Z}_2[\alpha_2]$

This is an isom. on H^1 : $\mathbb{Z}_2 \oplus \mathbb{Z}_2 = \{0, \alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$

So suffices to compute $w_1(L_1 \otimes L_2 \to G_1 \vee G_1)$

Over $G_1 \vee *$, L_2 trivial $\Rightarrow L_1 \otimes L_2 \cong L_1 \otimes 1 \cong L_1$

Similar for $* \vee G_1$

$\Rightarrow w_1(L_1 \otimes L_2) = \alpha_1 + \alpha_2 = w_1(L_1) + w_1(L_2)$.

use naturality of pullback via $G_1 \to G_1 \vee G_1$

Step 2. (Naturality) E_1, E_2 arbitrary line bundles

$$E_i = f_i^*(E_1) \quad f_i : X \to G_1.$$

$$\text{Let } F = (f_1, f_2) : X \to G_1 \times G_1$$

$$F^*(L_i) = f_i^*(E_1) = E_i$$

follow your nose...

$$\begin{aligned} w_1(E_1 \otimes E_2) &= w_1(F^*(L_1) \otimes F^*(L_2)) = w_1(F^*(L_1 \otimes L_2)) \\ &= F^*(w_1(L_1 \otimes L_2)) = F^*(w_1(L_1) + w_1(L_2)) \\ &= F^*(w_1(L_1)) + F^*(w_1(L_2)) \\ &= w_1(F^*(L_1)) + w_1(F^*(L_2)) \\ &= w_1(E_1) + w_1(E_2). \end{aligned}$$

∎

The isomorphism is
$$[X, G_1] \longrightarrow H^1(X; \mathbb{Z}_2)$$
$$[f] \longmapsto f^*(\alpha)$$

It factors as
$$[X, G_1] \longrightarrow \mathrm{Vect}^1(X) \longrightarrow H^1(X; \mathbb{Z}_2)$$
$$[f] \longmapsto f^*(E_1) \longmapsto w_1(f^*(E_1)) = f^*(w_1(E_1)) = f^*(\alpha)$$

First map is bij, comp is isom $\Rightarrow$ 2^{nd} map bij. ∎

We can unravel the last step. Want to define
$$H^1(X; \mathbb{Z}_2) \longrightarrow \mathrm{Vect}^1(X)$$
inverse to w_1. Given $\varphi \in H^1$, define an $\mathbb{R}$-bundle skeleton by skeleton. On 1-skeleton, use φ to decide between Möbius & trivial bundle. As φ is a cocycle, it is trivial on any loop bounding a 2-cell, so can extend over 2-skeleton and higher.

The Euler Class

$e \in H^n(\tilde{G}_n; \mathbb{Z})$

$\rightsquigarrow$ e is n-dim class for oriented $\mathbb{R}^n$-bundles

idea: given n-chain, put it in gen. pos wrt 0-section,
count intersection points with sign. ~~I really think of this as n-cochain dual to these pts.~~

The Euler class satisfies:

(1) $e(f^*(E)) = f^* e(E)$

(2) $e(\bar{E}) = -e(E)$

(3) $e(E_1 \oplus E_2) = e(E_1) \cup e(E_2)$

(4) $e(E) = -e(E)$ n odd (i.e. $e(E)$ is 2-torsion)

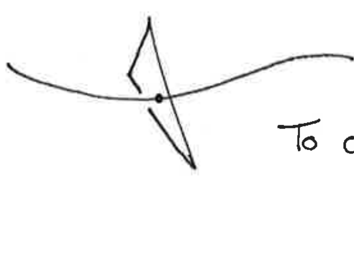
(5) $e(E) = 0$ if E has nonvan section

(6) $\langle e(M), [M] \rangle = \chi(M)$

Instability. Unlike w_i, c_i the class e is unstable:

$$e(E \oplus \text{trivial}) = 0 \qquad \text{(nonvan section)}$$

The construction of e requires one tool.
Let $E' = E - 0\text{-sec}$.
We'll show $\exists\ c \in H^n(E, E')$ restricting in each fiber to
a gen for $H^n(\mathbb{R}^n, \mathbb{R}^n \setminus 0)$. c = Thom class
Define e = restriction of c to 0-section: $H^*(E,E') \to H^*(E) \to H^*(B)$
This does just what we want:



To compute, perturb intersections
to lie in fibers.

Thom Isomorphism

Orientability. $\mathbb{R}^n \to E \to B \rightsquigarrow$ disk bundle $D^n \to D(E) \to B$ and
sphere bundle $S^{n-1} \to S(E) \to B$

Say $E, D(E)$ orientable if $S(E)$ is

$S(E)$ orientable if the map $H^{n-1}(S^{n-1};\mathbb{Z}) \circlearrowleft$ induced by any loop in B is id.

e.g. T^2 is orientable S^1 bundle over S^1, K.B. nonorientable.

Thom class. A Thom class is a $c \in H^n(D(E), S(E);\mathbb{Z})$ restricting to gen for $H^n(D^n, S^{n-1};\mathbb{Z})$ in each fiber.

<u>Thm</u> E orientable $\Rightarrow c$ exists.

Thom isomorphism. The map
$$H^i(B;\mathbb{Z}) \to H^{i+n}(D(E), S(E);\mathbb{Z})$$
$$b \mapsto p^*(b) \cup c$$
is isom. $\forall\, i \geq 0$, and $H^i(D(E), S(E);\mathbb{Z}) = 0 \quad i < n$.

Thom space. $T(E) = D(E)/S(E)$ disk fibers $\rightsquigarrow$ spheres S^n in $T(E)$,
all spheres meet at basept.

Thom class $\leftrightarrow$ elt of $H^n(T(E), x_0;\mathbb{Z}) \cong H^n(T(E);\mathbb{Z})$.
restricting to gen of $H^n(S^n;\mathbb{Z})$ in each "fiber"

Thom isom $\rightsquigarrow H^i(B;\mathbb{Z}) \cong \tilde{H}^{n+i}(T(E);\mathbb{Z})$

$T(E)$ central to Thom's work on cobordism.

THOM CLASS

* all coeffs = $\mathbb{Z}$

THM. Every orientable bundle $E \to B$ has a Thom class

Pf. Assume B = connected finite dim CW complex.

Claim. $H^i(D(E), S(E)) \xrightarrow{\cong} H^i(D^n, S^{n-1})$ $\forall$ fibers.

Say B is k-dim, assume true for smaller dim complexes.
For concreteness $i = n$. Other cases easier.
Set U = nbd of B^{k-1}, $V = \coprod$ open k-cells
Mayer-Vietoris:

$$0 \to H^n(D(E), S(E)) \to \underset{\substack{\cong \\ H^n(D^n, S^{n-1})}}{H^n(D(E)_U, S(E)_U)} \oplus \underset{\substack{\cong \\ \bigoplus_{|V|} H^n(D^n, S^{n-1})}}{H^n(D(E)_V, S(E)_V)} \xrightarrow[\text{diff map}]{\psi} H^n(D(E)_{U\cap V}, S(E)_{U\cap V})$$

induction

$\uparrow$ by induction & $U \cap V \simeq \coprod S^{k-1}$ & $A \hookrightarrow B$ weak h.e. $\Rightarrow E_A \hookrightarrow E$ weak h.e.

Orientability $\Rightarrow$ can choose the gens for the $\oplus$ in the middle consistently

for mod 2 version skip this step.

$\Rightarrow \ker \psi \cong \mathbb{Z} = \{(a, (a, \ldots, a))\}$

$\Rightarrow H^n(D(E), S(E)) \cong \mathbb{Z}$.

Can rewrite everything with $(E, E - (0\text{-sec}))$ & $(\mathbb{R}^n, \mathbb{R}^n - 0)$

Moreover the isom is given by restriction to fibers as

$$H^n(D(E), S(E)) \xrightarrow{\cong} \ker \psi \xrightarrow[\substack{\text{any factor} \\ \cong}]{\text{proj to}} H^n(D^n, S^{n-1})$$

this map is restriction to fibers.

∎

Relative LH $\Rightarrow$ $H^*(D(E), S(E))$ = free $H^*(B)$-module w/ basis ~~over~~ c

$\cong H^*(B)$

This is the Thom isomorphism.

Properties of the Euler Class

(1) Naturality. A pullback $f^*(E)$ comes with a map $f^*(E) \xrightarrow{\tilde{f}} E$ that is a lin. isom. on fibers. Thus $\tilde{f}$ pulls back the Thom class to a Thom class: $\tilde{f}^*(c(E)) = c(f^*(E))$

$\tilde{f}|_B = f$ so when we pass through

$$H^*(E, E') \longrightarrow H^*(E) \longrightarrow H^*(B)$$

we get the result.

(2) Negation. Basically obvious — negating the orientation of E negates all signs of intersection.

(3) Whitney sum. ~~Reduces~~ Consider $p_i : E_1 \oplus E_2 \longrightarrow E_i$. (linear on fibers)

Say $c(E_1) \in H^m(E_1, E_1')$ $c(E_2) \in H^n(E_2, E_2')$

Want: $p_1^*(c(E_1)) \cup p_2^*(c(E_2)) = c(E_1 \oplus E_2)$

Reduces to showing

$$H^m(\mathbb{R}^{m+n}, \mathbb{R}^{m+n} - \mathbb{R}^n) \times H^n(\mathbb{R}^{m+n}, \mathbb{R}^{m+n} - \mathbb{R}^m) \longrightarrow H^{m+n}(\mathbb{R}^{m+n}, \mathbb{R}^{m+n} \setminus 0)$$

takes (gen, gen) $\longmapsto$ gen.

(4) Odd dimensions. Use (2) plus the fact that negation is an orientation reversing automorphism

(5) Nonvanishing sections. Basically obvious — in the presence of a nonvan. section, any n-chain in B can be pushed completely off of B.

(6) Euler characteristic

We know $\langle e(M), M\rangle$ = self-int of M in TM

Step 1. $\langle e(M), M\rangle$ = self-int of Δ in $M\times M$.

Step 2. Latter = sum of indices of Lefschetz fixed pts of an $f: M\to M$

Step 3. Choose an f and compute

Step 1. Self-int of M in any $2n$-dim man. U equals $\langle e(N_U M), M\rangle$

Remains to show: $N_{M\times M}\Delta \cong TM$

A vector $(u,v)\in T_xM\times T_xM \cong T_{(x,x)} M\times M$

is tangent to $\Delta \iff u=v$

hence normal to $\Delta \iff u=-v$

The isomorphism $TM \longrightarrow N_{M\times M}\Delta$ is

$$(x,v)\longmapsto ((x,x),(v,-v)).$$

Step 2. $f: M\to M$ is Lefschetz if $Df-I$ invertible at each pt

The index of f at a fixed pt is $+1$ if $\det(Df-I)>0$, -1 o.w.

This number equals the sign of intersection of

Δ with graph $\Gamma(f)$ at $(x, f(x))$

Idea: Check sign of $\underbrace{(v_1,v_1),\dots,(v_n,v_n)}_{T\Delta},\ \underbrace{(v_1, Df(v_1)),\dots,(v_n, Df(v_n))}_{T\Gamma(f)}$

$\xrightarrow{\text{Gauss}} (v_1,v_1),\dots,(v_n,v_n),(0,(Df-I)(v_1)),\dots,(0,(Df-I)(v_n))$ ← last n span $0\times T\Delta$

$\xrightarrow{\text{Gauss}} (v_1,0),\dots,(v_n,0),(0,(Df-I)(v_1)),\dots,(0,(Df-I)(v_n))$

or

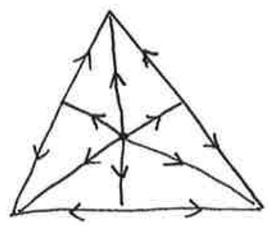
$$\begin{vmatrix} I & I \\ I & Df \end{vmatrix} = \begin{vmatrix} I & I \\ 0 & Df-I \end{vmatrix} = |Df-I|$$

Claim follows.

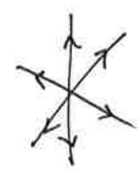
But $\Delta\cap\Gamma(f) = \Delta\cap\Delta$, so done.

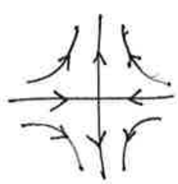
Step 3. Find a nice Lefschetz function.

Choose a vector field, say one pointing from barycenters of higher dim. simplices to barycenters of lower dim simplices (actually, gradient flow for any Morse fn will work).



At a vertex: 

face 

edge: 

Then f is time ε flow.

In the 3 cases, Df is $\begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$, $\begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}$, $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

So $\det(Df - I)$ is $+$ $\quad$ $-$ $\quad$ $+$

as desired.

Thom Isomorphism

The Thom Isom. reduces to a rel version of Leray-Hirsch.

Fiber bundle pairs. $F \to E \xrightarrow{p} B$ with $E' \subseteq E$ s.t. $E' \xrightarrow{p} B$
a bundle with fibers $F' \subseteq F$, compatible trivializations $\rightsquigarrow (E,E') \to B$
e.g. $S(E) \subseteq D(E)$

THM (Relative Leray-Hirsch). Say $(F,F') \to (E,E') \xrightarrow{p} B$ a f.b. pair
s.t. $H^*(F,F')$ f.g. free R-mod in each dim.
If $\exists\, c_j \in H^*(E,E')$ whose restrictions form a basis for $H^*(F,F')$
in each fiber then $H^*(E,E')$ = free $H^*(B)$-module w/ basis $\{c_j\}$.

