Diffeomorphism Groups of Surfaces

$S$ = compact, connected surface
Write Diff($S$) for Diff($S, \partial S$). $C^\infty$ topology.

Thm. If $S = S^2, \mathbb{RP}^2, T^2, KB$ then the components of Diff($S$) are contractible.

Note: Diff($S^2$) = Diff ($\mathbb{RP}^2$) = SO(3)
Diff ($T^2$) = $T^2$, Diff ($KB$) = $S^1$.

Proof has 3 steps. ① Reduction to case $\partial S \neq \emptyset$
will show $\pi_i$ (Diff($S$)) $\cong$ $\pi_i$ (Diff($S - D^2$)).
② Inductive step $S$ cut along $\alpha$
will show $\pi_i$ (Diff($S$)) $\cong$ $\pi_i$ (Diff($S\alpha$))
③ Base case
$\pi_i$ (Diff($D^2$)) = 0 $i > 1$.

Step 1. Reduction to case $\partial S \neq \emptyset$.

Fix $x_0 \in D \subseteq S$. Let $S_0 = S - \text{int} D$.
To show $\pi_i$ (Diff($S$)) = $\pi_i$ (Diff($S, x_0$)) = $\pi_i$ (Diff($S, D$)) = $\pi_i$ (Diff($S\alpha$))

Last equality easy. Remains to do other two.
First equality. There is a fiber bundle, \( \text{Diff}(S, x_0) \rightarrow \text{Diff}(S) \rightarrow S \).

\[ \uparrow \text{diffs fixing } x_0. \]

\[ \rightarrow \text{LES:} \]

\[ \pi_{i+1}(S) \rightarrow \pi_i(\text{Diff}(S, x_0)) \rightarrow \pi_i(\text{Diff}(S)) \rightarrow \pi_i(S) \]

But \( \pi_i(S) = 0 \) \( i > 1 \) (as \( S \neq \ast \)).

\[ \rightarrow \pi_i(\text{Diff}(S, x_0)) \cong \pi_i(\text{Diff}(S)) \quad i > 1. \]

\( i = 1 \) case:

\[ 0 \rightarrow \pi_1(\text{Diff}(S, x_0)) \rightarrow \pi_1(\text{Diff}(S)) \rightarrow \pi_1(S, x_0) \]

\[ \rightarrow \pi_0(\text{Diff}(S, x_0)) = \text{MCG}(S, x_0) \]

Suffices to show \( \ker d = 0. \)

But the composition

\[ \pi_1(S, x_0) \rightarrow \text{MCG}(S, x_0) \rightarrow \text{Aut} \pi_1(S, x_0) \]

is \( \alpha \mapsto \text{inner automorphism conj. by } \alpha \)

To show this is inj, suffices to show \( \mathbb{Z} \pi_1(S) = 1. \)

For latter:

\[ \mathbb{S} \cong \mathbb{H}^2 \]

\( \pi_1(S) \leftrightarrow \text{deck trans. in } \text{Isom}^+ \mathbb{H}^2 \)

& independent hyperbolic isometries do not commute.

Second equality. Another fiber bundle:

\( \text{Diff}(S, D) \rightarrow \text{Diff}(S, x_0) \rightarrow \text{Emb}(D, x_0), (S, x_0) \)

Claim: \( \text{Emb}(D, x_0), (S, x_0) \cong \text{GL}_2(\mathbb{R}) \cong O(2) \)

\[ f \mapsto D_{x_0} f \]

As above, LES \( \Rightarrow \pi_i \text{Diff}(S, x_0) \cong \pi_i \text{Diff}(S, D) \quad i > 1. \)
i=1 case: \[ 0 \rightarrow \pi_1 \text{Diff}(S,D) \rightarrow \pi_1 \text{Diff}(S,x_0) \rightarrow \pi_1 \text{Emb}(D,x_0), (S,x_0) \xrightarrow{\partial} \pi_0 \text{Diff}(S,D) = \text{MCG}(S_o). \]

\[ \mathbb{Z} \]

Again, need \( \ker \partial = 0 \).

