Diffeomorphism Groups of Surfaces

$S =$ compact, connected surface

Write $\mathrm{Diff}(S)$ for $\mathrm{Diff}(S, \partial S)$. C^∞ topology.

<u>Thm</u>. If $S \neq S^2, \mathbb{RP}^2, T^2, KB$ then the components of $\mathrm{Diff}(S)$ are contractible.

Note: $\mathrm{Diff}(S^2) \simeq \mathrm{Diff}(\mathbb{RP}^2) \simeq SO(3)$

$\mathrm{Diff}(T^2) \simeq T^2$, $\mathrm{Diff}(KB) \simeq S^1$.

Proof has 3 steps.

① Reduction to case $\partial S \neq \emptyset$

will show $\pi_i(\mathrm{Diff}(S)) \cong \pi_i(\mathrm{Diff}(S - \overset{\text{open}}{D^2}))$.

② Inductive step

will show $\pi_i(\mathrm{Diff}(S)) \cong \pi_i(\mathrm{Diff}(\overset{S \text{ cut along } \alpha}{S_\alpha}))$

③ Base case

$\pi_i(\mathrm{Diff}(D^2)) = 0 \quad i \geq 1$.

Step 1. Reduction to case $\partial S \neq \emptyset$.

Fix $x_0 \in D \subseteq S$. Let $S_0 = S - \mathrm{int}\, D$.

To show $\pi_i(\mathrm{Diff}(S)) = \pi_i(\mathrm{Diff}(S, x_0)) = \pi_i(\mathrm{Diff}(S, D)) = \pi_i(\mathrm{Diff}(S_0))$

Last equality easy. Remains to do other two.

First equality. There is a fiber bundle $\mathrm{Diff}(S, x_0) \longrightarrow \mathrm{Diff}(S) \longrightarrow S$.

$\uparrow$ diffeos fixing x_0.

$\rightsquigarrow$ L.E.S.:

$$\pi_{i+1}(S) \longrightarrow \pi_i(\mathrm{Diff}(S,x_0)) \longrightarrow \pi_i(\mathrm{Diff}(S)) \longrightarrow \pi_i(S)$$

But $\pi_i(S) = 0 \quad i > 1 \quad$ (as $\tilde{S} \simeq *$).

$\rightsquigarrow \pi_i(\mathrm{Diff}(S,x_0)) \cong \pi_i(\mathrm{Diff}(S)) \quad i > 1$.

$i=1$ case:

$$0 \longrightarrow \pi_1 \mathrm{Diff}(S,x_0) \longrightarrow \pi_1(\mathrm{Diff}(S)) \longrightarrow \pi_1(S, x_0)$$
$$\xrightarrow{\partial} \pi_0 \mathrm{Diff}(S, x_0) = MCG(S, x_0)$$

Suffices to show $\ker \partial = 0$.

But the composition

$$\pi_1(S,x_0) \longrightarrow MCG(S,x_0) \longrightarrow \mathrm{Aut}\, \pi_1(S,x_0)$$

is $\quad \alpha \longmapsto$ inner automorphism conj. by α

To show this is inj, suffices to show $Z\pi_1(S) = 1$.

For latter: $\quad \tilde{S} \cong \mathbb{H}^2$

$\pi_1(S) \leftrightarrow$ deck trans. in $\mathrm{Isom}^+ \mathbb{H}^2$

& independent hyperbolic isometries do not commute.

Second equality. Another fiber bundle: $\mathrm{Diff}(S,D) \longrightarrow \mathrm{Diff}(S, x_0) \longrightarrow \mathrm{Emb}((D,x_0),(S,x_0))$

D fixed

Claim: $\mathrm{Emb}((D,x_0),(S,x_0)) \simeq GL_2(\mathbb{R}) \simeq O(2)$

$$f \longmapsto D_{x_0} f$$

As above, LES $\Rightarrow \pi_i \mathrm{Diff}(S,x_0) \cong \pi_i \mathrm{Diff}(S,D) \quad i > 1$.

$i=1$ case: $\quad 0 \to \pi_1 \mathrm{Diff}(S,D) \to \pi_1 \mathrm{Diff}(S,x_0)$

$$\to \pi_1 \mathrm{Emb}((D,x_0),(S,x_0)) \xrightarrow{\partial} \pi_0 \mathrm{Diff}(S,D) = MCG(S_0).$$

$$\overset{\shortparallel}{\mathbb{Z}}$$

Again, need $\ker \partial = 0$.

$\rho \in \partial S_0$

But $\quad \mathbb{Z} \to MCG(S_0) \longrightarrow \mathrm{Aut}\, \pi_1(S_0, \rho)$

is $\quad 1 \longmapsto$ conj. by ∂-element.

Since $\pi_1(S_0)$ is free, we are done.