Pf Main ~~step~~ idea: Construct a related bundle $\hat{E}$, apply absolute LH to $\hat{E}$.

Construction of $\hat{E}$. Let M = mapping cyl of $p : E' \to B$ note $E' \subseteq M$

$$\hat{E} = M \amalg_{E'} E$$

$$\hat{F} = \text{Cone on } \hat{E} = \text{mapping cyl. of const. map}$$

Key isomorphism. $H^*(\hat{E}) \cong H^*(\hat{E}, B) \oplus H^*(B)$ as $H^*(B)$ modules

$H^*(\hat{E},B) \cong H^*(E,E') \leftarrow$ killing E' in E same as killing M in $\hat{E}$, same as killing B in M in $\hat{E}$.

* splitting from retraction $p : \hat{E} \to B$.

Let $\hat{c}_j$ correspond to $(c_j, 0)$. The c_j & 1 restrict to basis
for $H^*(\hat{F}) \cong H^*(F,F')$

LH $\Rightarrow H^*(\hat{E})$ free $H^*(B)$-modules, basis $\{1, \hat{c}_j\}$

$\Rightarrow c_j$ free basis for $H^*(E,E')$. ∎

Euler Class via Poincaré Duality

Fix some oriented $\mathbb{R}^n \longrightarrow E \longrightarrow B$ = smooth, oriented, k-manifold.

Let D = disk bundle of E.

D is an oriented $(n+k)$-manifold with ∂, so it has Poincaré duality

$$H^i(M,\partial M) \xrightarrow{\cong} H_{n+k-i}(M)$$
$$\alpha \longmapsto \underbrace{[M]}_{\text{relative fundamental class}} \frown \alpha \;= \alpha^*$$

Regard the fundamental class $[B]$ as elt of $H_k(D)$

via the map on H_* induced by $B \hookrightarrow D$.

<u>Prop.</u> $[B] = \underset{\text{Thom class}}{c}{}^*$ in $H_k(D)$.

So: An explicit cochain $\{2\text{-cells of } B\} \longrightarrow \mathbb{Z}$ representing u is given by counting intersections of a section with 2-cells of B (assuming gen. pos.). Actually, can replace the section with any subspace homotopic/homologous to B.

Pf. Apply three isomorphisms (WLOG B connected):

$$\mathbb{Z} = H^0(B) \xrightarrow{\text{Thom}} H^n(D, \overset{\text{sphere bundle}}{S}) \xrightarrow{\text{P.D.}} H_k(D) \longrightarrow H_k(B) = \mathbb{Z}$$
$$H^n(D,S) = H^n(D,\partial D)$$

$$1 \longmapsto c \longmapsto c^*$$

Since the composition $\mathbb{Z} \to \mathbb{Z}$ is an iso, $c^* = \pm[B]$.
(Must work harder to get the sign.) $\blacksquare$

CIRCLE BUNDLES AND THE EULER CLASS

There are correspondences:

$$\mathbb{C}^1\text{-bundles} \longleftrightarrow \text{oriented } \mathbb{R}^2\text{-bundles} \longleftrightarrow \text{oriented } S^1\text{-bundles}$$

Both $\rightarrow$ are easy.

First $\leftarrow$ via Euc. metric. $\mathbb{C}$-structure is rotation by π.

Second $\leftarrow$ uses $\text{Diff}^+(S^1) \simeq \text{Isom}^+(S^1) \cong S^1$.

This implies we can modify the local trivializations so they remember distance on S^1. Then build $\mathbb{R}^2$-fibers by coning off S^1-fibers.

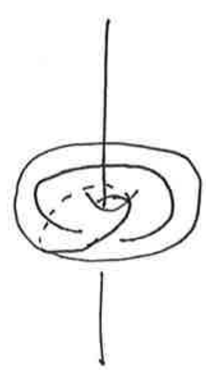
Key example. (Hopf bundle $S^1 \to S^3 \to S^2$) $\longleftrightarrow$ ($CLB \to \mathbb{CP}^1$)

$\mathbb{C}$-description $\quad S^3 = \{(z,w) \in \mathbb{C}^2 : |z|^2+|w|^2 = 1\}$

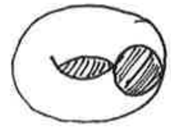
$(z,w) \longmapsto w/z \in \hat{\mathbb{C}}$

or $(z,w) \longmapsto$ line spanned by (z,w) $\in \mathbb{CP}^1$

Topological description



There are two $D^2 \times S^1$



The bundles over the two ∂D^2 are equal as sets

$\rightsquigarrow$ a map $S^3 \to S^2$

Euler class via sections of S^1-bundles

A bundle $S^1 \to E \to X$ is trivial iff it has a section.
For X = CW complex, can try to build a section inductively over skeleta.
Say s_i = section over $X^{(i)}$
s_i extends over D^{i+1} iff $S^i \cong \partial D^{i+1} \xrightarrow{\text{attach}} X^{(i)} \xrightarrow{s_i} S^1$ is homot. trivial (exercise)
But we know: $\pi_i(S^1) = \begin{cases} \mathbb{Z} & i=1 \\ 0 & \text{o.w.} \end{cases}$

So only obstruction is over 2-skeleton.

Can use this idea to build a cochain $\{2\text{-cells of } X\} \to \mathbb{Z}$.

Step 1. Choose any section s_1 over $X^{(1)}$

Step 2. Take degrees of maps $\partial D^2 \to S^1$ as above.

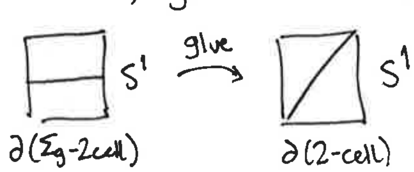
Can check directly this is a cocycle. It vanishes $\Leftrightarrow$ trivial bundle.
(see Candel-Conlon).
It turns out this is the *Euler class*. See below.

We will show:

c_1 for $\mathbb{C}^1$-bundles $\longleftrightarrow$ e for or. $\mathbb{R}^2$-bundles $\longleftrightarrow$ e for or. S^1-bundles

We already showed: $c_1 : \mathrm{Vect}^1_{\mathbb{C}}(X) \xrightarrow{\cong} H^2(X;\mathbb{Z})$

For $X = \Sigma_g$ can build explicitly the unique E_k s.t. $e(E_k) = k \in \mathbb{Z} \cong H^2(\Sigma_g;\mathbb{Z})$.
Idea: Remove a 2-cell. Take trivial bundle over complement, trivial over 2-cell, glue with a twist on $\partial = T^2$



Dehn surgery on $\Sigma_g \times S^1$
Dehn twist in fiber direction.

use Dehn surgery description.

Exercise. $g=0$ $\quad E_k = L(k,1)$ $\quad L(0,1) = S^2 \times S^1$

note $L(2,1) = UTS^2$ since have same Euler class.

$\ \| $

$\mathbb{R}P^3$

$g=1$ $\quad E_k = M_{\left[\begin{smallmatrix}1 & k\\ 0 & 1\end{smallmatrix}\right]}$

<u>Prop.</u> For $\mathbb{C} \to E \to X$, $c_1 = e = e$.

<u>Pf.</u> First compare e for S^1-bundles with c_1.

If we believe e is a char class, then we know it is a deg 1 poly in the c_i $\Rightarrow$ it is a multiple of c_1.

So suffices to check on $CLB \to \mathbb{C}P^1$.

By defn $c_1(CLB) = \alpha = 1 \in \mathbb{Z} \cong H^2(\mathbb{C}P^1)$.

We choose trivializations of the circle bundle $S^1 \to S^3 \to S^2$ over Δ, Δ^c and show corresponding sections over $S^1 = \partial\Delta$ intersect in one pt. This means (up to sign) $e=1$.

Over Δ: $\alpha \mapsto (\alpha, 1)/\text{norm}$

Δ^c: $\alpha \mapsto (1, \alpha)/\text{norm}$ $\quad (\infty \mapsto (1,0))$

On $\partial\Delta$ these equal only for $\alpha = 1$.

exercise: check e for top. description.

We'll also show the two e's ~~have~~ are same, ~~image in~~ for X a manifold. ~~$H^4(X;\mathbb{Z}) = \mathbb{Z}$~~).

Idea: Suppose have a section of E over ∂D^2 of degree 1.

i.e. $(1, \theta) \mapsto \theta$.

Can try to extend to a section of assoc. $\mathbb{R}^2$-bundle.

$(r, \theta) \mapsto (r, \theta)$

There is one zero, at origin. So the cocycle we constructed for S^1-bundles counts intersection pts (with sign) of elts of $H^2(X;\mathbb{Z})$ with themselves.

Using this, and axioms for c_i can again show $e = c_1$.

Milnor-Wood Inequality

Thm. If $E \to \Sigma_g$ is oriented S^1-bundle with $g \geq 1$ and has a foliation transverse to the fibers, then
$$|e(E)| \leq |\chi(\Sigma_g)|.$$

Will show: $UT(\Sigma_g)$ realizes this bound.

There is a correspondence:
$$\left\{\begin{array}{l}\text{oriented } S^1\text{-bundles} \\ \text{over } M \text{ with} \\ \text{transverse foliation}\end{array}\right\} \longleftrightarrow \{\pi_1(M) \longrightarrow \mathrm{Homeo}^+(S^1)\}$$

$\rightarrow$ is monodromy (the foliation identifies pts of fibers).

$\leftarrow$ is: $\tilde{M} \times S^1 / \pi_1(M)$ by diag action gives the bundle, foliation by $\tilde{M} \times pt$ descends.

Unit tangent bundle of Σ_g. We already know $e(UT(\Sigma_g)) = \chi(\Sigma_g)$.

Need to find foliation.

Setup: $\tilde{\Sigma}_g = \mathbb{H}^2$ $\quad UT(\mathbb{H}^2) \cong \mathbb{H}^2 \times S^1$ (triv. given by proj. to $\partial_\infty \mathbb{H}^2 = S^1$)

$\pi_1(\Sigma_g) \longrightarrow \mathrm{Isom}^+(\mathbb{H}^2)$ via deck trans.

induces action on $UT(\mathbb{H}^2)$.

Quotient is precisely $UT(\Sigma_g)$, as desired.

$\downarrow$ So leaves are unit vectors with asymptotic rays.

Above theorem due to Wood. Milnor showed if the bundle admits a flat connection (curvature$=0$) then $|e(E)| \leq |\chi(\Sigma_g)|/2$. (This is a strictly stronger assumption.)

Later we'll use this to prove $\mathrm{Diff}^+(\Sigma_{g,1}) \longrightarrow \mathrm{MCG}(\Sigma_{g,1})$ has no section.

Pontryagin Classes

Complexification. $E \to B \rightsquigarrow E^{\mathbb{C}} \to B$

$E^{\mathbb{C}} = E \otimes \mathbb{C}$ or $E \oplus E$ with $i(x,y) = (-y,x)$.

Pontryagin classes. $p_i(E) = (-1)^i c_{2i}(E^{\mathbb{C}}) \in H^{4i}(B;\mathbb{Z})$

Why only even c_i? The $c_{2i+1}(E^{\mathbb{C}})$ are determined by the w_i:

$$c_{2i+1}(E^{\mathbb{C}}) = \beta(w_{2i}(E)\, w_{2i+1}(E))$$

β ← Bockstein: $H^*(G_n;\mathbb{Z}_2) \circlearrowleft$

Relations to other classes. (1) $p_i(E) \mapsto w_{2i}(E)^2$ via $H^{4i}(B;\mathbb{Z}) \to H^{4i}(B;\mathbb{Z}_2)$

(2) $p_n(E) = e(E)^2$ E = orient. $\mathbb{R}^{2n}$-bundle.

Pf. Whitney sum, $c_{2i} \mapsto w_{4i}$, $c_{2n} = e$.

Later: $p_1(M^4) = \sigma(M^4)$

We can now describe all $\mathbb{Z}$ char classes for real (oriented) bundles.

Thm. (1) $H^*(G_n;\mathbb{Z})/\text{torsion} \cong \mathbb{Z}[p_1, \ldots, p_{\lfloor n/2 \rfloor}]$

(2) $H^*(\tilde{G}_n;\mathbb{Z})/\text{torsion} \cong \begin{cases} \mathbb{Z}[\tilde{p}_1, \ldots, \tilde{p}_{\lfloor n/2 \rfloor}] & n = 2k+1 \\ \mathbb{Z}[\tilde{p}_1, \ldots, \tilde{p}_{\frac{n}{2}-1}, e] & n = 2k \end{cases}$

where $p_i = p_i(E_n)$, $\tilde{p}_i = p_i(\tilde{E}_n)$, $e = e(\tilde{E}_n)$.

All torsion is 2-torsion, so lies in $H^*(G_n;\mathbb{Z}_2)$. It is the image of the Bockstein homomorphism $\beta: H^*(G_n;\mathbb{Z}_2) \circlearrowleft$

Quick idea: Start with $0 \to \mathbb{Z}_2 \to \mathbb{Z}_4 \to \mathbb{Z}_2 \to 0$

Apply $\mathrm{Hom}(C_n(G_n), -) \rightsquigarrow$ LES in H^*

Get $\beta: H^n(G_n;\mathbb{Z}_2) \to H^{n+1}(G_n;\mathbb{Z}_2)$

(notice $\deg c_{2i+1} = \deg w_{2i}w_{2i+1} + 1$).