But \[ \mathbb{Z} \rightarrow \text{MCG}(S_o) \rightarrow \text{Aut} \pi_1(S_o, p) \]

is \[ 1 \rightarrow \text{conj. by } \partial \text{-element}. \]

Since \( \pi_1(S_o) \) is free, we are done.

Another point of view. We could have combined the two steps.

There is a fiber bundle

\[ \text{Diff}(S,(p,v)) \rightarrow \text{Diff}(S) \rightarrow \text{UT}(S) \]

with fiber \( \partial \text{Diff}(S_o) \).

Apply same argument.
Step 3. Base step: $\text{Diff}_0(D^2)$ contractible

$D^2_+ = \text{top half of } D^2$

$\text{Emb}(D^2_+, D^2) =$ space of embeddings $D^2_+ \to D^2$ fixing $D^2_+ \cap \partial D^2$ and taking rest of $D^2_+$ to int $D^2$.

$x = D^1 =$ equator of $D^2$

$A(D^2, x) =$ embeddings of proper arcs in $D^2$ with same endpts as $x$.

$\sim$ Fibration $\text{Diff}(D^2_+) \to \text{Emb}(D^2_+, D^2)$

$\downarrow$

$A(D^2, x)$

Claim 1. $\text{Emb}(D^2_+, D^2) \cong \ast$. Uses: the space of tubular nbds of a submanifold is contractible.

Claim 2. $A(D^2, x) \cong \ast$. More generally, $A(S, x) \cong \ast$. Proven below.

$\text{LES} \Rightarrow \text{Diff}(D^2_+) \cong \ast$. But $D^2_+ \cong D^2$.

Step 2. Induction step.

Induction on $-\kappa(S)$.

$x =$ proper arc in $S$.

$A(S, x) =$ emb's of proper arcs in $S$, iso to $x$, same endpts

$\sim$ fiber bundle $\text{Diff}_0(S, x) \to \text{Diff}_0(S) \to A(S, x)$

$\downarrow$ diffeos fixing $x$ ptwise., $\cong \text{Diff}_0(S \text{ cut along } x)$.

$\text{LES + induction + Claim 2 } \Rightarrow \text{Diff}_0(S) \cong \ast$. 
SMALE'S Proof.  (Original version of Step 3)

Ihm. The space of $C^\infty$ diffeos of $I^2$ that are id in nbd of $\partial I^2$

is contractible.

Some ideas. Given $f: I^2 \rightarrow I^2 \rightsquigarrow$ vector field $V$:

$$V(x,y) = df^{-1}(x,y)(1,0).$$

There is a homotopy $V_t$ s.t. $V_0 = V$, $V_1 =$ const. vector field $(1,0)$,

$V_t =$ nonvan. vector field since $V_0, V_1 : I^2 \rightarrow \mathbb{R}^2 - (0,0)$.

id in nbd of $\partial I^2$.

Then define, $f_t : I^2 \rightarrow \mathbb{R}^2 \times [0,1]$:

$$f_t(x,y) = \text{flow along } V_t, \text{ start at } (0,y), \text{ for time } t.$$  

Clearly $f_1 = \text{id}$, $f_0 = f$. (n.b. no spiralling, for then there would

be a singularity).

Problem: $\text{Im } f_t$ maybe not $= \mathbb{R}^2$.

Solution: Precompose each $f_t$ with a reparameterization

in the $x$-dir. Result is a homotopy of $f$ to id through diffeos.

By fixing once and for all a retraction of $\mathbb{R}^2 - (0,0)$ to a point, get a consistent way

of deforming an arbitrary diffeo to id, at

all times $= \text{id}$ in nbd of $\partial I^2$.