Another point of view. We could have combined the two steps. There is a fiber bundle

$$\mathrm{Diff}(S,(p,v)) \longrightarrow \mathrm{Diff}(S) \longrightarrow UT(S)$$

with fiber $\simeq \mathrm{Diff}(S_0)$.

Apply same argument.

Step 3. Base step: $\mathrm{Diff}_0(D^2)$ contractible

D^2_+ = top half of D^2

$\mathrm{Emb}(D^2_+, D^2)$ = space of embeddings $D^2_+ \to D^2$ fixing $D^2_+ \cap \partial D^2$ and taking rest of D^2_+ to $\mathrm{int}\, D^2$.

$\alpha = D^1$ = equator of D^2

$A(D^2, \alpha)$ = embeddings of proper arcs in D^2 with same endpts as α. ← intersect ∂D^2 only at endpts.

⇝ fibration

$$\begin{array}{ccc} \mathrm{Diff}(D^2_+) & \longrightarrow & \mathrm{Emb}(D^2_+, D^2) \\ & & \downarrow \\ & & A(D^2, \alpha) \end{array}$$

Claim 1. $\mathrm{Emb}(D^2_+, D^2) \simeq *$.

Uses: the space of tubular nbds of a submanifold is contractible.

Claim 2. $A(D^2, \alpha) \simeq *$. *More generally*, $A(S, \alpha) \simeq *$. Proven below.

LES $\Rightarrow \mathrm{Diff}(D^2_+) \simeq *$. But $D^2_+ \cong D^2$.

Step 2. Induction step.

Induction on $-\chi(S)$.

α = proper arc in S.

$A(S, \alpha)$ = emb's of proper arcs in S, iso to α, same endpts

⇝ fiber bundle $\mathrm{Diff}_0(S, \alpha) \longrightarrow \mathrm{Diff}_0(S) \longrightarrow A(S, \alpha)$

↑ diffeos fixing α ptwise, $\simeq \mathrm{Diff}_0(S \text{ cut along } \alpha)$.

LES + induction + Claim 2 $\Rightarrow \mathrm{Diff}_0(S) \simeq *$.

SMALE'S PROOF. (Original version of Step 3)

<u>Thm</u> The space of C^∞ diffeos of I^2 that are id in nbd of ∂I^2 is contractible.

Note: $\widetilde{\mathbb{R}^n - \{0\}}$ not contractible, $n \neq 2$.

Some ideas. Given $f : I^2 \to I^2 \rightsquigarrow$ vector field V:

$$V(x,y) = df_{f^{-1}(x,y)}(1,0).$$

There is a homotopy V_t s.t. $V_0 = V$, $V_1 =$ const. vector field $(1,0)$, $V_t =$ nonvan. vector field id in nbd of ∂I^2 since $V_0, V_1 : I^2 \to \mathbb{R}^2 - \{0\}$.

Then define $f_t : I^2 \to [0,\infty) \times [0,1]$

$f_t(x,y) =$ flow along V_t, start at $(0,y)$, for time x.

Clearly $f_1 = \mathrm{id}$, $f_0 = f$. (n.b. no spiralling, for then there would be a singularity).

Problem: $\mathrm{Im}\, f_t$ maybe not $= I^2$.

Solution: Precompose each f_t with a reparameterization in the x-dir. Result is a homotopy of f to id through diffeos.

By fixing once and for all a def. retraction of $\widetilde{\mathbb{R}^2 - \{0\}}$ to a point, get a consistent way of deforming an arbitrary diffeo to id, at all times = id in nbd of ∂I^2.

(See Lurie's notes for an *Earle-Eells*-style approach.)

CERF'S STRAIGHTENING TRICK. (Toy case for Claim 2).

We'll need to know that some basic spaces of embeddings are contractible. We start with a warmup.

<u>Prop</u>. The space of smooth embeddings of arcs in $\mathbb{R} \times [0,\infty)$ based at 0 is contractible.

<u>Pf</u>. The space of linear arcs is clearly contractible — it is homeo to $\mathbb{R} \times [0,\infty)$.

Here is a canonical isotopy from an arbitrary arc ~~to~~ f to a linear one:

$$F_t(x) = \begin{cases} \dfrac{f((1-t)x)}{1-t} & t<1 \\ f'(0)x & t=1 \end{cases}$$

Can soup this up:

<u>Prop</u>. The space of smooth embeddings of arcs in S based at $p \in \partial S$ is contractible.

<u>Pf</u>. By previous prop, need a canonical isotopy of an arbitrary arc into a fixed tubular nbd of p.

For any compact set of arcs, can use

$$F_t(x) = f(\alpha x) \qquad \alpha = \max\{\varepsilon, (1-tx)\}.$$

i.e. $F_t(x)$ traces out shorter & shorter subarcs.