Gysin Sequence

The computation of $H^*(G_n;\mathbb{Z})$ needs one final tool:

$$\cdots \longrightarrow H^{i-n}(B) \xrightarrow{\cup e} H^i(B) \xrightarrow{p^*} H^i(S(E)) \longrightarrow H^{i-n+1}(B) \longrightarrow \cdots$$

This sequence is the LES for $(D(E),S(E))$ in disguise:

$$\begin{tikzcd}
\cdots \arrow[r] & H^i(D(E),S(E)) \arrow[r, "j^*"] & H^i(D(E)) \arrow[r] & H^i(S(E)) \arrow[r] & H^{i+1}(D(E),S(E)) \arrow[r] & \cdots \\
\cdots \arrow[r] & H^{i-n}(B) \arrow[r, "\cup e"] \arrow[u, "\cong", "\Phi = \text{Thom}"'] & H^i(B) \arrow[r, "p^*"] \arrow[u, "\cong", "p^*"'] & H^i(S(E)) \arrow[r] \arrow[u, "="] & H^{i-n+1}(B) \arrow[r] \arrow[u, "\cong", "\Phi = \text{Thom}"'] & \cdots
\end{tikzcd}$$

Commutativity of first square.
$$\begin{aligned} j^*\Phi(b) &= j^*(p^*(b)\cup c) \\ &= p^*(b)\cup j^*(c) \\ &= p^*(b)\cup p^*(e) \\ &= p^*(b\cup e). \end{aligned}$$

The map $H^i(S(E)) \to H^{i-n+1}(B)$ is called the Gysin map.
It is defined s.t. the third square commutes.
For B a k-manifold, it can also be defined by:

$$H^i(S(E)) \xrightarrow{P.D.} H_{k+(n-1)-i}(S(E)) \xrightarrow{p_*} H_{k+(n-1)-i}(B) \xrightarrow{PD} H^{i-n+1}(B).$$

Or: given an i-cochain φ on $S(E)$ we evaluate on an $(i-n+1)$-chain σ in B by taking the pullback S^{n-1} bundle over σ and applying φ to this.

Computing with Gysin

The computation of $H^*(G_n;\mathbb{Z})$ is modeled on the following argument for $H^*(G_n;\mathbb{Z}_2)$.

$E_n \xrightarrow{\pi} G_n$ universal bundle

$S(E_n) = \{(v,\ell)\}$ ℓ = n-plane in $\mathbb{R}^\infty$, $v \in \ell$ unit.

Define $p: S(E_n) \longrightarrow G_{n-1}$

$(v,\ell) \longmapsto v^\perp \subseteq \ell$

This is a fiber bundle, with fiber S^∞ = unit vectors in $\mathbb{R}^\infty$ $\perp$ to given $(n-1)$-plane.

S^∞ contractible $\Rightarrow p^*$ is $\cong$ on H^*.

Gysin: $\cdots \longrightarrow H^i(G_n) \xrightarrow{\cup e} H^{i+n}(G_n) \xrightarrow{\eta} H^{i+n}(G_{n-1}) \longrightarrow H^{i+1}(G_n) \longrightarrow \cdots$

Key step. $\eta(w_j(E_n)) = w_j(E_{n-1})$.

By defn η is the composition $H^*(G_n) \xrightarrow{\pi^*} H^*(S(E_n)) \xleftarrow[\cong]{p^*} H^*(G_{n-1})$

induced by $G_{n-1} \xleftarrow{p} S(E_n) \xrightarrow{\pi} G_n$

Take pullback $\pi^*(E_n) = \{(v,w,\ell) : \ell \in G_n,\ v,w \in \ell,\ v \text{ unit}\}$

$\cong L \oplus p^*(E_{n-1})$

where L is subbundle with $w \in \mathrm{span}(v)$.

$p^*(E_{n-1})$ is subbundle with $w \perp v$.

But L is trivial: it has section (v,v,ℓ)

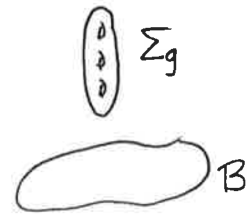
So: $\pi^* w_j(E_n) = w_j \pi^*(E_n) = w_j(L \oplus p^*(E_{n-1}))$

$= w_j\, p^*(E_{n-1}) = p^* w_j(E_{n-1})$ as desired.

Thus η surjective. Now induct on n!

Characteristic Classes for Surface Bundles: An Overview

Surface bundles. These are smooth fiber bundles



$$\begin{array}{c} \Sigma_g \longrightarrow E \\ \downarrow \\ B \end{array}$$

i.e. B covered by U s.t. $p^{-1}(U) \cong U \times \Sigma_g$ (restriction to fibers smooth)

Examples. $B \times \Sigma_g$

M_φ = mapping torus of $\varphi : \Sigma_g \to \Sigma_g$. $B = S^1$

$M_\varphi \times S^1 \to T^2$

Isomorphism. As before, a homeo $E \xrightarrow{p} B$ to $E' \xrightarrow{p'} B$ taking $p^{-1}(b)$ to $(p')^{-1}(b)$ by diffeo.

Pullback. As before, given $f : A \to B$, we set

$$f^*(E) = \{(a,x) : f(a) = p(x)\}$$

Characteristic classes. Fix g, R. A char class is a fn

$$\chi : \{\Sigma_g\text{-bundles}\}/\cong \;\longrightarrow H^*(\text{Base}; R)$$

that is natural:

$$\chi(f^*(E)) = f^*\chi(E).$$

Why? Surface bundles are basic fiber bundles / manifolds.

Want invariants.

There are other applications to mapping class groups.

We study surface bundles in analogy with vector bundles.

- A Grassmannian for surface bundles

$C(\Sigma_g, \mathbb{R}^\infty)$ = space of smooth (oriented) submanifolds of $\mathbb{R}^\infty$ diffeo to Σ_g.

$\mathcal{E}(\Sigma_g, \mathbb{R}^\infty) = \{(x, S) \in \mathbb{R}^\infty \times C(\Sigma_g, \mathbb{R}^\infty) : x \in S\}$

$\mathcal{E}(\Sigma_g, \mathbb{R}^\infty) \longrightarrow C(\Sigma_g, \mathbb{R}^\infty)$ is a Σ_g-bundle.

We will show:

$$\{\Sigma_g\text{-bundles over } B\}/_{\cong} \longleftrightarrow [B, C(\Sigma_g, \mathbb{R}^\infty)]$$

and so (fixing g, $\mathbb{R}$):

$$\{\text{char. classes for } \Sigma_g\text{-bundles}\} \longleftrightarrow H^* C(\Sigma_g, \mathbb{R}^\infty).$$

- The mapping class group

In vector bundle case, can reduce structure group to $O(n)$ i.e. transition maps can be taken to be isometries on fibers. Have an analogous reduction here.

$$\begin{aligned} MCG(\Sigma_g) &= \pi_0 \mathrm{Diff}^+(\Sigma_g) \\ &= \mathrm{Diff}^+(\Sigma_g) / \text{isotopy} \end{aligned}$$

We'll show: $\mathrm{Diff}(\Sigma_g)$ has contractible components, i.e.

$$\mathrm{Diff}^+(\Sigma_g) \simeq MCG(\Sigma_g)$$

From this we can deduce:

$$\left\{ \begin{array}{c} \Sigma_g\text{-bundles} \\ \text{over } B \end{array} \right\} \longleftrightarrow [B, K(MCG(\Sigma_g), 1)]$$

$$\longleftrightarrow \mathrm{Hom}(\pi_1(B), MCG(\Sigma_g)) \Big/ \underbrace{MCG(\Sigma_g)}_{\text{conj.}}$$

and so:

$$\left\{ \begin{array}{c} \text{Char. classes} \\ \text{for } \Sigma_g\text{-bundles} \end{array} \right\} \longleftrightarrow H^* MCG(\Sigma_g).$$

- Morita-Mumford-Miller classes.

$$\boxed{\begin{array}{l} \text{Here Gysin means:} \\ H^{2i+2}(E) \xrightarrow{PD} H_{n-2i}(E) \\ \xrightarrow{proj_*} H_{n-2i}(B) \xrightarrow{PD} H^{2i}(B) \end{array}}$$

Given $\Sigma_g \to E \to M$ = smooth manifold

Let V = vertical 2-plane bundle on E

Define $e_i(E) = \mathrm{Gysin}(e^{i+1}) \in H^{2i}(M)$.

We'll see: e_1 is proportional to: signature, WP form, 1^{st} Pontryagin class.

<u>Thm</u> $\lim\limits_{g \to \infty} H^*(MCG(\Sigma_g^1); \mathbb{Q}) \cong \mathbb{Q}[e_1, e_2, \ldots]$

i.e. the e_i exactly describe the stable rational char. classes.

- Unstable classes

We know $\chi(MCG(\Sigma_g)) = \zeta(1-2g)/2-2g$. So there are lots of other char. classes. Almost nothing is known.

Cohomology of Mapping Class Groups

coeff $= \mathbb{Q}$

Thm. $\mathrm{vcd}(MCG(\Sigma_g)) = 4g-5 \Rightarrow H^i(MCG(\Sigma_g)) = 0 \quad i > 4g-5$ (any coeff)

(although $H^{4g-5}(MCG(\Sigma_g)) = 0$).

Low dim's:

$H^1(MCG(\Sigma_g)) = 0 \quad g \geq 0.$

$H^2(MCG(\Sigma_g)) = \mathbb{Q} \quad g \geq 4$

$H^3(MCG(\Sigma_g)) = 0 \quad g \geq 6$

$H^4(MCG(\Sigma_g)) = \mathbb{Q}^2 \quad g \geq 10.$

Low genus:

$H^*(MCG(T^2)) = 0.$

$H^*(MCG(\Sigma_2)) = 0$

$H^*(MCG(\Sigma_3)) = \mathbb{Q}[C_2, C_6]$

$H^*(MCG(\Sigma_4)) = \mathbb{Q}[C_2, C_4, C_5]$

C_5, C_6 unstable.

Stability. $H^i(MCG(\Sigma_g^1))$ indep of g, $g \geq 3i/2 + 1$.

$\boxed{\text{Also, } H^i(MCG(\Sigma_g^1)) \cong H^i(MCG(\Sigma_g)) \text{ in this case.}}$

Mumford Conjecture. $H^i(MCG(\Sigma_\infty^1)) = \mathbb{Q}[e_1, e_2, \ldots] \quad e_i \in H^{2i}$ i^{th} MMM class

Euler char. $\chi(MCG(\Sigma_g)) = \zeta(1-2g)/2-2g \sim (-1)^g (2g-1)!/2^{2g-1}\pi^{2g}$

$\Rightarrow > 2^g$ unstable classes. use: $p(n) \sim \frac{1}{n} e^{\pi\sqrt{2n/3}}$

Applications.

① $\mathrm{Diff}^+(\Sigma_g) \xrightarrow{\pi} MCG(\Sigma_g)$ has no section

pf: $\pi^*(e_3) = 0$.

② Odd e_i are geometric, cobordism invar, vanish on handlebody group.

A Classifying Space for Surface Bundles

Goal: $\{\Sigma_g\text{-bundles over } B\}/\cong \;\leftrightarrow\; [B, K(MCG(\Sigma_g),1)]$ $\quad$ B = conn CW

$\leftrightarrow \operatorname{Hom}(\pi_1(B), MCG(\Sigma_g))/MCG(\Sigma_g)$

$\Rightarrow$ Ring of char. classes for Σ_g-bundles $\cong H^*(MCG(\Sigma_g))$

We first construct a direct analogue of G_n. Then use contractibility of $\operatorname{Diff}_0(\Sigma_g)$ to show this is a $K(MCG(\Sigma_g),1)$ ← this part special to Σ_g bundles.

The Grassmannian. G_{Σ_g} = set of smooth submanifolds of $\mathbb{R}^\infty$ diffeo to Σ_g.
$G_{\Sigma_g}(\mathbb{R}^n)$ topologized as quotient $\operatorname{Emb}(\Sigma_g, \mathbb{R}^n)/\operatorname{Diff}(\Sigma_g)$
and $G_{\Sigma_g} = \varinjlim G_{\Sigma_g}(\mathbb{R}^n)$ $\quad$ ↳ C^∞ topology

Canonical bundle. $E_{\Sigma_g} = \{(x,S) \in \mathbb{R}^\infty \times G_{\Sigma_g} : x \in S\}$
Need to check $E_{\Sigma_g} \to G_{\Sigma_g}$ is a Σ_g-bundle
i.e. if $S \in G_{\Sigma_g}$ and $S' \in G_{\Sigma_g}$ is sufficiently close,
need a canonical diffeo $S' \to S$.
First for $G_{\Sigma_g}(\mathbb{R}^n)$.
Main idea: if S' close to S then S' is a section of
normal bundle N of S = tubular nbd;
then $S' \to S$ is projection in N.
This is because S is transverse to fibers, which is an open condition, so nearby S' is transverse to any given fiber, hence to all nearby fibers, hence to all fibers by compactness.
For S' close enough to S there is an isotopy of S to S' preserving transversality, hence $S' \cap \text{fiber} = 1$ pt
$\Rightarrow S'$ a section.
The result follows by defn of topology on G_{Σ_g}.

