CHARACTERISTIC CLASSES IN SEVEN PAGES

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Suppose you want to tell the difference between two vector bundles. If the two bundles are an annulus and a Möbius band, you probably don't need any help. But what if the two bundles are the tangent bundle of a sphere and the trivial bundle? Or the tangent bundles for two different smooth structures on the same topological manifold? The theory of characteristic classes is one very helpful tool for distinguishing between such bundles.

The goal of this note is to serve as a concise and accessible overview of the theory of characteristic classes for topologists. Hopefully it will inspire the reader to undertake a more in-depth study of this beautiful theory. Some references are provided at the end for this purpose.

1. Characteristic Classes

A vector bundle over a base space B is a space E together with a map $p : E \to B$ so that each fiber, that is, each $p^{-1}(b)$, has the structure of a fixed vector space V, and so that B is covered by open sets U where $p^{-1}(U)$ is homeomorphic to $U \times V$ and the homeomorphism respects the vector space structure on each fiber.

A major reason why vector bundles are so useful is that every smooth manifold has a canonically associated vector bundle, the tangent bundle. Also, any manifold embedded in another manifold has the associated normal bundle. Therefore, we can hope to distinguish between different manifolds and different embeddings of manifolds by studying the invariants of their associated vector bundles.

A characteristic class for vector bundles is a function

$$\chi : \{\text{vector bundles } V \to E \to B\} \to H^k(B; G),$$

for some fixed choices of V, k, and G, that is natural under pullbacks in the sense that

$$\chi(f^\star(E)) = f^\star(\chi(E))$$

for any pullback bundle $f^\star(E)$. We emphasize that the base B is allowed to vary, that is, χ is a function defined on all vector bundles over all spaces, as long as the fibers are isomorphic to V. We also allow ourselves to restrict to special types of bundles, for instance oriented bundles.

We immediately ask: do characteristic classes exist? If so, can we list them all? To what extent do they help us distinguish between different bundles? We will address all of these questions.

In the case where $V = \mathbb{R}^n$, $k = n$ and $G = \mathbb{Z}$, it turns out that there is a characteristic class defined on oriented bundles called the Euler class e. If B is a manifold and E is the tangent bundle TB, the Euler class $e(E) \in H^n(B;\mathbb{Z}) \cong \mathbb{Z}$ is equal to the Euler characteristic $\chi(B)$. In other words, the familiar Euler characteristic is a special case of a characteristic class, the grandfather of them all.

One reason why the Euler characteristic is interesting is that it has varied definitions, including:

(1) Combinatorial: $\chi(M) = \sum (-1)^i$(number of i-cells)
(2) Geometric: $\chi(M) = (1/\mathrm{vol}\, S^n) \int_M \kappa(x)\, \mathrm{dvol}_M$
(3) Homological: $\chi(M) = \sum \mathrm{rank}\, H_i(M;\mathbb{Z})$
(4) Cohomological: $\chi(M) =$ self-intersection of M in TM

The last description has an important consequence: if M has a nonvanishing vector field, then $\chi(M) = 0$. In this sense, we can think of $\chi(M)$ as an obstruction to having a nonvanishing vector field. This point of view will be echoed by the other characteristic classes.

By the work of Stiefel, Whitney, Pontryagin, and Chern we have a complete list of all characteristic classes in the following cases.

vector bundle	coefficients	characteristic classes
real	$\mathbb{Z}_2$	Stiefel–Whitney
complex	$\mathbb{Z}$	Chern
real	$\mathbb{Z}$	Pontryagin, Stiefel–Whitney
oriented real	$\mathbb{Z}$	Pontryagin, Stiefel–Whitney, Euler

Below we will give more detailed descriptions of each of these kinds of characteristic classes. In a word, all of the characteristic classes tell you something about how twisted the vector bundle is. For instance the first Steifel–Whitney class is zero if and only if the bundle is orientable.