This implies weak contractibility.

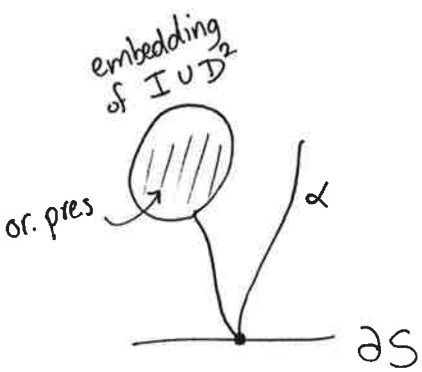
Claim 2: Contractibility of arc spaces

α = proper arc in S

$A(S,\alpha)$ = space of proper arcs $\simeq \alpha$, same endpts as α.

Case 1. α connects distinct components of ∂S.

T = surface obtained from S by capping with disk at one end of α



$\rightsquigarrow$ fiber bundle

$$\begin{array}{ccc} \mathrm{Emb}(I,S) & \longrightarrow & \mathrm{Emb}(I \cup D^2, S) \\ \uparrow & & \downarrow \\ \text{both endpts fixed} & & \mathrm{Emb}(D^2, T - \partial T) \end{array}$$

Claim. $\mathrm{Emb}(I \cup D^2, S) \simeq *$. $\qquad p \in \partial D^2,\ x \in \mathrm{int}\, S$

Pf of claim. Another fiber bundle

$$\begin{array}{ccc} \mathrm{Emb}((D^2,p),(S,x)) & \longrightarrow & \mathrm{Emb}(I \cup D^2, S) \\ & & \downarrow \\ \text{one endpt fixed} & \longrightarrow & \mathrm{Emb}(I, S) \end{array}$$

Base, fiber contractible by variations on Cerf's straightening.

Claim. $\pi_i \mathrm{Emb}(D^2, T - \partial T)$ $\quad i > 0$

Pf. Yet another fiber bundle:

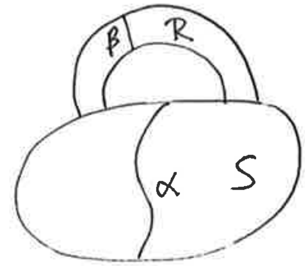
$$\begin{array}{c} \mathrm{Emb}(D^2, T - \partial T) \\ \big\downarrow \text{eval@}0 \\ T - \partial T \end{array}$$

By two claims, plus LES for main fiber bundle, $\mathrm{Emb}(I, S)$ has contractible components, one of which is $A(S,\alpha)$.

Case 2. α joins a component of ∂S to itself

Idea: add a handle $T = S \cup R$ s.t.

α joins distinct comp's of ∂T



T

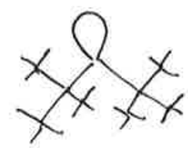
Suffices to show

$\pi_i A(T-\beta, \alpha) \longrightarrow \pi_i A(T, \alpha)$ injective.

Key: there is a cov. space $\tilde{T}$ of T hom. eq. to S.

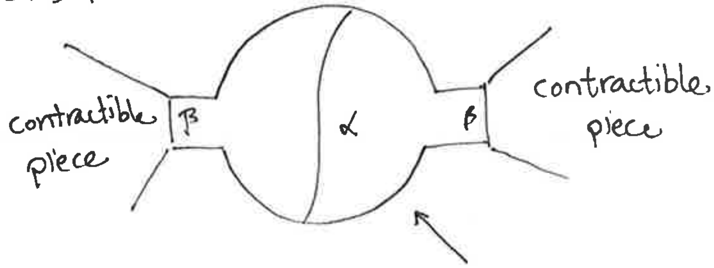
because $\pi_1(T) = \pi_1(S) * \mathbb{Z}$

so $\tilde{T}$ = cover corr to $\pi_1(S)$

Toy case 

$\mathbb{Z} \leq \mathbb{Z} * \mathbb{Z}$

$\tilde{T}$ looks like T cut along β



Identify $A(T-\beta, \alpha)$ with space of arcs in this region of $\tilde{T}$.

$A(T, \alpha)$ with a space of arcs in $\tilde{T}$: ~~subspace of $A(\tilde{T},\alpha)$~~

lifts of arcs in T $\rightarrow \tilde{A}(T, \alpha) \subseteq A(\tilde{T}, \alpha) \leftarrow$ arcs in $\tilde{T}$

Suffices to show composition $A(T-\beta, \alpha) \xhookrightarrow{i} A(\tilde{T}, \alpha)$ is inj on π_*.

Need a retraction $r: A(\tilde{T}, \alpha) \longrightarrow A(T-\beta, \alpha)$

s.t. $r \circ i \sim id$.

The r is induced by shrinking the two contractible pieces.