Universality. To show $\{\Sigma_g\text{-bundles over } B\}/\cong \longleftrightarrow [B, G_{\Sigma_g}]$ $\quad B =$ paracompact

Essentially same as v.b. case. Basic idea: Realizing $E \to B$ as $f^*(E_{\Sigma_g})$ equiv. to finding $E \xrightarrow{g} \mathbb{R}^\infty$ smooth emb. on fibers. Such g induces $f, \tilde{f}$ s.t.

$$\begin{tikzcd} E \arrow[r, "\tilde{f}"] \arrow[d] & E_{\Sigma_g} \arrow[d] \\ B \arrow[r, "f"] & G_{\Sigma_g} \end{tikzcd}$$

Fix some $E \xrightarrow{p} B \leftarrow$ compact. Want to find g, hence f.
Choose $U_i \subseteq B$ s.t. $p^{-1}(U_i) \cong U_i \times \Sigma_g$, subord. part of 1 $\{\varphi_i\}$

$$g_i : p^{-1}(U_i) \longrightarrow U_i \times \Sigma_g \longrightarrow \Sigma_g \xrightarrow[\text{emb.}]{\text{any}} \mathbb{R}^n$$

$$g : E \longrightarrow \mathbb{R}^n \times \cdots \times \mathbb{R}^n \subseteq \mathbb{R}^\infty$$

$$p \longmapsto (\varphi_1 g_1(p), \ldots, \varphi_N g_N(p))$$

Any two g's are homotopic: $g_0 \searrow$ even coords $\xrightarrow[\text{hom.}]{\text{str line}}$ odd coords $\swarrow g_1$

$\rightsquigarrow$ resulting f unique up to homotopy.

Relation to MCG. Step 1: There is a bundle $\mathrm{Diff}^+(\Sigma_g) \to P_{\Sigma_g} \to G_{\Sigma_g}$

(use tubular nbds / sections as above)

Step 2: $\mathrm{Emb}(\Sigma_g, \mathbb{R}^\infty) \simeq *$

Enough to find canonical, continuously varying paths to some basept. S

Choose S in even coords.

For any S', apply $\mathbb{R}^\infty \longrightarrow \mathbb{R}^{\text{odd coords}}$

then straight line homotopy to S.

Step 3: Apply L.E.S. for fiber bundle (or, fibration)

$$\cdots \longrightarrow \pi_n(F) \longrightarrow \pi_n(E) \longrightarrow \pi_n(B) \longrightarrow \pi_{n-1}(F) \longrightarrow \cdots$$

(comes from L.E.S. in π_* for (E,F) and $\pi_*(E,F) \cong \pi_*(B)$).

<u>Thm</u> (Earle-Eells). $\mathrm{Diff}(\Sigma_g)$ has contractible components.

$$\rightsquigarrow \quad \pi_i(G_{\Sigma_g}) \cong \pi_{i-1}(\mathrm{Diff}(\Sigma_g)) \quad \forall i.$$

$$\pi_1(G_{\Sigma_g}) \cong \pi_0(\mathrm{Diff}(\Sigma_g)) = MCG^{\pm}(\Sigma_g)$$

$$\pi_i(G_{\Sigma_g}) = 0 \quad i > 1.$$

Diffeomorphism Groups of Surfaces

$S =$ compact, connected surface

Write $\mathrm{Diff}(S)$ for $\mathrm{Diff}(S, \partial S)$. $\quad C^\infty$ topology.

<u>Thm</u>. If $S \neq S^2, \mathbb{RP}^2, T^2, KB$ then the components of $\mathrm{Diff}(S)$ are contractible.

Note: $\mathrm{Diff}(S^2) \simeq \mathrm{Diff}(\mathbb{RP}^2) \simeq SO(3)$

$\mathrm{Diff}(T^2) \simeq T^2$, $\mathrm{Diff}(KB) \simeq S^1$.

Proof has 3 steps.

① Reduction to case $\partial S \neq \emptyset$

will show $\pi_i(\mathrm{Diff}(S)) \cong \pi_i(\mathrm{Diff}(S - \overset{\text{open}}{D^2}))$.

② Inductive step

will show $\pi_i(\mathrm{Diff}(S)) \cong \pi_i(\mathrm{Diff}(\overset{S \text{ cut along } \alpha}{S_\alpha}))$

③ Base case

$\pi_i(\mathrm{Diff}(D^2)) = 0 \quad i \geqslant 1.$

Step 1. Reduction to case $\partial S \neq \emptyset$.

Fix $x_0 \in D \subseteq S$. Let $S_0 = S - \mathrm{int}\, D$.

To show $\pi_i(\mathrm{Diff}(S)) = \pi_i(\mathrm{Diff}(S, x_0)) = \pi_i(\mathrm{Diff}(S, D)) = \pi_i(\mathrm{Diff}(S_0))$

Last equality easy. Remains to do other two.

First equality. There is a fiber bundle $\mathrm{Diff}(S, x_0) \longrightarrow \mathrm{Diff}(S) \longrightarrow S$.

$\uparrow$ diffeos fixing x_0.

$\rightsquigarrow$ L.E.S.:

$$\pi_{i+1}(S) \longrightarrow \pi_i(\mathrm{Diff}(S, x_0)) \longrightarrow \pi_i(\mathrm{Diff}(S)) \longrightarrow \pi_i(S)$$

But $\pi_i(S) = 0$ $i > 1$ (as $\tilde{S} \simeq *$).

$\rightsquigarrow$ $\pi_i(\mathrm{Diff}(S, x_0)) \cong \pi_i(\mathrm{Diff}(S))$ $i > 1$.

$i = 1$ case:

$$0 \longrightarrow \pi_1 \mathrm{Diff}(S, x_0) \longrightarrow \pi_1(\mathrm{Diff}(S)) \longrightarrow \pi_1(S, x_0)$$
$$\xrightarrow{\partial} \pi_0 \mathrm{Diff}(S, x_0) = MCG(S, x_0)$$

Suffices to show $\ker \partial = 0$.

But the composition

$$\pi_1(S, x_0) \longrightarrow MCG(S, x_0) \longrightarrow \mathrm{Aut}\, \pi_1(S, x_0)$$

is $\alpha \longmapsto$ inner automorphism conj. by α

To show this is inj, suffices to show $Z\, \pi_1(S) = 1$.

For latter: $\tilde{S} \cong \mathbb{H}^2$

$\pi_1(S) \longleftrightarrow$ deck trans. in $\mathrm{Isom}^+ \mathbb{H}^2$

& independent hyperbolic isometries do not commute.

Second equality. Another fiber bundle: $\underset{D \text{ fixed}}{\mathrm{Diff}(S, D)} \longrightarrow \mathrm{Diff}(S, x_0) \longrightarrow \mathrm{Emb}((D, x_0), (S, x_0))$

Claim: $\mathrm{Emb}((D, x_0), (S, x_0)) \simeq GL_2(\mathbb{R}) \simeq O(2)$

$$f \longmapsto D_{x_0} f$$

As above, LES $\Rightarrow$ $\pi_i \mathrm{Diff}(S, x_0) \cong \pi_i \mathrm{Diff}(S, D)$ $i > 1$.

$i=1$ case: $\quad 0 \to \pi_1 \mathrm{Diff}(S,D) \to \pi_1 \mathrm{Diff}(S,x_0)$

$$\to \underset{\overset{\|}{\mathbb{Z}}}{\pi_1 \mathrm{Emb}((D,x_0),(S,x_0))} \xrightarrow{\partial} \pi_0 \mathrm{Diff}(S,D) = MCG(S_0).$$

Again, need $\ker \partial = 0$.

$p \in \partial S_0$

But $\quad \mathbb{Z} \longrightarrow MCG(S_0) \longrightarrow \mathrm{Aut}\, \pi_1(S_0, p)$

is $\quad 1 \longmapsto$ conj. by ∂-element.

Since $\pi_1(S_0)$ is free, we are done.

Another point of view. We could have combined the two steps.

There is a fiber bundle

$$\mathrm{Diff}(S,(p,v)) \longrightarrow \mathrm{Diff}(S) \longrightarrow UT(S)$$

with fiber $\simeq \mathrm{Diff}(S_0)$.

Apply same argument.

Step 3. Base step: $\mathrm{Diff}_0(D^2)$ contractible

D^2_+ = top half of D^2

$\mathrm{Emb}(D^2_+, D^2)$ = space of embeddings $D^2_+ \to D^2$ fixing $D^2_+ \cap \partial D^2$ and taking rest of D^2_+ to int D^2.

$\alpha = D^1$ = equator of D^2

$A(D^2, \alpha)$ = embeddings of proper arcs in D^2 with same endpts as α. ← intersect ∂D^2 only at endpts.

$\rightsquigarrow$ fibration

$$\begin{array}{ccc} \mathrm{Diff}(D^2_+) & \longrightarrow & \mathrm{Emb}(D^2_+, D^2) \\ & & \downarrow \\ & & A(D^2, \alpha) \end{array}$$

Uses: the space of tubular nbds of a submanifold is contractible.

Claim 1. $\mathrm{Emb}(D^2_+, D^2) \simeq *$.

Claim 2. $A(D^2, \alpha) \simeq *$. *More generally*, $A(S,\alpha) \simeq *$. Proven below.

LES $\Rightarrow$ $\mathrm{Diff}(D^2_+) \simeq *$. But $D^2_+ \cong D^2$.

Step 2. Induction step.

Induction on $-\chi(S)$.

α = proper arc in S.

$A(S, \alpha)$ = emb's of proper arcs in S, iso to α, same endpts

$\rightsquigarrow$ fiber bundle $\mathrm{Diff}_0(S,\alpha) \longrightarrow \mathrm{Diff}_0(S) \longrightarrow A(S,\alpha)$

↑ diffeos fixing α ptwise, $\simeq \mathrm{Diff}_0(S \text{ cut along } \alpha)$.

LES + induction + Claim 2 $\Rightarrow$ $\mathrm{Diff}_0(S) \simeq *$.

SMALE'S PROOF. (Original version of Step 3)

<u>Thm</u> The space of C^∞ diffeos of I^2 that are id in nbd of ∂I^2 is contractible.

Some ideas. Given $f : I^2 \to I^2 \rightsquigarrow$ vector field V:

$$V(x,y) = df_{f^{-1}(x,y)}(1,0).$$

Note: $\widetilde{\mathbb{R}^n - \{0\}}$ not contractible, $n \neq 2$.

There is a homotopy V_t s.t. $V_0 = V$, $V_1 =$ const. vector field $(1,0)$, $V_t =$ nonvan. vector field id in nbd of ∂I^2 since $V_0, V_1 : I^2 \to \mathbb{R}^2 - \{0\}$.

Then define $f_t : I^2 \to [0,\infty) \times [0,1]$

$f_t(x,y) =$ flow along V_t, start at $(0,y)$, for time x.

Clearly $f_1 = \mathrm{id}$, $f_0 = f$. (n.b. no spiralling, for then there would be a singularity).

Problem: $\mathrm{Im}\, f_t$ maybe not $= I^2$.

Solution: Precompose each f_t with a reparameterization in the x-dir. Result is a homotopy of f to id through diffeos.

By fixing once and for all a def. retraction of $\widetilde{\mathbb{R}^2 - \{0\}}$ to a point, get a consistent way of deforming an arbitrary diffeo to id, at all times $=$ id in nbd of ∂I^2.

(See Lurie's notes for an *Earle-Eells*-style approach.)

CERF'S STRAIGHTENING TRICK. (Toy case for Claim 2).

We'll need to know that some basic spaces of embeddings are contractible. We start with a warmup.

<u>Prop</u>. The space of smooth embeddings of arcs in $\mathbb{R} \times [0,\infty)$ based at 0 is contractible.

<u>Pf</u>. The space of linear arcs is clearly contractible – it is homeo to $\mathbb{R} \times [0,\infty)$.

Here is a canonical isotopy from an arbitrary arc ~~to~~ f to a linear one:

$$F_t(x) = \begin{cases} \dfrac{f((1-t)x)}{1-t} & t<1 \\ f'(0)x & t=1 \end{cases}$$

Can soup this up:

<u>Prop</u>. The space of smooth embeddings of arcs in S based at $p \in \partial S$ is contractible.

<u>Pf</u>. By previous prop, need a canonical isotopy of an arbitrary arc into a fixed tubular nbd of p.

For any compact set of arcs, can use

$$F_t(x) = f(\alpha x) \qquad \alpha = \max\{\varepsilon, (1-tx)\}.$$

i.e. $F_t(x)$ traces out shorter & shorter subarcs.

This implies weak contractibility.

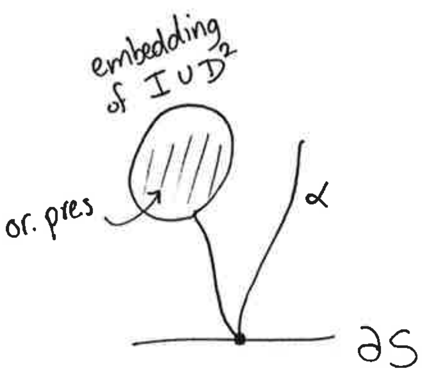
Claim 2: Contractibility of arc spaces

α = proper arc in S

$A(S,\alpha)$ = space of proper arcs $\simeq \alpha$, same endpts as α.

Case 1. α connects distinct components of ∂S.

T = surface obtained from S by capping with disk at one end of α



$\rightsquigarrow$ fiber bundle

$$\begin{array}{ccc} \text{both endpts fixed} \to \mathrm{Emb}(I,S) & \longrightarrow & \mathrm{Emb}(I \cup D^2, S) \\ & & \downarrow \\ & & \mathrm{Emb}(D^2, T - \partial T) \end{array}$$

Claim. $\mathrm{Emb}(I \cup D^2, S) \simeq *$.

Pf of claim. Another fiber bundle ($p \in \partial D^2$, $x \in \operatorname{int} S$)

$$\begin{array}{ccc} \mathrm{Emb}((D^2,p),(S,x)) & \longrightarrow & \mathrm{Emb}(I \cup D^2, S) \\ & & \downarrow \\ \text{one endpt fixed} & \longrightarrow & \mathrm{Emb}(I,S) \end{array}$$

Base, fiber contractible by variations on Cerf's straightening.