2. Grassmann manifolds and Characteristic Classes

The Grassmann manifold G_n is the space of all n-dimensional linear subspaces of $\mathbb{R}^\infty$ (note $G_1 = \mathbb{RP}^\infty$). It has a canonical $\mathbb{R}^n$-bundle $E_n \to G_n$. This is the subspace of $G_n \times \mathbb{R}^\infty$ consisting of all pairs

$$(n\text{-plane in } \mathbb{R}^\infty, \text{vector in that plane}).$$

For a paracompact base space B, there is a bijective correspondence:

$$\{\mathbb{R}^n\text{-bundles over } B\}/\text{isomorphism} \longleftrightarrow \{\text{maps } B \to G_n\}/\text{homotopy}.$$

The map in one direction is easy: given a map $B \to G_n$, just take the pullback of the canonical bundle E_n. Amazingly, all bundles come from this construction. This is similar to the familiar fact about group homomorphisms and maps to $K(G,1)$ spaces.

If χ is a characteristic class for $\mathbb{R}^n$-bundles, then we can evaluate it on the universal bundle $E_n \to G_n$ to obtain $\chi(E_n) \in H^k(G_n;G)$. Conversely, given

$\chi \in H^k(G_n; G)$, we can use the naturality property and the correspondence between bundles and pullbacks of E_n in order to define $\chi(E) = f^\star(\chi(E_n))$, where $E \cong f^\star(E_n)$ for $f : B \to G_n$. Thus the characteristic classes of $\mathbb{R}^n$-bundles are in bijective correspondence with cohomology classes of G_n.

We can now reinterpret one of our statements from the previous section. When we said that we know all characteristic classes for, say, real vector bundles with $\mathbb{Z}_2$-coefficients, what this means is that we have a complete description of $H^\star(G_n; \mathbb{Z}_2)$. We will describe this ring precisely as a polynomial ring in the next section.

Why are characteristic classes the right thing to look at? Well, we have no better chance of understanding maps $B \to G_n$ than we do of understanding $\mathbb{R}^n$-bundles over B. However, we do have a chance of understanding the induced map on cohomology $H^\star(G_n) \to H^\star(B)$, and that is exactly what the characteristic classes capture.

The downside, of course, is that cohomological data is in general strictly weaker than topological data. That holds true here: we can readily find examples of nonisomorphic bundles with the same characteristic classes. For instance the characteristic classes of the tangent bundle of S^5 are all trivial but the bundle is not. However, we will see that the characteristic classes can give lots of interesting information about vector bundles.

There are similar stories for characteristic classes of complex vector bundles and characteristic classes of oriented real vector bundles. For those cases we need to replace $G_n = G_n(\mathbb{R})$ with $G_n(\mathbb{C})$, the space of complex n-dimensional linear subspaces of $\mathbb{C}^n$, or $\tilde{G}_n$, the space of oriented n-dimensional linear subspaces of $\mathbb{R}^\infty$.

3. Stiefel–Whitney Classes

The first kind of characteristic class we should try to understand is the case of real vector bundles and coefficient group $G = \mathbb{Z}_2$. We will see that even with such a simple coefficient group, we will be able to obtain a lot of information about our bundles.

As per the previous section, we can list all characteristic classes of real vector bundles with $\mathbb{Z}_2$ coefficients if we can compute $H^\star(G_n; \mathbb{Z}_2)$. It turns out this ring is isomorphic to a polynomial algebra in n variables:

$$H^\star(G_n; \mathbb{Z}_2) \cong \mathbb{Z}_2[w_1, \ldots, w_n].$$

This computation makes use of a specific, explicit cell structure on G_n; the cells are called Schubert cells.

Each w_i has degree i in the cohomology of G_n, and the w_i are in natural bijection with the symmetric polynomials in the n generators for $H^\star((\mathbb{R}\mathrm{P}^\infty)^n; \mathbb{Z}_2)$. The characteristic class w_i is called the ith Stiefel–Whitney class.

The first Stiefel–Whitney class has a concrete interpretation: we can identify the class w_1 with the homomorphism $H_1(B; \mathbb{Z}_2) \to \mathbb{Z}_2$ that records whether the pullback bundle is trivial over a given element of $H_1(B; \mathbb{Z}_2)$. In particular, w_1 is equal to 0 if and only if the bundle is orientable. So this is precisely the class that distinguishes between the annulus and the Möbius band.

It turns out that the higher Stiefel–Whitney classes have similar interpretations: w_i measures the obstruction to finding $n-k+1$ independent sections over the i-skeleton of B. (For a 4-manifold M, it follows that $w_2(TM)$ is an obstruction to finding a spin structure on M.)

