A Classifying Space for Surface Bundles

Goal: $\{\Sigma_g\text{-bundles over } B\}/\cong \;\longleftrightarrow\; [B, K(MCG(\Sigma_g),1)]$ $\quad$ B = conn CW

$\qquad\qquad\qquad\qquad\qquad\longleftrightarrow\; \mathrm{Hom}(\pi_1(B), MCG(\Sigma_g))/MCG(\Sigma_g)$

$\Rightarrow$ Ring of char. classes for Σ_g-bundles $\cong H^*(MCG(\Sigma_g))$

We first construct a direct analogue of G_n. Then use contractibility of $\mathrm{Diff}_0(\Sigma_g)$ to show this is a $K(MCG(\Sigma_g),1)$ $\leftarrow$ this part special to Σ_g bundles.

The Grassmannian. $\quad G_{\Sigma_g}$ = set of smooth submanifolds of $\mathbb{R}^\infty$ diffeo to Σ_g.

$G_{\Sigma_g}(\mathbb{R}^n)$ topologized as quotient $\mathrm{Emb}(\Sigma_g, \mathbb{R}^n)/\mathrm{Diff}(\Sigma_g)$ $\quad\uparrow C^\infty$ topology

and $G_{\Sigma_g} = \varinjlim G_{\Sigma_g}(\mathbb{R}^n)$

Canonical bundle. $\quad E_{\Sigma_g} = \{(x,S) \in \mathbb{R}^\infty \times G_{\Sigma_g} : x \in S\}$

Need to check $E_{\Sigma_g} \to G_{\Sigma_g}$ is a Σ_g-bundle

i.e. if $S \in G_{\Sigma_g}$ and $S' \in G_{\Sigma_g}$ is sufficiently close,

need a canonical diffeo $S' \to S$.

First for $G_{\Sigma_g}(\mathbb{R}^n)$.

Main idea: if S' close to S then S' is a section of

normal bundle N of S = tubular nbd;

then $S' \to S$ is projection in N.

This is because S is transverse to fibers, which is an open condition, so nearby S' is transverse to any given fiber, hence to all nearby fibers, hence to all fibers by compactness.

For S' close enough to S there is an isotopy of S to S' preserving transversality, hence $S' \cap \text{fiber} = 1$ pt

$\Rightarrow S'$ a section.

The result follows by defn of topology on G_{Σ_g}.

Universality. To show $\{\Sigma_g\text{-bundles over } B\}/\cong \;\longleftrightarrow\; [B, G_{\Sigma_g}]$ $\quad B=$ paracompact

Essentially same as v.b. case. Basic idea: Realizing $E \to B$ as $f^*(E_{\Sigma_g})$ equiv. to finding $E \xrightarrow{g} \mathbb{R}^\infty$ smooth emb. on fibers. Such g induces $f, \tilde{f}$ s.t.

$$\begin{tikzcd} E \arrow[r, "\tilde{f}"] \arrow[d] & E_{\Sigma_g} \arrow[d] \\ B \arrow[r, "f"] & G_{\Sigma_g} \end{tikzcd}$$

Fix some $E \xrightarrow{p} B \leftarrow$ compact. Want to find g, hence f.
Choose $U_i \subseteq B$ s.t. $p^{-1}(U_i) \cong U_i \times \Sigma_g$, subord. part of 1 $\{\varphi_i\}$

$$g_i : p^{-1}(U_i) \longrightarrow U_i \times \Sigma_g \longrightarrow \Sigma_g \xrightarrow[\text{emb.}]{\text{any}} \mathbb{R}^n$$

$$g : E \longrightarrow \mathbb{R}^n \times \cdots \times \mathbb{R}^n \subseteq \mathbb{R}^\infty$$

$$p \longmapsto (\varphi_1 g_1(p), \ldots, \varphi_N g_N(p))$$

Any two g's are homotopic:

$$\begin{tikzcd} g_0 \arrow[dr] & & g_1 \arrow[d] \\ & \text{even coords} \arrow[r, "\text{str line}", "\text{homo.}"'] & \text{odd coords} \end{tikzcd}$$

$\rightsquigarrow$ resulting f unique up to homotopy.

Relation to MCG. Step 1: There is a bundle $\mathrm{Diff}^+(\Sigma_g) \to P_{\Sigma_g} \to G_{\Sigma_g}$

(use tubular nbds / sections as above)

~~[illegible]~~.

Step 2: $\mathrm{Emb}(\Sigma_g, \mathbb{R}^\infty) \simeq *$

Enough to find canonical, continuously varying paths to some basept. S

Choose S in even coords.

For any S', apply $\mathbb{R}^\infty \longrightarrow \mathbb{R}^{\text{odd coords}}$ then straight line homotopy to S.

Step 3: Apply L.E.S. for fiber bundle (or, fibration)

$$\cdots \longrightarrow \pi_n(F) \longrightarrow \pi_n(E) \longrightarrow \pi_n(B) \longrightarrow \pi_{n-1}(F) \longrightarrow \cdots$$

(comes from L.E.S. in π_* for (E,F) and $\pi_*(E,F) \cong \pi_*(B)$).

<u>Thm</u> (Earle-Eells). $\mathrm{Diff}(\Sigma_g)$ has contractible components.

$$\rightsquigarrow \quad \pi_i(G_{\Sigma_g}) \cong \pi_{i-1}(\mathrm{Diff}(\Sigma_g)) \quad \forall i.$$

$$\pi_1(G_{\Sigma_g}) \cong \pi_0(\mathrm{Diff}(\Sigma_g)) = MCG^{\pm}(\Sigma_g)$$
$$\pi_i(G_{\Sigma_g}) = 0 \quad i > 1.$$