Claim. $\pi_i \mathrm{Emb}(D^2, T - \partial T)$ $i > 0$

Pf. Yet another fiber bundle:

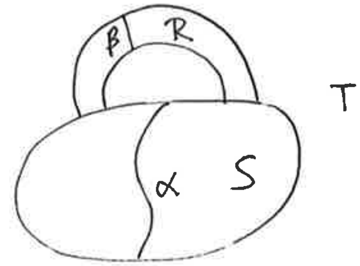
$$\begin{array}{c} \mathrm{Emb}(D^2, T - \partial T) \\ \downarrow \text{eval@}0 \\ T - \partial T \end{array}$$

By two claims, plus LES for main fiber bundle, $\mathrm{Emb}(I,S)$ has contractible components, one of which is $A(S,\alpha)$.

Case 2. α joins a component of ∂S to itself

Idea: add a handle $T = S \cup R$ s.t.

α joins distinct comp's of ∂T



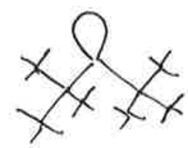
Suffices to show

$\pi_i\, A(T-\beta, \alpha) \longrightarrow \pi_i\, A(T, \alpha)$ injective.

Key: there is a cov. space $\tilde{T}$ of T hom. eq. to S.

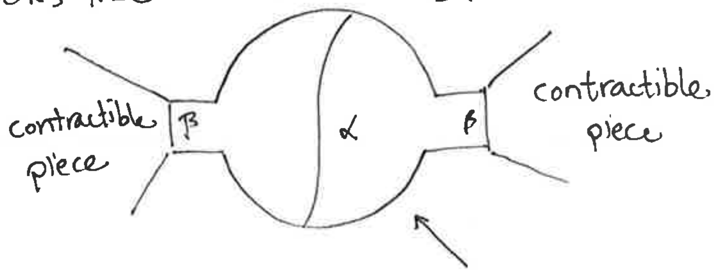
because $\pi_1(T) = \pi_1(S) * \mathbb{Z}$

so $\tilde{T}$ = cover corr to $\pi_1(S)$

Toy case 

$\mathbb{Z} \leq \mathbb{Z} * \mathbb{Z}$

$\tilde{T}$ looks like T cut along β



Identify $A(T-\beta, \alpha)$ with space of arcs in this region of $\tilde{T}$.

$A(T, \alpha)$ with a space of arcs in $\tilde{T}$:

lifts of arcs in T $\rightarrow \tilde{A}(T, \alpha) \subseteq A(\tilde{T}, \alpha) \leftarrow$ arcs in $\tilde{T}$

Suffices to show composition $A(T-\beta, \alpha) \overset{i}{\hookrightarrow} A(\tilde{T}, \alpha)$ is inj on π_*.

Need a retraction $r : A(\tilde{T}, \alpha) \longrightarrow A(T-\beta, \alpha)$

s.t. $r \circ i \sim id$.

The r is induced by shrinking the two contractible pieces.

Characteristic Classes in Degree One

We know now: $H^*(MCG(S_g)) \cong$ Ring of char classes for Σ_g-bundles

<u>Thm</u>. $H^1(MCG(S_g); \mathbb{Z}) = 0 \quad g \geq 1.$

<u>Pf</u>. We'll do $g \geq 3$. Ingredients:

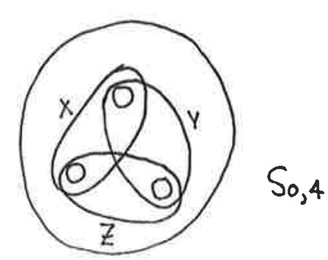
1. $MCC(S_g)$ is gen. by Dehn twists about nonseparating curves

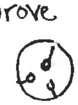
2. Any two such Dehn twists are conjugate in $MCG(S_g)$
3. There is a relation among such twists of the form
$$T_x T_y T_z = T_a T_b T_c T_d$$

It follows that $H_1(MCG(S_g); \mathbb{Z}) \cong MCG(S_g)^{ab}$ is trivial. hence $H^1(MCG(S_g); \mathbb{Z}) = 0$.

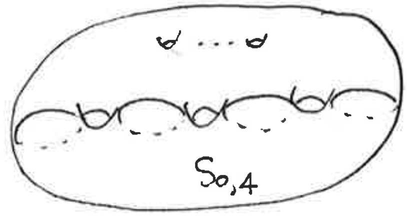
<u>Ingredient 2</u>. Follows from: $f T_a f^{-1} = T_{f(a)}$ and classification of surfaces.

<u>Ingredient 3</u>. Follows from: Lantern relation
$$T_x T_y T_z = \prod T_{\partial_i}$$



(prove by checking action on  and using $Mod(D^2) = 1$)

and the embedding:



GENERATING MCG (Ingredient 1).

Two (sub) ingredients: ① The complex of curves $C(S_g)$ is connected $g \geq 2$.

vertices: isotopy classes of simple closed curves

edges: disjoint representatives

② The Birman exact sequence $\chi(S) < 0$.

$$1 \to \pi_1(S,p) \to MCG(S,p) \to MCG(S) \to 1.$$

<u>Outline of proof.</u> ① $\Rightarrow$ complex of nonsep. curves $N(S_g)$ is connected.

$\Rightarrow$ given any two isotopy classes of nonsep s.c.c. in S_g $\exists \prod T_{c_i}$ c_i nonsep taking one to other.*

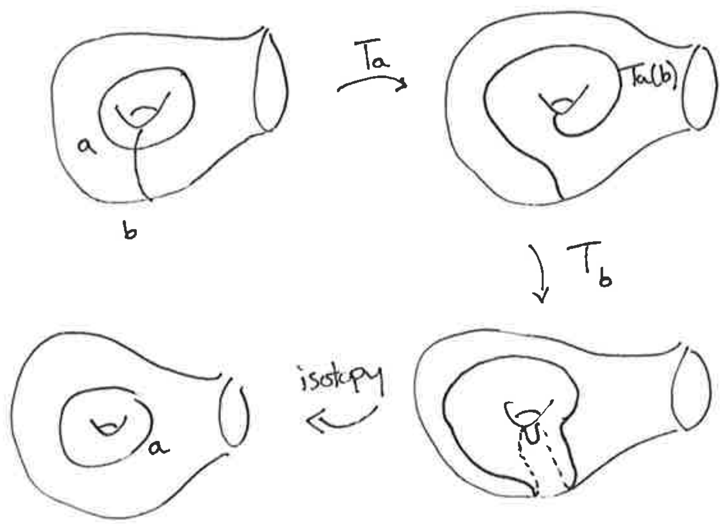
$\Rightarrow$ $MCG(S_g)$ gen. by nonsep twists if $MCG(S_g - c)$ is.

But $MCG(S_g - c) \cong MCG(S_{g-1}, 2)$

② (applied twice) ~~& induction~~ $\Rightarrow$ $MCG(S_{g-1}, 2)$ is gen by nonsep twists if $MCG(S_{g-1})$ is.

Done by induction. Base case is $MCG(S_1) \cong SL_2\mathbb{Z}$ gen by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$.

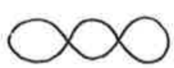
* Use the relation $T_b T_a(b) = a$ for $i(a,b) = 1$.



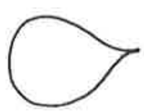
Connectivity of $C(S_g)$

Take two vertices of $C(S_g)$, represent them by s.c.c. a, b in S_g.
Choose smooth fns f_0, f_1 s.t. a is a level set of f_0, b of f_1.
Connect f_0 to f_1 by a path $f_t \in C^\infty(S_g, \mathbb{R})$.

Cerf Lemma. Any path $f_t \in C^\infty(S_g, \mathbb{R})$ can be approx. by $g_t \in C^\infty(S_g, \mathbb{R})$ so each g_t is in one of following classes:

① Morse functions with at most 2 coincident critical values ← crit. values passing each other



② functions with distinct crit vals and exact one degen. crit pt of the form $x^3 \pm y^2 + c$ ← crit vals merging/splitting

Claim. Each g_t has a level set rep. a vertex of $C(S_g)$.

Nearby curves are isotopic $\Rightarrow \{t : v \in C(S_g) \text{ is rep by a level set of } g_t\}$ is open in $\mathbb{R}$
Also, level sets of the same g_t are disjoint.
Result follows from compactness of $[0,1]$.

Remains to prove claim. Take nbd of crit set: 

If two circles bound disks, modify the function to get rid of this crit pt.
Look at another crit pt.

Or: Given $f: S_g \to \mathbb{R} \rightsquigarrow$ graph Γ_f by crushing conn. comp. of level sets.
easy Euler char. count. → $\operatorname{rk}(\Gamma_f) = g$. except in case ② above where $\operatorname{rk}(\Gamma_f) = g-1$. ← this is where $g \geq 2$ used!
Any nontrivial cocycle (= pt) in Γ_f corresponds to a nontrivial level set in S_g. (this shows $N(S_g)$ connected!)

MMM CLASSES

$S_g \to E \to B$

$\rightsquigarrow$ V = vertical 2-plane bundle on E.

$$e_1(E) = \text{Gysin}\left(e(V)^2\right) \in H^4(B).$$

For $B = S_h$ compute by intersecting 2 generic sections with 0-section, since ① e is P. dual to section $\cap$ 0-section

② $\cup$ is P. dual to $\cap$

③ Gysin is P. dual to projection.

We will see: if E_1 diffeo. E_2 then $e_1(E_1) = e_1(E_2)$

e.g. Atiyah-Kodaira: $\begin{array}{ccc} S_4 & \to & M \\ & & \downarrow \\ & & S_{17} \end{array}$ $\qquad$ $\begin{array}{ccc} S_{49} & \to & M \\ & & \downarrow \\ & & S_2 \end{array}$

Say e_1 geometric.

More generally: $e_i(E) = \text{Gysin}\left(e(V)^{i+1}\right) \in H^{2i}(B)$

Compute by intersecting $i+1$ sections with 0-section.

Thm. (Church-Farb-Thibault) e_{2i+1} geometric.

Want to show $e_i \neq 0$. Need $S_g \to M^{2i+2} \to B^{2i}$ with $e_i(M) \neq 0$ $\forall\, g, i$.

Will use branched covers.

SIGNATURE

M = closed, oriented $4k$-manifold

$$\rightsquigarrow \quad H^{2k}(M;\mathbb{Q}) \otimes H^{2k}(M;\mathbb{Q}) \longrightarrow H^{4k}(M;\mathbb{Q}) \approx \mathbb{Q}$$
$$\alpha \otimes \beta \longmapsto \alpha \cup \beta$$

bilin. form, <u>symmetric</u> since $2k$ even.

$\sigma(M)$ = signature of this form : # pos. eigen vals − # neg. eigenvals

Rochlin: $\sigma(M^4) = 0 \iff M^4 = \partial W^5$

Hirzebruch: $p_1(M^4) = 3\sigma(M^4)$ (baby case of H. σ formula)

<u>Prop.</u> $S_g \to E \to S_h$
$$\Rightarrow \langle e_1(E), S_h \rangle = \langle p_1(E), E \rangle \ (= 3\sigma(E))$$

<u>Cor.</u> e_1 is geometric.

<u>Pf of Prop.</u> $TE \cong V \oplus \pi^* S_h$

$$\rightsquigarrow p_1(E) = p_1(V \oplus \pi^* S_h)$$
$$= p_1(V) + \pi^* p_1(S_h)$$
$$= e(V)^2 + 0$$

in general $p_1 = e^2$

$$\Rightarrow \langle e_1(E), S_h \rangle = \langle \mathrm{Gysin}(e(V)^2), S_h \rangle$$
$$= \langle e(V)^2, E \rangle$$
$$= \langle p_1(E), E \rangle$$

exercise:
① $\mathrm{Gysin}(\alpha)(\sigma) = \alpha(\pi^* \sigma)$
② $\pi^* S_h = E$

$\blacksquare$

BRANCHED COVERS

A cyclic branched cover is a map $\tilde{M} \xrightarrow{p} M$ that is a cyclic covering away from a codim 2 subman of M = ramification locus
(can allow more complicated ram. locus, but we won't)

$\forall\, p \in M\ \exists$ nbd U s.t. $p^{-1}(U) \to U$ is
① trivial m-fold cover (m copies of U), or
② quotient by order m rotation (m = degree of cover)

e.g.

Can sometimes get cyclic branched covers via group actions: Say
$\mathbb{Z}/m \curvearrowright N$ by or. pres. diffeos s.t. ① fixed set has codim 2, F = mnfld
② action free outside F

Then $\bar{N} = N/\mathbb{Z}/m$ is a manifold (check!) and $N \to \bar{N}$ is cyclic b. cover
Near F, proj looks like $F \times \mathbb{C} \to \bar{F} \times \mathbb{C}$

$$(p, z) \mapsto (p, z^m)$$

<u>Thm</u>. Every closed, or. 3-man is a 3-fold simple branched cover over S^3.

Existence of Branched Covers

Prop. M = closed or. smooth n-man.
$B \subseteq M$ or. subman of codim 2.
If $[B] \in H_{n-2}(M)$ divis. by m. in $H_{n-2}(M;\mathbb{Z})$.
then $\exists$ m-fold cyclic branched cover over M ramified along B.