There is also an axiomatic description of the Stiefel–Whitney classes. The w_i form the unique sequence of functions $w_1, w_2, \dots$ assigning to each real vector bundle $E \to B$ a class $w_i(E) \in H^i(B; \mathbb{Z}_2)$, depending only on the isomorphism type of E, such that:

(1) $w_i(f^\star(E)) = f^\star(w_i(E))$ for a pullback $f^\star(E)$
(2) $w(E_1 \oplus E_2) = w(E_1) \smile w(E_2)$, where $w = 1 + w_1 + \cdots \in H^\star(B : \mathbb{Z}_2)$
(3) $w_i(E) = 0$ if $i > \dim E$
(4) For the canonical line bundle $E \to \mathbb{R}\mathrm{P}^\infty$, $w_1(E)$ is a generator of $H^1(\mathbb{R}\mathrm{P}^\infty; \mathbb{Z}_2)$.

One more thing before we end: given an $\mathbb{R}^n$ bundle $E \to B$, the total Stiefel–Whitney class is $1 + w_1(E) + \cdots + w_n(E)$. This gadget organizes all of the Stiefel–Whitney classes into a single object.

4. Applications and Computations

Whitney duality says that if M is a submanifold of Euclidean space, then

$$w_i(NM) = \bar{w}_i(TM),$$

where NM denotes the normal bundle and $\bar{w}_i$ denotes the multiplicative inverse of w_i

We can use Whitney duality to compute the total Stiefel–Whitney class w of several bundles. Below, γ_1^n is the canonical line bundle over $\mathbb{R}\mathrm{P}^n$ and $\gamma_n^\perp$ is orthogonal complement of γ_1^n in the $(n+1)$-dimensional trivial bundle.

(1) $w(S^n) = 1$
(2) $w(\gamma_n^1) = 1 + a$
(3) $w(\gamma_n^\perp) = 1 + a + \cdots + a^n$
(4) $w(\mathbb{R}\mathrm{P}^n) = (1 + a)^{n+1}$

In particular, Stiefel–Whitney classes do not distinguish nontrivial bundles from trivial bundles, and there are bundles with all Stiefel–Whitney classes nonzero. Also, the last computation shows that if $\mathbb{R}\mathrm{P}^n$ is parallelizable, then $n + 1 = 2^k$ (it turns out to be parallelizable exactly when $n = 1, 3, 7$).

Some further applications: (1) If $\mathbb{R}\mathrm{P}^{2^r}$ immerses in $\mathbb{R}^{2^r+k}$, then $k \geq 2^r - 1$, and (2) any n-dimensional division algebra must have $n = 2^k$.

Wu's formula gives one description of the Stiefel–Whitney classes in terms of the Steenrod squares of a particular cohomology class, the total Wu class. In the case of a simply-connected, orientable 4-manifold M, Wu's formula has the pretty interpretation that $\langle w_2(TM), [S] \rangle$ is equal to the self-intersection of S modulo two.

Say we have a manifold M. If we take a collection of non-negative integers r_i with $\sum ir_i = n$, then we get an element of $H^n(M;\mathbb{Z}/2)$, namely $\sum w_i(TM)^{r_i}$. We can pair this with the fundamental class of the manifold M to get a number associated to the monomial $\sum w_i(TM)^{r_i}$. These are the Stiefel–Whitney numbers of M. We have the following remarkable theorem due to Pontryagin and Thom: two smooth closed n-manifolds belong to the same cobordism class if and only if their corresponding Stiefel–Whitney numbers are equal.