(See Lurie's notes for an Earle-Eells-style approach.)
CERF'S STRAIGHTENING TRICK. (Toy case for Claim 2).

We'll need to know that some basic spaces of embeddings are contractible. We start with a warmup.

**Prop.** The space of smooth embeddings of arcs in \( \mathbb{R} \times [0, \infty) \) based at 0 is contractible.

**Pf.** The space of linear arcs is clearly contractible — it is homeo to \( \mathbb{R} \times [0, \infty) \).

Here is a canonical isotopy from an arbitrary arc \( f \) to a linear one:

\[
F_t(x) = \begin{cases} 
  f((1-t)x) & t < 1 \\
  \frac{1}{1-t} & t = 1 
\end{cases}
\]

Can soup this up:

**Prop.** The space of smooth embeddings of arcs in \( S \) based at \( p \in \partial S \) is contractible.

**Pf.** By previous prep, need a canonical isotopy of an arbitrary arc into a fixed tubular nbd of \( p \).

For any compact set of arcs, can use

\[ F_t(x) = f(ax) \quad a = \max \{ \epsilon, (1-t)x \} \]

i.e. \( F_t(x) \) traces out shorter \& shorter subarcs.

This implies weak contractibility.
Claim 2: Contractibility of arc spaces

\[ \alpha = \text{proper arc in } S \]
\[ A(S, \alpha) = \text{space of proper arcs } \preceq \alpha, \text{ same endpoints as } \alpha. \]

Case 1. \( \alpha \) connects distinct components of \( \partial S \).

\[ T = \text{surface obtained from } S \text{ by capping with disk at one end of } \alpha \]
\[ \sim \text{fiber bundle } \quad \begin{array}{c}
\text{Emb}(I,S) \to \text{Emb}(I\cup D^2, S) \\
\uparrow \quad \downarrow \\
\text{both endpoints fixed} \quad \text{Emb}(D^2, T-\partial T)
\end{array} \]

Claim. \( \text{Emb}(I \cup D^2, S) \cong \ast \).

PF of claim. Another fiber bundle:
\[ \text{Emb}((I,p),(S,x)) \to \text{Emb}(I \cup D^2, S) \]

Base, fiber contractible by variations on Cerf's straightening.

Claim. \( \forall \text{Emb}(D^2, T-\partial T) \quad i > 0 \)

PF. Yet another fiber bundle:
\[ \text{Emb}(D^2, T-\partial T) \]
\[ \downarrow \text{eval}\circ \zeta \]
\[ T-\partial T \]

By two claims, plus LES for main fiber bundle, \( \text{Emb}(I,S) \) has contractible components, one of which is \( A(S, \alpha) \).
Case 2. $\alpha$ joins a component of $\partial S$ to itself

Idea: add a handle $T = S \cup R$ s.t.
$\alpha$ joins distinct comps of $\partial T$
Suffices to show
$\pi_1 A(T-\beta) \to \pi_1 A(T)$ injective.

Key: there is a cov. space of $T$ hom. eq. to $S$.
because $\pi_1(T) = \pi_1(S) \ast \mathbb{Z}$
so $\tilde{T} = \text{cover corr to } \pi_1(S)$

$\tilde{T}$ looks like $T$ cut along $\beta$

Identify $A(T-\beta, \alpha)$ with space of arcs in this region of $\tilde{T}$.
$A(T, \alpha)$ with a space of arcs in $\tilde{T}$.

Lifts of arcs in $T$ \to $A(T, \alpha) \subseteq A(\tilde{T}, \alpha)$ \to arcs in $\tilde{T}$

Suffices to show composition $A(T-\beta, \alpha) \subseteq A(T, \alpha)$ is inj on $\pi_1$. Need a retraction $r: A(\tilde{T}, \alpha) \to A(T-\beta, \alpha)$ s.t. $r \circ i = \text{id}$.
The $r$ is induced by shrinking the two contractible pieces.