Proof for $M = S^3$, $B = K$. Let S = Seifert surface
$\rightsquigarrow [S] \in H_2(S^3, K)$
$\cong H^1(S^3 - K)$

(via $H_2(S^3, K) \longrightarrow H_2(S^3 - K, N(K) - K) \longrightarrow H_2(S^3 - N(K), \partial N(K))$
$\xrightarrow{P.D.} H^1(S^3 - N(K)) \longrightarrow H^1(S^3 - K)$)

The elt of H^1 is signed intersection with S.
An elt of $H^1(S^3 - K)$ is a map $H_1(S^3 - K) \longrightarrow \mathbb{Z}$.
Reduce mod any m, get a cover over $S^3 - K$.
Glue K into the cover.

This works in general. There is no Seifert surface per se, but there is a class A in $H_{n-1}(M, \mathbb{Z}_m)$ with boundary B. Then, elts of $H^1(M; \mathbb{Z}_m)$ are maps $H_1(M;\mathbb{Z}) \longrightarrow \mathbb{Z}_m$, so can proceed as above.

We know the elt of H^1 is nontrivial by considering a small loop around B in M. It intersects A in one pt.

EXISTENCE OF BRANCHED COVERS II

Vector Bundle Version.

Suppose $[B] = m[A]$ in $H_{n-2}(M;\mathbb{Z})$

Let $[B]^*$, $[A]^*$ be P. duals.

We know:

$$\begin{array}{l}\text{Group of } \mathbb{C}^1\text{-bundles} \\ \text{on } M \text{ under } \otimes\end{array} \cong H^2(M;\mathbb{Z})$$

Let E_B be $\mathbb{C}^1$-bundle corr. to $[B]^*$. This means

E_B has a section $s: M \to E_B$ s.t.

$$\mathrm{Im}(s) \cap M = B.$$

Similarly, $E_A \leftrightarrow [A]^*$. By above isomorphism:

$$E_A^{\otimes m} \cong E_B$$

Define

$$f: E_A \longrightarrow E_B$$
$$v \longmapsto v \otimes \cdots \otimes v = v^m$$

Set

$$\tilde{M} = f^{-1}(\mathrm{Im}(s)).$$

Each pt of $M - B$ has m preimages: the mth roots.

Branched Covers and Euler Classes

A cyclic branched cover $\tilde{E} \xrightarrow{p} E$ is a fiberwise cyclic branched cover of surface bundles if the restriction of p to a (surface) fiber is a branched cover of surfaces onto a fiber of E.

Equivalently $\tilde{E}$ is a cyclic branched cover over E s.t. ramification locus intersects each fiber of E in a 0-manifold.

(use: the restriction of a (branched) cover to a subman. of base is a branched cover.)

Prop. Let $\tilde{E} \xrightarrow{p} E$ be a 2-fold fiberwise cyclic branched covers over M with fiber genus $2g$ & g. Then

$$(1)\quad p^*[D]^* = 2[\tilde{D}]^* \qquad D = \text{ram. locus}.$$

$$(2)\quad e(\tilde{V}) = p^* e(V) - [\tilde{D}]$$

Note: (1) is just a fact about branched covers.

Pf of (1). ~~$p^*[D]^*$ computed~~ [illegible]. Clear when D is a 0-manifold. In general, replace fundamental class with Thom class of normal bundle.

Pf of (2). Clearly:

$$\begin{tikzcd} H^2(E) \arrow[r, "p^*"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & H^2(\tilde{E}) \arrow[d] \\ H^2(E \setminus \operatorname{Int} N(D)) \arrow[r] & H^2(\tilde{E} \setminus \operatorname{Int} N(\tilde{D})) \end{tikzcd}$$

$N(D)$ = tub. nbd.

(check on the level of bundles).

$\Rightarrow$ $e(V)$, $e(\tilde{V})$ have same image in lower right.

Consider LES of pair:

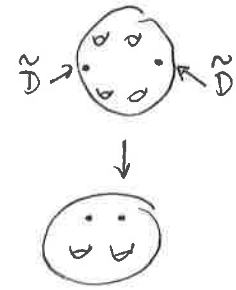
$$\cdots \longrightarrow H^2(\tilde{E}, \tilde{E} \setminus \operatorname{Int} N(\tilde{D})) \longrightarrow H^2(\tilde{E}) \longrightarrow H^2(\tilde{E} \setminus \operatorname{Int} N(\tilde{D})) \longrightarrow \cdots$$

Since $p^* e(V)$, $e(\tilde{V})$ have same image in

they differ by elt of

$$H^2(\tilde{E}, \tilde{E} \setminus \operatorname{Int} N(\tilde{D})) \cong H^2(N(\tilde{D}), \partial N(\tilde{D}))$$
$$\cong H_{n-2}(\tilde{D}) \cong \mathbb{Z}.$$

Remains to compute this integer. Evaluate $p^*(e(V)) + k[\tilde{D}]^*$ and $e(\tilde{V})$ on fiber S_{2g} of $\tilde{E}$:



$$e(\tilde{V})(S_{2g}) = 2 - 2(2g) = 2 - 4g.$$

since fibers map with degree 2. $\longrightarrow$ $p^*(e(V))(S_{2g}) = 2(2-2g) = 4 - 4g$

$k[\tilde{D}]^*(S_{2g}) = 2k$ $\leftarrow$ $\tilde{D}$ intersects each fiber in 2 pts

$$\rightsquigarrow \quad 2 - 4g = 4 - 4g + 2k$$
$$\Rightarrow \quad k = -1, \text{ as desired.}$$

Thm. $\tilde{E} \xrightarrow{p} E$ as above. Then:

$$e_1(\tilde{E}) = 2e_1(E) - 3i(\tilde{D}, \tilde{D})$$

Pf. By Prop (2):

$$e(\tilde{V}) = p^*(e(V)) - [\tilde{D}]^*$$

Squaring:

$$e(\tilde{V})^2 = p^*(e(V)^2) - 2p^*(e(V))[\tilde{D}]^* + [\tilde{D}]^{*2}$$

Use Prop(1) $\longrightarrow$

$$e_1(\tilde{E}) = 2e_1(E) - 2\left(e(\tilde{V})[\tilde{D}]^* + [\tilde{D}]^{*2}\right) + [\tilde{D}]^{*2}$$

$$= 2e_1(E) - i(\tilde{D}, \tilde{D}) - 2e(\tilde{V})[\tilde{D}]^*$$

Remains to show: $e(\tilde{V})[\tilde{D}]^* = i(\tilde{D}, \tilde{D})$.

But since $\tilde{V}$ is transverse to $\tilde{D}$ at all points, its restriction to $\tilde{D}$ is isomorphic to the normal bundle $N\tilde{D}$

$$\Rightarrow \quad e(\tilde{V})[\tilde{D}]^* = e(\tilde{V})(\tilde{D})$$

$$= e(N\tilde{D})(\tilde{D})$$

$$= i(\tilde{D}, \tilde{D}). \qquad \blacksquare$$

Atiyah's Construction

Will form a 2-fold branched cover over $S_{129} \times S_3$.

$\rightsquigarrow$ need a D with $[D]$ even.

Start with two covers:

$$\begin{tikzcd} S_{129} \arrow[d, "f"'] \\ S_3 \arrow[d, "h"'] \\ S_2 \end{tikzcd}$$

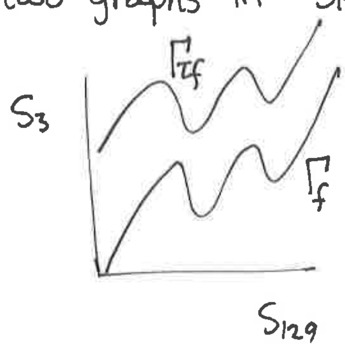
cover corr. to $\pi_1(S_3) \longrightarrow H_1(S_3; \mathbb{Z}_2)$

quotient by $\langle \tau \rangle$

key: $f^* = 0$ on $H^1(S_3; \mathbb{Z}_2)$

$h^* = 0$ on $H^1(S_2; \mathbb{Z}_2)$

D is union of two graphs in $S_{129} \times S_3$:



"key" will $\Rightarrow$ $[D]$ is even

Some features: ① $\Gamma_f \cap \Gamma_{\tau f} = \emptyset$ since τ has no fixed pts

② Vertical bundle V ($=$ pullback of TS_3 via proj to S_3) is transverse to D

③ Projection $D \longrightarrow S_3$ is a covering map (namely f).

④ Each S_3-fiber intersects D in two pts.

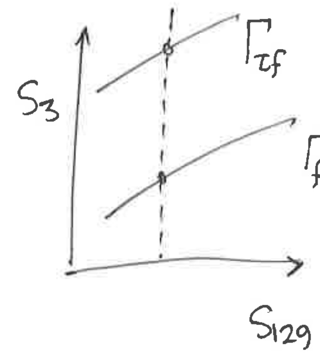
② $\Rightarrow$ $V|_D \cong ND$ normal bundle

③ $\Rightarrow$ $V|_D \cong TD$ tangent bundle.

① $\Rightarrow$ $i(D,D) = 2\,i(\Gamma_f, \Gamma_f)$

④ $\Rightarrow$ when we take the branched cover over D, fibers are S_6.

Claim ① $[D]$ is even.

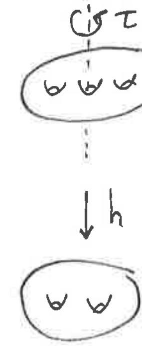


these graphs should be wavy— proj. to S_3 is the covering map!

Let $[D]^*$ be P. dual,

$[D]^*_2 \in H^2(S_{129} \times S_3 ; \mathbb{Z}_2)$ the mod 2 reduction

Need $[D]^*_2 = 0$.



$$\begin{array}{ccccc} D & & & & \Delta \\ \cap & & & & \cap \\ S_{129} \times S_3 & \xrightarrow{f \times id} & S_3 \times S_3 & \xrightarrow{h \times h} & S_2 \times S_2 \end{array}$$

$$[D]^*_2 = (f \times id)^* (h \times h)^* [\Delta]^*_2$$

But $H^2(S_2 \times S_2) \cong H^2(S_2 \times pt) \oplus (H^1(S_2) \otimes H^1(S_2)) \oplus H^2(pt \times S_2)$

and $(h \times h)^*$ kills H^2 factors since h has deg 2

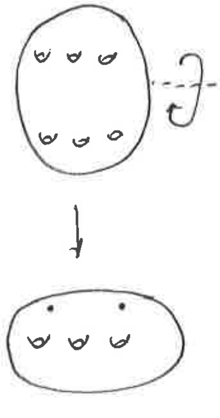
$(f \times id)^*$ kills middle factor since

$f_*(H_1(S_{129}; \mathbb{Z})) \subseteq 2 H_1(S_3; \mathbb{Z})$ by defn.

Thus $\exists$ 2-fold cyclic branched cover $E \to S_{129} \times S_3$ with ram. locus D.

E has the structure of a surface bundle over S_{129}

Fiber is S_6:



Thm. $e_1(E) = 768 \neq 0$.

Pf. By previous Thm:
$$\begin{aligned} e_1(E) &= 2e_1(S_{129} \times S_3) - 3\, i(\tilde{D}, \tilde{D}) \\ &= -3\, i(\tilde{D}, \tilde{D}) \\ &= -3/2\, i(D, D) \qquad \text{by Prop (1)} . \\ &= -3\, i(\Gamma_f, \Gamma_f) \end{aligned}$$

Recall from above that the normal bundle $N\Gamma_f$ is isomorphic to the tangent bundle $T\Gamma_f$ (both are $\cong$ to $V|_{\Gamma_f}$).

So:
$$i(\Gamma_f, \Gamma_f) = e(N\Gamma_f) = e(T\Gamma_f) = \chi(\Gamma_f) = \chi(S_{129}).$$

$\blacksquare$

Construction of S_g-Bundles with $e_i \neq 0$

$$\begin{tikzcd}
 & D \arrow[d, phantom, "\cap"] & & & & \Delta \arrow[d, phantom, "\cap"] & & \\
\tilde{E}^* \arrow[r] \arrow[d, "S_{g''}"] & E_3^* \arrow[r] \arrow[d, "S_{g'}"] & E_2^* \arrow[r] \arrow[d, "S_{g'}"] & (E_1^*)' \arrow[r] \arrow[d, "S_{g'}"] & E_1^* \arrow[r] \arrow[d, "S_{g*}"] & E^* \arrow[d, "S_g"] & E \arrow[d, "S_g"] \\
\tilde{\tilde{E}} \arrow[r] & \tilde{E} \arrow[r] \arrow[l, bend left] & E_2 \arrow[r] \arrow[l, bend left] & E_1 \arrow[r] \arrow[l, bend left] & E_1 \arrow[r] \arrow[l, bend left] & E \arrow[l, bend left] & M \arrow[l, bend left]
\end{tikzcd}$$

Induction: Start with $e_i(E) \neq 0$. $\dim(M) = 2i$

fiberwise cyclic branched cover over D

now can take cover of base so preimage of Δ is divis. by m

3

kill action of $\pi_1(\text{base})$ on $H^1(\text{fiber}; \mathbb{Z}_m)$ and ensure $H^1(E_1) \to H^1(E_2)$ is 0 mod m.

take the m-fold fiberwise cover

take cover over base so pullback bund. has m-fold fiberwise cover

2

diagonal construction: bump up dim by 2, new bundle has diagonal section Δ.

1

The steps labeled 1, 2, 3 are the new ones.

Morita calls this the m-construction on $E \to M$.
Atiyah's construction is 2-construction on $S_g \to pt$.