The last theorem is a gateway to another of Thom's landmark theorems. The set of cobordism classes of smooth, oriented manifolds forms a ring $\Omega_\star^{SO}$ where addition is disjoint union and multiplication is given by the direct product. Thom gives an amazingly simple description of the tensor product of this ring with the rationals:

$$\Omega_\star^{SO} \otimes \mathbb{Q} \cong \mathbb{Q}[\mathbb{C}\mathrm{P}^1, \mathbb{C}\mathrm{P}^2, \dots].$$

5. Chern classes

The integral cohomology for the complex Grassmannian looks a lot like the $\mathbb{Z}_2$-cohomology for the real Grassmannian:

$$H^\star(G_n(\mathbb{C});\mathbb{Z}) \cong \mathbb{Z}_2[c_1, \dots, c_n].$$

The c_i are called Chern classes. They live in $H^{2i}(G_n(\mathbb{C});\mathbb{Z})$, and they correspond exactly to the elements of $H^\star(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$. Like Stiefel–Whitney classes, the Chern classes can be described axiomatically. The axioms are the same, with w_i replaced by c_i, $\mathbb{Z}_2$ replaced by $\mathbb{Z}$, $\mathbb{R}\mathrm{P}^\infty$ replaced by $\mathbb{C}\mathrm{P}^\infty$, and $H^1(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}_2)$ replaced by $H^2(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z})$. The proof of existence and uniqueness of Chern classes is almost exactly the same as for the Stiefel–Whitney classes.

So far, there is one relationship between the characteristic classes worth noting. Given any complex vector bundle E, we can regard it as a real vector bundle and as such it has the associated Stiefel–Whitney classes. Under the map $H^\star(B;\mathbb{Z}) \to H^\star(B;\mathbb{Z}_2)$, we have

$$c(E) \mapsto w(E).$$

Therefore, we can think of c as a refinement of w in the complex case.

A sample computation for Chern classes is

$$c(T\mathbb{C}\mathrm{P}^n) = (1+a)^{n+1}$$

where a is a generator for $H^2(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$.

6. The Euler class

The Euler class for oriented vector bundles lies in $H^n(\tilde{G}_n)$. While it is not the only characteristic class for oriented vector bundles (see the next section), it is interesting (and special) enough to deserve separate attention.

The image of the class e in $\mathrm{Hom}(H_n(B;\mathbb{Z}),\mathbb{Z})$ can be described as follows: given an n-chain x in B, we place it in general position with the 0-section

B; the intersection will be a finite collection of points with sign and the sum is $e(x)$.

The Euler class further satisfies the following properties:

(1) $e(\bar{E}) = -e(E)$, where $\bar{E}$ is E with the opposite orientation
(2) $e(E_1 \oplus E_2) = e(E_1) \smile e(E_2)$
(3) $e(E) = -e(E)$ if the fibers are odd-dimensional (i.e. $e(E)$ has order two)
(4) $e(E) = 0$ if E has a nonvanishing section
(5) $\langle e(TM), [M] \rangle = \chi(M)$, where $[M]$ is the fundamental class.

In the course of proving the last property, another beautiful fact is established: both quantities are equal to the self-intersection number of M in $M \times M$.

Amazingly, vector bundles with two-dimensional fibers are completely characterized by their Euler classes. In the case where the base is a surface, e lies in $H^2(B; \mathbb{Z}) \cong \mathbb{Z}$, and it is not hard to realize each element of $\mathbb{Z}$ as an Euler class—so this is a very tidy classification!

The Euler class is related to two other classes we have seen so far. If we have a complex vector bundle E, then we can also think of E as a real vector bundle, and we have

$$e(E) = c(E).$$

This is consistent with the fact that

$$e(E) \mapsto w_n(E)$$

under the map $H^n(B; \mathbb{Z}) \to H^n(B; \mathbb{Z}_2)$. In other words, the Euler class is obtained from the Chern class by ignoring the complex structure, and the top Stiefel–Whitney class is obtained from the Euler class by ignoring orientation.

One more word about why the Euler class is special. The Stiefel–Whitney classes and Chern classes are stable in the sense that $w_i(E \oplus E') = w_i(E)$ when E' is trivial (similar for c_i). However, $e(E \oplus E')$ is always equal to zero, since $E \oplus E'$ has an obvious nonvanishing section, so unlike the other classes, the Euler class is unstable.

7. Pontryagin classes

We would be remiss not to discuss the last player in the story, the Pontryagin classes, as they describe the integral homology of oriented and non-oriented vector bundles.

The Pontryagin class $p_i(E) \in H^{4i}(B; \mathbb{Z})$ can be defined as $(-1)^i c_{2i}(E^{\mathbb{C}})$, where $E^{\mathbb{C}}$ is the complexification of E (if we think of a real vector space V as the formal $\mathbb{R}$-span of some vectors, then $V^{\mathbb{C}}$ is the formal $\mathbb{C}$-span).

