Math 8803: Characteristic Classes of Vector Bundles and Surface Bundles

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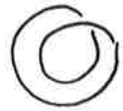
Theory of Characteristic classes:

$$\boxed{\text{Bundles over } B} \longrightarrow \boxed{H^*(B)}$$

so as to distinguish bundles, e.g.

 vs 

This course: Vector bundles, surface bundles.

VECTOR BUNDLES

$$\begin{array}{c} E \\ p\big\downarrow \\ B \end{array}$$

B = base

$p^{-1}(B)$ = fiber $\leftarrow$ struct. of vector space V.

B covered by U s.t.

$p^{-1}(U) \longrightarrow U \times V$ homeo respecting v.s. structure of fibers

Important because smooth manifolds have tangent bundles, submanifolds have normal bundles.

e.g. can distinguish two smooth structures on a manifold if we can distinguish their tangent bundles using characteristic classes.

<u>Thm</u> (Milnor) $\exists$ exotic 7-spheres.

Characteristic Classes

A char. class for vect. bundles is a function

$$\chi : \{ V\text{-bundles over } B \} \longrightarrow H^k(B; G)$$

for fixed V, k, G (B allowed to vary!)
that is natural:

$$\chi(f^*(E)) = f^* \chi(E)$$

Euler Class

Take $V = \mathbb{R}^n$, $k = n$, $G = \mathbb{Z}$, restrict to oriented bundles.

$\rightsquigarrow$ *Euler class* e.

$$B = M \quad E = TM \rightsquigarrow \underset{\displaystyle \chi(M)}{\overset{\shortparallel}{e(TM)}} \in H^n(M; \mathbb{Z}) \cong \mathbb{Z}$$

Euler char is a char. class. It has many interpretations, e.g:

(1) Combinatorial: $\chi(M) = \Sigma (-1)^i (\#\ i\text{-cells})$
(2) Geometric: $\chi(M) = \frac{1}{\mathrm{vol}\, S^n} \int_M K(x)\, d\mathrm{vol}_M$
(3) Homological: $\chi(M) = \Sigma (-1)^i \operatorname{rank} H_i(M; \mathbb{Z})$
(4) Cohomological $\chi(M) =$ self-intersection of M in TM.

(4) implies $\chi(M)$ is obstruction to nonvanishing vector field (recall Thurston's proof).

Grassmann Manifolds

Euler class is so beautiful, we want to find all other char classes.

G_n = space of n-planes in $\mathbb{R}^\infty$.
E_n = canonical bundle over G_n:

$$(n\text{-plane in } \mathbb{R}^\infty,\ \text{vector in that plane}) \subseteq G_n \times \mathbb{R}^\infty.$$

We will show:

$$\left\{\begin{matrix}\mathbb{R}^n\text{-bundles} \\ \text{over } B\end{matrix}\right\} \Big/ \text{isomorp.} \longleftrightarrow \left\{\begin{matrix}\text{maps} \\ B \to G_n\end{matrix}\right\} \Big/ \text{homotopy}.$$

$$f^*(E_n) \longleftarrow\!\shortmid\ f$$

This gives:

$$\left\{\begin{matrix}\text{char. classes for } \mathbb{R}^n\text{-bundles} \\ G\text{-coeff}\end{matrix}\right\} \longleftrightarrow H^*(G_n; G).$$

Goal: compute the latter.

If we care about:

complex bundles $\rightsquigarrow G_n(\mathbb{C})$
oriented real bundles $\rightsquigarrow \tilde{G}_n$

Stiefel-Whitney Classes

We will show: $H^*(G_n;\mathbb{Z}_2) \cong \mathbb{Z}_2[w_1,\dots,w_n]$

w_i called i^{th} SW class.

w_1 is very concrete $\in H^1(B;\mathbb{Z}_2) \cong \mathrm{Hom}(H_1(B;\mathbb{Z}_2);\mathbb{Z})$
It records whether the bundle is orientable over an element of H_1.

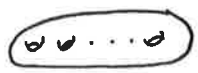
w_i = obstruction to finding $n-k+1$ indep. sections over the i-skeleton of B.

<u>Thm</u> (Thom). Two manifolds are cobordant iff the SW numbers of their tangent bundles are equal.

Other Characteristic Classes

vector bundle	coeff.	characteristic classes
real	$\mathbb{Z}_2$	SW
complex	$\mathbb{Z}$	Chern
real	$\mathbb{Z}$	Pontryagin, SW
oriented real	$\mathbb{Z}$	Pont., SW, Euler.

Surface Bundles

$S_g =$ 

S_g-bundle $\begin{array}{c} E \\ p \downarrow \\ B \end{array}$ $\qquad p^{-1}(U) \cong U \times S_g$

Important class of manifolds (also, they are the next-simplest bundles).

Characteristic class

$$\chi : \left\{ \begin{array}{c} \text{oriented} \\ S_g\text{-bundles} \\ \text{over } B \end{array} \right\} / \text{isom.} \longrightarrow H^k(B; G)$$

naturality $\chi(f^*(E)) = f^*(\chi(E))$

Classifying space

$$\left\{ \begin{array}{c} \text{oriented} \\ S_g\text{-bundles} \\ \text{over } B \end{array} \right\} / \text{isom.} \longleftrightarrow \left\{ \begin{array}{c} \text{maps} \\ B \to B\mathrm{Homeo}^+(S_g) \end{array} \right\} / \text{hom.}$$

$B\mathrm{Homeo}^+(S_g) =$ space of S_g-submanifolds of $\mathbb{R}^\infty$
$= K(MCG(S_g), 1)$

So: Char. classes for orient. S_g-bundles $\Longleftrightarrow H^*(MCG(S_g); G)$.

We do <u>not</u> have a full list, but

$e_i \in H^{2i}(MCG(S_g); \mathbb{Z})$ — Morita-Mumford-Miller classes

generate $H^*(MCG(S_g))$ <u>stably</u> (Madsen-Weiss).

Morita's Theorem

$\pi : \mathrm{Diff}^+(S_g) \longrightarrow MCG(S_g)$ has no section $g \gg 0$.

Proof: $e_3 \neq 0$, $\pi^*(e_3) = 0$.

Odd MMM classes are geometric.

$e_1 \in H^2(B; \mathbb{Z})$ WLOG: B = surface.

$\Rightarrow E$ = 4-manifold M

Hirzebruch: $e_1(E) = \sigma(M)$ signature.

But σ (hence e_1) ignores bundle structure even though e_1 defined via bunde structure.

Say e_1 is geometric.

<u>Thm</u> (Church-Farb-Thibault) e_{2i+1} is geometric.

e.g. $\exists$ S_4-bundle over $S_{17} \cong S_{49}$ bundle over S_2.

Pf that e_1 is geometric: $e_1(E) = p_1(M) \leftarrow$ 1st Pontryagin class.

$= \sigma(M)$ (Hirzebruch).