Prop $e_{i+1}(\tilde{E}^*) = -dm^2(1 - m^{-(i+2)})\, e_i(E)$ $\qquad$ d = degree of $\tilde{E} \to E$

The proof is analogous to that of: $e_1(\tilde{M}) = 2e_1(M) - i(\tilde{D}, \tilde{D})$ above.

So $e_i(E) \neq 0 \Rightarrow e_{i+1}(\tilde{E}^*) \neq 0$.

Higher Dimensional Surface Bundles

Goal. $e_i \neq 0 \ \forall i.$

Iterated surface bundles. $C_0 = \{*\}$

$C_{i+1} = \{$finite covers of S_g-bundles over elts of C_i, $g \geq 2\}$

e.g. $C_1 = \{S_g : g \geq 2\}$

Choose $E \in C_i$ surf. bundle with $e_i(E) \neq 0$.
Will use to construct $\tilde{E} \in C_{i+1}$ with $e_{i+1}(\tilde{E}) \neq 0$.

note: e_0 always $\neq 0$, which is why you can use the trivial bundle in Atiyah's construction.

Step 1. $C_i \to C_{i+1}$

Given S_g-bundle $\pi : E \to M$

$\rightsquigarrow E^* = \pi^*(E) = \{(u,u') \in E \times E : \pi(u) = \pi(u')\}$

Bundle structure: $\pi' : E^* \to E$

$(u,u') \mapsto u$

Have a bundle map:

$$\begin{tikzcd} E^* \arrow[r, "q"] \arrow[d, "\pi'"'] & E \arrow[d, "\pi"] \\ E \arrow[r, "\pi"] & M \end{tikzcd}$$

$q(u,u') = u'$

E^* comes with a section $\Delta = \{(u,u)\}$, which intersects each fiber in one point.

Write ν for $\Delta^* \in H^2(E^*; \mathbb{Z})$

$\nu_m \in H^2(E^*; \mathbb{Z}_m)$ the mod m reduction example. $E = S_g$, $M = *$.

$\rightsquigarrow E^* = S_g \times S_g$, Δ = usual diagonal.

Step 2. Given an S_g-bundle $E \to M$

$\exists$ finite cover $M_1 \xrightarrow{p} M$

s.t. $p^*(E)$ admits m-fold (unbranched) cover along fibers.

Note. Step 2 not needed in e_1 case since $S_g \times S_g \to S_g$ admits m-fold cover over fibers for any m.

Pf. Pick any m-fold $\tilde{S}_g \to S_g$

Denote $h: M \to MCG(S_g)$ the monodromy.

Goal: Construct a cover $\tilde{M} \to M$ and a monodromy

$\tilde{h}: \tilde{M} \to MCG(\tilde{S}_g)$ s.t.

$\tilde{h}(\alpha)$ is a lift of $h(\alpha)$ $\forall \alpha \in \pi_1(\tilde{M})$.

Then check: the combination of the two covering maps (of base and fiber) give a covering map of bundles.*

Need two facts about MCG: ① $\mathrm{Out}\,\pi_1(S_g) = MCG^{\pm}(S_g)$

② $MCG(S_g)$ has torsion free subgp of finite index, eg.

$\mathrm{Ker}\left(MCG(S_g) \longrightarrow Sp(2g, \mathbb{Z}_3)\right)$

* In general, monodromy of pullback f^* is given by composition of f_* (on π_1) with original monodromy.

Cover along fibers given by lifting monodromy to MCG of cover.

Choose $\tilde{\Gamma}_1 \leq \mathrm{Aut}\,\pi_1(S_g)$ finite index, preserves $\pi_1(\tilde{S}_g)$

$\rightsquigarrow$ $r: \tilde{\Gamma}_1 \longrightarrow \mathrm{Aut}\,\pi_1(\tilde{S}_g) \longrightarrow MCG(\tilde{S}_g)$

note: $r(\tilde{\Gamma}_1 \cap \mathrm{Inn}\,\pi_1(S_g))$ consists of torsion since any $\alpha \in \pi_1(S_g)$ has a power in $\pi_1(\tilde{S}_g)$, which then is an inner aut of $\pi_1(\tilde{S}_g)$.

$\Rightarrow \exists\ \tilde{\Gamma}_2 < \tilde{\Gamma}_1$ finite index s.t. $\tilde{\Gamma}_2 \cap \mathrm{Inn}\,\pi_1(S_g) = 1$.

(using ② above).

$\Rightarrow \Gamma_2 = \pi(\tilde{\Gamma}_2)$ finite index in $MCG(S_g)$

$\rightsquigarrow \Gamma_3 \lhd MCG(S_g)$ finite index (intersect all conjugates of Γ_2)

Γ_3 not needed unless we want a reg. cover.

and $\Gamma_3 \longrightarrow MCG(\tilde{S}_g)$ is well defined.

Let $\tilde{M} \to M$ be the cover given by

$$\pi_1(M) \longrightarrow MCG(S_g)/_{\Gamma_3}$$

Then $\tilde{h}: \tilde{M} \longrightarrow MCG(\tilde{S}_g)$ given by

$$\pi_1(\tilde{M}) \longrightarrow \Gamma_3 \longrightarrow MCG(\tilde{S}_g). \qquad \blacksquare$$

In other words, we showed: Given $\tilde{S}_g \to S_g$, $\exists$ finite index $\Gamma < MCG(S_g)$ and a $\Gamma \to MCG(\tilde{S}_g)$ where each $f \in \Gamma$ maps to a lift of f.

Then if the original bundle E has monodromy $\rho: \pi_1(M) \to MCG(S_g)$ the ~~monodromy~~ cover of M is the one corresponding to $\rho^{-1}(\Gamma)$ and the monodromy after taking the fiberwise cover is $\rho^{-1}(\Gamma) \hookrightarrow \pi_1(M) \longrightarrow \Gamma \longrightarrow MCG(\tilde{S}_g)$.

Step 3. $E \in C_n$, $\Delta \in H^2(E)$ all coeff $= \mathbb{Z}/m\mathbb{Z}$

Then $\exists$ finite cover $\tilde{E} \xrightarrow{p} E$ s.t. $p^*(\Delta) = 0$.

Induct on n.

Reduce to case $E = S_g$-bundle by taking pullbacks.

Apply Step 2, then take m-fold fiberwise cover.

Take another pullback to kill action on H^1(fiber)

and kill H^1(base)

$$
\begin{tikzcd}
 & & & \Delta \arrow[d, phantom, "\rotatebox{90}{$\in$}"] \\
E_2^* \arrow[r] \arrow[d, "\pi"', "S_{g'}"] & (E_1^*)' \arrow[r] \arrow[d, "S_{g'}"] & E_1^* \arrow[r] \arrow[d, "S_g"] & E \arrow[d, "S_g"] \\
E_2 & \tilde{E}_1 \arrow[l, phantom, ""] \arrow[r] & E_1 \arrow[r] & M
\end{tikzcd}
\qquad v \in E_2 \longrightarrow \tilde{E}_1
$$

Denote $E_2^* \to E$ by p_0^*

Claim: $\exists$ $v \in H^2(E_2)$ s.t. $p_0^*(\Delta) = \pi^*(v)$

Pf: Serre spectral seq (below)

By induction, $\exists$ finite cover $\tilde{E} \to E_2$ s.t. $v \mapsto 0$ in $H^2(\tilde{E})$:

$$
\begin{tikzcd}
E_3^* \arrow[r] \arrow[d] & E_2^* \arrow[d] \\
\tilde{E} \arrow[r] & E_2
\end{tikzcd}
$$

By commutativity, the result follows.

Serre Spectral Sequence

Want to prove claim. Write $F \to E \to B$ for $S_g \to E_2^* \to E_2$

Page 2 of Serre SS:

By construction, all $\mathbb{Z}/m$ coeffs are trivial.

$$\begin{array}{|lll}
H^0(B;H^2(F)) & H^1(B;H^2(F)) & H^2(B;H^2(F)) \\
H^0(B;H^1(F)) & H^1(B;H^1(F)) & H^2(B;H^1(F)) \\
H^0(B;H^0(F)) & H^1(B;H^0(F)) & H^2(B;H^0(F)) \\
\hline
\end{array}$$

The Serre SS package gives ~~three~~ two things

① There is a filtration $F_2 \subseteq F_1 \subseteq F_0 = H^2(E)$ s.t.

$$F_i/F_{i+1} \cong E_\infty^{i,2-i}$$

② The map

$$H^2(E) \to E_\infty^{0,2} \to E_2^{0,2} = H^2(F)$$

is the one induced by $F \hookrightarrow E$.

(the map $H^2(E) \to E_\infty^{0,2}$ comes from ①, the other map comes from the SS)

What are the F_i?

- $F_2/F_3 = F_2 = E_\infty^{2,0}$
- $F_1/E_\infty^{2,0} = E_\infty^{1,1}$
- $H^2(E)/F_1 = E_\infty^{0,2}$

Still need to determine F_1. Have:

$$1 \to F_1 \to H^2(E) \to E_\infty^{0,2} \to 1$$

The term $E_\infty^{0,2}$ is a subgp of $E_2^{0,2}$ (it is the kernel of the differential shown above). So by ②,

$$F_1 = K = \ker\left(H^2(E) \to H^2(F)\right)$$

In other words, we have two short exact seqs:

$$1 \to K \to H^2(E) \to E_\infty^{0,2} \to 1$$

$$1 \to E_\infty^{2,0} \to K \to E_\infty^{1,1} \to 1 \qquad \leftarrow \text{typo in Morita!}$$

Recall, we have $p_0^*(\Delta) \in H^2(E)$, we want to show it lives in $E_\infty^{2,0} = H^2(B)$.

Step 1. Image of $p_0^*(\Delta)$ in $E_\infty^{0,2}$ is 0, i.e. $p_0^*(\Delta) \in K$.

Recall we took an m-fold fiberwise cover

$$\begin{tikzcd} S_{g'} \arrow[r] \arrow[d] & S_g \arrow[d] & \Delta \\ E_2^* \arrow[r] & E^* & \end{tikzcd}$$

The map $H^2(S_g) \to H^2(S_{g'})$ is zero.

The map $H^2(E_2^*) \to E_\infty^{0,2}$ is the map $H^2(E_2^*) \to H^2(S_{g'})$

Use commutativity.

Step 2. Image of $p_0^*(\Delta)$ in $E_\infty^{1,1}$ is 0, i.e. $p_0^*(\Delta) \in E_\infty^{2,0} = H^2(B)$

Recall we arranged s.t. $H^1(E) \to H^1(E_2)$ is zero.

Algebraic Independence of the MMMs

Thm. Fix n. $\exists\, g$ s.t.

$$\mathbb{Q}[e_1, e_2, \ldots] \longrightarrow H^*(MCG(S_g^1); \mathbb{Q})$$

is injective up to degree $2n$ (in fact $g = 3n$).

i.e. $\mathbb{Q}[e_1, e_2, \ldots] \hookrightarrow H^*(MCG(S_\infty^1))$

Pf. Choose $g_1, \ldots, g_n$ s.t. $e_i \in MCG(S_{g_i}^1)$ is nonzero $i = 1, \ldots, n$.
(i.e. do our bundle construction for surfaces with boundary)
Choose d_j s.t. $j d_j \geq n$, set $g = \sum d_j g_j$

$$\rightsquigarrow \iota : MCG(S_{g_1}^1)^{d_1} \times \cdots \times MCG(S_{g_n}^1)^{d_n} \hookrightarrow MCG(S_g^1)$$

Fact: $\iota^*(e_i) = \sum_{j=1}^{n} p_j^*(e_i)$ $\quad p_j$ = proj to j^{th} factor

(the point is that the euler classes live in separate subbundles).

Now just apply the Künneth formula. The image of any polynomial of deg $\leq 2n$ will have one term in the direct sum of the form

$$e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_N} \otimes 1 \otimes \cdots \otimes 1$$

which is $\neq 0$ by construction.

Computing H_2.

- First show e_1 generates a $\mathbb{Z}$ in $H^2(MCG(S_g))$ $g \geq 3$.
- Then use Hopf formula, to show $H^2(MCG(S_g))$ is a quotient of $\mathbb{Z}$ for $g \geq 4$ and of $\mathbb{Z} \oplus \mathbb{Z}_2$ for $g = 3$.
- Remains to show $H^2(MCG(S_3)) = \mathbb{Z} \oplus \mathbb{Z}_2$.

There is: $1 \to \mathcal{I}(S_3) \to MCG(S_3) \to Sp_6(\mathbb{Z}) \to 1$

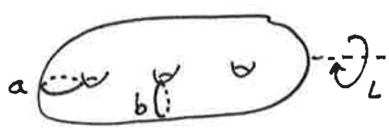
$\rightsquigarrow$ 5-term sequence:

$$H_2(MCG(S_3)) \to H_2(Sp_6(\mathbb{Z})) \to H_1(\mathcal{I}(S_3))_{Sp_6(\mathbb{Z})} \to H_1(MCG(S_3)) \to H_1(Sp_6(\mathbb{Z}))$$

But: $H_1(MCG(S_3)) = 0$.

$H_2(Sp_6(\mathbb{Z})) = \mathbb{Z} \oplus \mathbb{Z}_2$ Stein '75.

Remains: $H_1(\mathcal{I}(S_3))_{Sp_6(\mathbb{Z})} \cong \mathcal{I}(S_3)/[MCG(S_3), \mathcal{I}(S_3)] \cong 1$. Johnson '79

Pf. 

and choose $h \in MCG(S_3)$ s.t. $h(b) = a$.

