Characteristic Classes in Degree One

We know now: \( H^*(\text{MCG}(S_g)) \cong \text{Ring of char classes for } \Sigma_g\text{-bundles} \)

**Thm.** \( H^1(\text{MCG}(S_g); \mathbb{Z}) = 0 \quad g \geq 1 \).

**Pf.** We'll do \( g \geq 3 \). Ingredients:

1. \( \text{MCG}(S_g) \) is gen. by Dehn twists about nonseparating curves
2. Any two such Dehn twists are conjugate in \( \text{MCG}(S_g) \)
3. There is a relation among such twists of the form
   \[ T_x T_y T_z = T_a T_b T_c T_d \]
   It follows that \( H^1(\text{MCG}(S_g); \mathbb{Z}) \cong \text{MCG}(S_g)^{ab} \) is trivial.
   hence \( H^1(\text{MCG}(S_g); \mathbb{Z}) = 0 \).

**Ingredient 2.** Follows from: \( f T a f^{-1} = T_{f(a)} \) and classification of surfaces.

**Ingredient 3.** Follows from: Lantern relation
   \[ T_x T_y T_z = T_T T_a T_c \]
   (prove by checking action on \( \mathbb{S}^3 \) and using \( \text{Mod}(D^2) = 1 \)) and the embedding:
Generating MCG (Ingredient 1).

Two (sub)ingredients:

1. The complex of curves \( C(S_g) \) is connected \( g \geq 2 \).
   Vertices: isotopy classes of simple closed curves
   Edges: disjoint representatives

2. The Birman exact sequence \( \chi(S) < 0 \).

\[ 1 \rightarrow \pi_1(S, p) \rightarrow \text{MCG}(S, p) \rightarrow \text{MCG}(S) \rightarrow 1. \]

Outline of proof:

1. \( \Rightarrow \) complex of nonsep. curves \( N(S_g) \) is connected.
   \( \Rightarrow \) given any two isotopy classes of nonsep s.c.c. in \( S_g \)
   \( \exists \text{TTc}_i \) ci nonsep taking one to other.*
   \( \Rightarrow \) \( \text{MCG}(S_g) \) gen. by nonsep twists if
   \( \text{MCG}(S_g - c) \) is.

   But \( \text{MCG}(S_g - c) \cong \text{MCG}(S_g/2) \)

2. \( \Rightarrow \) by induction, \( \text{MCG}(S_g, 2) \) is gen by nonsep twists
   if \( \text{MCG}(S_g - 1) \) is.

Done by induction. Base case is \( \text{MCG}(S_1) \cong \text{SL}_2 \mathbb{Z} \)
   gen by \( (\sigma^1), (-1, 0) \).

* Use the relation \( T_b T_a (b) = a \) for \( i(a, b) = 1 \).
Connectivity of $C(S_g)$

Take two vertices of $C(S_g)$, represent them by s.c.c. in $S_g$.
Choose smooth funs $f_0, f_1$ s.t. $a$ is a level set of $f_0$, $b$ of $f_1$.
Connect $f_0$ to $f_1$ by a path $f_t \in C^\infty(S_g, \mathbb{R})$.

Cert Lemma. Any path $f_t \in C^\infty(S_g, \mathbb{R})$ can be approx. by $g_t \in C^\infty(S_g, \mathbb{R})$
so each $g_t$ is in one of following classes:

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1. Morse functions with at most 2 coincident
   critical values $\leftrightarrow$ crit. values passing each other

2. functions with distinct crit vals and exact one
degen. crit pt of the form: $x^2 + y^2 + c$ $\leftrightarrow$ crit vals
   merging/splitting

Claim. Each $g_t$ has a level set rep. a vertex of $C(S_g)$.

Not by curves are isotopic $\Rightarrow \{ t : v \in C(S_g) \text{ is rep by a level set of } g_t \}$
is open in $\mathbb{R}$

Also, level sets of the same $g_t$ are disjoint.
Result follows from compactness of $[0,1]$.

Remains to prove claim. Take nbd of crit set:

If two circles bound disks, modify the function to
get rid of this crit pt.
Look at another crit pt.

Or: Given $f : S_g \to \mathbb{R} \rightsquigarrow$ graph $\Gamma_f$ by crushing level sets.
$\text{rk}(\Gamma_f) = 9$, except in case 2 above where $\text{rk}(\Gamma_f) = 9 - 1$.
Any nontrivial cocycle (= pt) in $\Gamma_f$ corresponds to a
nontrivial level set in $S_g$. (this shows $N(S_g)$ connected!)