The integral cohomology of the Grassmannian G_n is described as follows: the torsion in $H^\star(G_n; \mathbb{Z})$ consists of elements of order two, and

$$H^\star(G_n; \mathbb{Z})/\text{torsion} \cong \mathbb{Z}\left[p_1, \ldots, p_{\lfloor n/2 \rfloor}\right].$$

Moreover, the generators of the torsion subgroup correspond to the images of the Stiefel–Whitney classes under a certain Bockstein homomorphism. Similarly,

$$H^\star(\tilde{G}_n;\mathbb{Z})/\text{torsion} \cong \mathbb{Z}\left[\tilde{p}_1,\ldots,\tilde{p}_{\lfloor n/2\rfloor}\right]/\!\sim$$

where $\tilde{p}_i$ is the evaluation of p_i on the canonical oriented bundle over $\tilde{G}_n$, where $\sim$ represents the relation $e = 0$ if n is odd and $e^2 = p_{n/2}$ if n is even. Again the torsion is generated by (images of) the Stiefel–Whitney classes. The proof of this theorem uses two important tools, namely the Thom isomorphism and the Gysin sequence.

We will mention a few applications of the Pontryagin classes. First, we can define Pontryiagin numbers in the same way we defined Stiefel–Whitney numbers. It is not hard to show that a manifold admitting an orientation-preserving diffeomorphism must have all of its Pontryagin numbers equal to zero (such a diffeomorphism preserves the Pontryagin classes and reverses the fundamental class). In particular, $\mathbb{C}\mathrm{P}^n$ does not admit an orientation-reversing diffeomorphism when n is even, since

$$p(\mathbb{C}\mathrm{P}^n) = \sum_{k}^{\lfloor n/2\rfloor} \binom{n+1}{k} a^{2k}.$$

Also, Wall proved the following analogue of Thom's theorem mentioned earlier: a smooth, compact, oriented manifold is an oriented boundary if and only if all of its Pontryiagin and Stiefel–Whitney numbers are zero.

For a smooth 4-manifold M, the first Pontryagin class has a particularly useful interpretation in terms of the signature $\sigma(M)$. Specicially, the first case of the Hirzebruch signature theorem says that

$$p_1(M) = 3\sigma(M).$$

The Pontryagin classes are certainly the most mysterious of the characteristic classes discussed in this overview, and it would be most useful to have geometric interpretations of the other Pontryagin classes.

We will end our discussion by mentioning one of the most spectacular applications of characteristic classes. A basic question in topology is: if two manifolds are homeomorphic, then must they be diffeomorphic? Milnor showed that the answer is no: there are exotic 7-spheres, that is, manifolds that are homeomorphic to S^7 but not diffeomorphic to S^7. This is hard to believe! But Milnor gives an explicit example a topological 7-sphere M and proves that it is exotic by studying the Pontryagin numbers of the closed 8-manifolds obtained by filling in M on both sides. Novikov showed that homeomorphic manifolds have the same Pontryagin classes, thus giving another method for detecting non-diffeomorphic manifolds. For these works, Milnor and Novikov were both awarded the Fields Medal.

8. Further reading

The theory of characteristic classes was developed in the first half of the 20th century and has been continuously active since that time. As a result, there are now a number of excellent references, of which we will mention just a few. The standard text is Characteristic Classes, written by Milnor and Stasheff. Hatcher has an (unfinished) textbook, Vector Bundles and K-theory, available for free on his web site. The obstruction theory point of view is detailed in Steenrod's book, The Topology of Fibre Bundles. Differential Forms in Algebraic Topology by Bott and Tu develops the theory from the point of view of de Rham cohomology. We also recommend Morita's book, Geometry of Characteristic Classes, which treats the theory of vector bundles and then segues into the analogous theory for surface bundles. Finally, many of the original sources in this subject are very accessible, for instance Milnor's construction of exotic spheres in "On manifolds homeomorphic to the 7-sphere" and Hirzebruch's Topological Methods in Algebraic Geometry, and so we encourage the reader to seek out these classics.