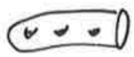
In $\mathcal{I}/[MCG, \mathcal{I}]$:

$$\begin{aligned}
[T_b, \iota] &= h[T_b, \iota]h^{-1} \quad \text{since } [T_b, \iota] \in \mathcal{I}(S_3) \\
&= [hT_bh^{-1}, h\iota h^{-1}] \\
&= [T_a, \iota[\iota^{-1}, h]] \\
&= [T_a, \iota]\, \iota [T_a, [\iota^{-1}, h]] \iota^{-1} \\
&= 1 \quad \text{since } T_a \leftrightarrow \iota \\
&\qquad \text{and } \iota \leftrightarrow h \text{ in } Sp. \\
&\qquad (\text{so } [\iota^{-1}, h] \in \mathcal{I}).
\end{aligned}$$

Benson-Cohen: $H_2(MCG(S_2))$ consists of 2,3,5-torsion only.

Madsen-Weiss Theorem

We know $\mathbb{Q}[e_1, e_2, \dots] \hookrightarrow H^*(MCG(S_\infty^1))$
Want to show this is surjective
Will do this by relating $H^*(MCG(S_\infty^1))$ to a "familiar" space.

$S_\infty =$  $\cdots$

$G_{S_g^1} =$ space of subsurfaces of $(0,g] \times \mathbb{R}^\infty$ diffeo to S_g^1 and that agree on ∂S_g^1 with a fixed embedding of S_∞.
$= K(MCG(S_g^1), 1)$

$G_{S_g^1} \hookrightarrow G_{S_{g+1}^1} \rightsquigarrow G_{S_\infty} = \bigcup G_{S_g^1}$

Harer stability $\Rightarrow H_i(G_{S_\infty}) = \lim_g H_i(G_{S_g^1}) = \lim_g H_i(MCG(S_g^1))$

$AG_{n,d}$ = affine Grassmannian of d-planes in $\mathbb{R}^n$
$\cong G_{n,d}^\perp$ since affine plane determined by plane thru 0 & $\perp$ vector

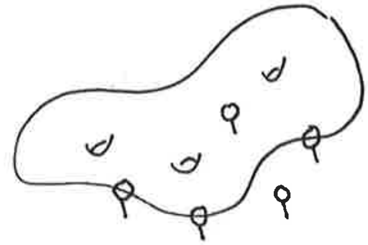
$AG_{n,d}^+$ = 1-pt comp
$\cong$ Thom space for $G_{n,d}^\perp$ when $n < \infty$.

Theorem. $H_*(G_{S_\infty}) \cong H_*(\Omega_0^\infty AG_{\infty,2}^+)$ basept @ ∞

In general, the $\mathbb{Q}$-cohomology of a loop space is a tensor product of a polynomial algebra on even-dim gens and an exterior alg. on odd-dim gens (assuming the loop space is path conn and has f.g. $\mathbb{Z}$-homology in each dim).

Scanning Map

Take some point in G_{n,S_g} :



With a small lens we either see an almost-flat 2-plane or $\emptyset$.
If we identify the lens with $\mathbb{R}^n$, get a pt in $AG^+_{n,2}$ (slope is same as in lens but position of plane given by lens $\to \mathbb{R}^n$).
Near ∞, lens sees $\emptyset$ $\rightsquigarrow$

$$S^n = \mathbb{R}^n \cup \{\infty\} \longrightarrow AG^+_{n,2}$$

i.e. a point in $\Omega^n AG^+_{n,2}$

As we move in G_{n,S_g} can vary the size of the lens continuously.
As we let n increase, have :

$$\begin{tikzcd} G_{n,S_g} \arrow[r, hook] \arrow[d] & G_{n+1,S_g} \arrow[d] \\ \Omega^n AG^+_{n,2} \arrow[r] & \Omega^{n+1} AG^+_{n+1,2} \end{tikzcd}$$

where bottom row obtained by applying Ω^n to the map

$$AG^+_{n,2} \longrightarrow \Omega AG^+_{n+1,2}$$

obtained by translating a plane from $-\infty$ to ∞ in $n+1^{st}$ coord.
Taking limit over n:

$$G_{S_g} \longrightarrow \Omega^\infty AG^+_{\infty,2} \qquad \text{"Scanning map"}$$

Note that the target does not depend on g, which is why we should expect to consider some limit over g in order to get an isomorphism.

A First Outline

Fix d (for us $d=2$)

C^n = space of all smooth, oriented d-dim submanifolds of $\mathbb{R}^n$ that are properly embedded (maybe disconn, open, empty).

Topology: pts are close if they are close in C^∞ top. on a large ball

Note C^n is path conn: radial expansion from a pt not on the manifold gives a path to the empty manifold.

Prop. $C^n \simeq AG^+_{n,d}$

Pf. Want to rescale from 0, but this is not continuous since we can push a manifold off 0, changing image from nonempty plane to empty plane.

Fix: For $M \in C^n$ choose tub. nbd $N = N(M)$ continuously.

If $0 \notin N$, rescale as above.

If $0 \in N$, rescale in tangent dir from $1 \longrightarrow \infty$ as before

in normal dir $1 \longrightarrow \lambda$ where

$\lambda = 1$ near 0-sec, $\lambda = \infty$ near frontier.

This takes $AG^+_{n,d}$ to itself. $\blacksquare$

Filter C^n by $C^{n,0} \subseteq C^{n,1} \subseteq \cdots \subseteq C^{n,n} = C^n$

where $C^{n,k}$ = subspace of C^n consisting of manifolds lying in $\mathbb{R}^k \times (0,1)^{n-k}$

i.e. manifolds that extend to ∞ in only k directions.

There is: $C^{n,k} \longrightarrow \Omega C^{n,k+1}$ by translating from $-\infty$ to ∞ in $(k+1)$st coord.

Putting these together:

$$C^{n,0} \longrightarrow \Omega C^{n,1} \longrightarrow \Omega^2 C^{n,2} \longrightarrow \cdots \longrightarrow \Omega^n C^n$$

The composition takes a compact manifold and translates it to ∞ in all directions. (can think of this as scanning with an ∞'ly large lens; shrinking the lens gives a homotopy to the original scanning map).

Would like: $C^{n,k} \longrightarrow \Omega C^{n,k+1}$ is a homotopy equivalence.

Easier: $k>0$ case. works for any $d \geq 0$.

Harder: $k=0$ case. when $d=2$, works after passing to limits where $n, g \to \infty$. uses group completion theorem. only get a homology equivalence:

$$H_*(C_\infty) \cong H_*(\Omega_0 C^{\infty,1}).$$

So the main thread for the MW Thm is:

$$\begin{aligned}
H_*(C_\infty) &\cong H_*(\Omega_0 C^{\infty,1}) && \text{harder delooping} \\
&\cong \lim H_*(\Omega_0 C^{n,1}) \\
&\cong \lim H_*(\Omega_0^n C^n) && \text{easier delooping} \\
&\cong \lim H_*(\Omega_0^n AG^+_{n,2}) && \text{above Prop.} \\
&\cong H_*(\Omega_0^\infty AG^+_{\infty,2})
\end{aligned}$$

DELOOPING – THE EASIER CASE

Want: $C^{n,k} \simeq \Omega C^{n,k+1}$ $\quad k > 0.$

Road map: $C^{n,k} \simeq M^{n,k} \simeq \Omega B M^{n,k} \simeq \Omega C_0^{n,k+1}$

Step 1. $M^{n,k} = \{ (M,a) \in C^n \times [0,\infty) : M \subseteq \mathbb{R}^k \times (0,a) \times (0,1)^{n-k-1} \}$

This is a monoid version of $C^{n,k}$, analogous to the Moore loopspace, which is a monoid version of ΩX.

The map
$$\begin{aligned} C^{n,k} &\longrightarrow M^{n,k} \\ M &\longmapsto (M,1) \end{aligned}$$
is a homotopy equivalence.

Step 2. $M^{n,k} \simeq \Omega B M^{n,k}$

A topological monoid M has a classifying space BM

Construction is analogous to group case: p-simplices $\longleftrightarrow (m_1,\ldots,m_p)$

faces obtained by dropping m_1, m_p

& multiplying $m_i m_{i+1}$

There is a space of p-simplices with topology from $\coprod_p \Delta^p \times M^p$ and face identifications.

There is a map
$$\begin{aligned} M &\longrightarrow \Omega BM \\ m &\longmapsto (m) \end{aligned}$$

General fact: This is a hom. eq. when $\pi_0 M$ is a group with mult. coming from mult. in M.

So we want: $\pi_0 M^{n,k}$ is a group.

Prop. $\pi_0 C^{n,k} = \begin{cases} 0 & k > d \\ \Omega^{SO}_{d-k,n-k} & k \leq d \end{cases}$

$\uparrow$ Cobordism group of closed, oriented $(d-k)$-manifolds in $\mathbb{R}^{n-k}$.

Pf. A point of $C^{n,k}$ is a d-mnfld $M \subseteq \mathbb{R}^n$ with $p: M \longrightarrow \mathbb{R}^k$ proper.

Can perturb M s.t. p is transverse to $0 \in \mathbb{R}^k$.

$k > d$: $p(M)$ misses 0. Expand radially from 0 in $\mathbb{R}^k$ to get path to empty manifold.

$k \leq d$: $p^{-1}(0) = M \cap (\{0\} \times \mathbb{R}^{n-k}) = M_0 \rightsquigarrow [M_0] \in \Omega^{SO}_{d-k,n-k}$

$\rightsquigarrow \varphi: \pi_0 C^{n,k} \longrightarrow \Omega^{SO}_{d-k,n-k}$

$[M] \longmapsto [M_0]$

This is a homom since both group ops are disj. union.

and surjective since $[\mathbb{R}^k \times M_0] \longmapsto [M_0]$

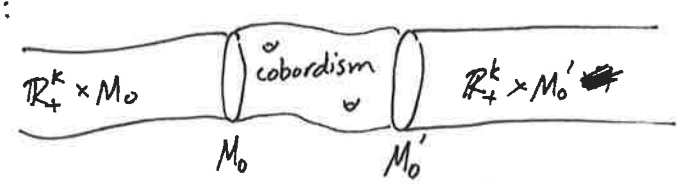
Remains: φ injective.

First we claim any M is path conn to $\mathbb{R}^k \times M_0$ (first make M agree with $\mathbb{R}^k \times M_0$ on a nbd of M_0, then expand radially)

Now if $\varphi([M]) = [M_0]$ equals $\varphi([M']) = [M_0']$

can assume $M = \mathbb{R}^k \times M_0$, $M' = \mathbb{R}^k \times M_0'$ and $M_0 \sim M_0'$

Build a manifold:



Translating right gives path to $\mathbb{R}^k \times M_0$,

and left gives path to $\mathbb{R}^k \times M_0'$ so $[M] = [M']$ in $\pi_0 C^{n,k}$. $\blacksquare$

STEP 3. $BM^{n,k} \simeq C_0^{n,k+1}$

We will define a natural map $\nabla : BM^{n,k} \longrightarrow C_0^{n,k+1}$

A point in $BM^{n,k}$ is given by $(m_1,\dots,m_p) \in (M^{n,k})^p$, $(w_0,\dots,w_p)$
A stupid map (ignoring the w_i) is:

$(m_1,\dots,m_p) \longmapsto m_1 m_2 \cdots m_p = \bigcup M_i$ where M_i is a manifold with $(k+1)^{st}$ coord in $[a_{i-1}, a_i]$

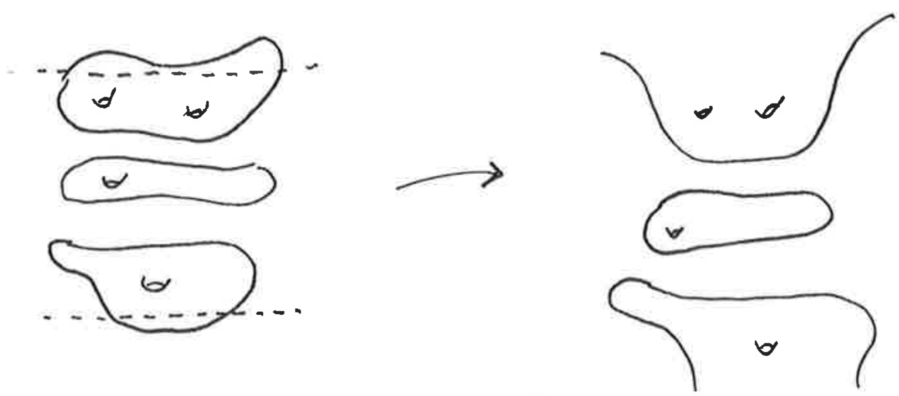
This map is not continuous upon passage to faces:

① When w_0 or $w_p \to 0$, M_1 or M_p suddenly deleted.

② When $w_0 \longrightarrow 0$ $m_2 \cdots m_p$ suddenly shifts by $a_1 - a_0$ in $(k+1)^{st}$ coord

Can easily address ②: translate in $(k+1)^{st}$ coord so barycenter $b = \sum w_i a_i$ equals 0.

Idea for ①: truncate M_1, M_p a little at a time.



precisely: $a_i^+ = \max\{a_i, b\}$ $b^+ = \sum w_i a_i^+$ "upper & lower
$a_i^- = \min\{a_i, b\}$ $b^- = \sum w_i a_i^-$ barycenters"

$\nabla(m_1 \cdots m_p)$ obtained by stretching $\mathbb{R}^k \times (b^-, b^+) \times \mathbb{R}^{n-k-1}$ to $\mathbb{R}^k \times \mathbb{R} \times \mathbb{R}^{n-k-1}$

Need to check ∇ is $\cong$ on π_q $\forall$ q.