Characteristic Classes in Degree One

We know now: $H^*(MCG(S_g)) \cong$ Ring of char classes for Σ_g-bundles

<u>Thm</u>. $H^1(MCG(S_g);\mathbb{Z}) = 0 \quad g \geq 1.$

<u>Pf</u>. We'll do $g \geq 3$. Ingredients:

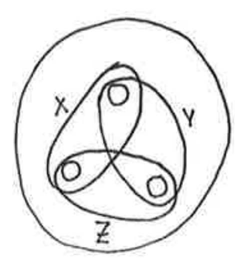
1. $MCG(S_g)$ is gen. by Dehn twists about nonseparating curves

2. Any two such Dehn twists are conjugate in $MCG(S_g)$
3. There is a relation among such twists of the form
$$T_x T_y T_z = T_a T_b T_c T_d$$

It follows that $H_1(MCG(S_g);\mathbb{Z}) \cong MCG(S_g)^{ab}$ is trivial.
hence $H^1(MCG(S_g);\mathbb{Z}) = 0$.

<u>Ingredient 2</u>. Follows from: $f T_a f^{-1} = T_{f(a)}$ and classification of surfaces.

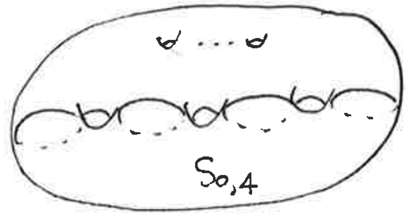
<u>Ingredient 3</u>. Follows from: Lantern relation
$$T_x T_y T_z = \prod T_{\partial_i}$$



$S_{0,4}$

(prove by checking action on  and using $Mod(D^2) = 1$)

and the embedding:



Generating MCG (Ingredient 1).

Two (sub) ingredients: ① The complex of curves $C(S_g)$ is connected $g \geq 2$.

vertices: isotopy classes of simple closed curves

edges: disjoint representatives

② The Birman exact sequence $\chi(S) < 0$.

$$1 \to \pi_1(S,p) \to MCG(S,p) \to MCG(S) \to 1.$$

<u>Outline of proof.</u> ① $\Rightarrow$ complex of nonsep. curves $N(S_g)$ is connected.

$\Rightarrow$ given any two isotopy classes of nonsep s.c.c. in S_g $\exists$ $\prod T_{c_i}$ c_i nonsep taking one to other.*

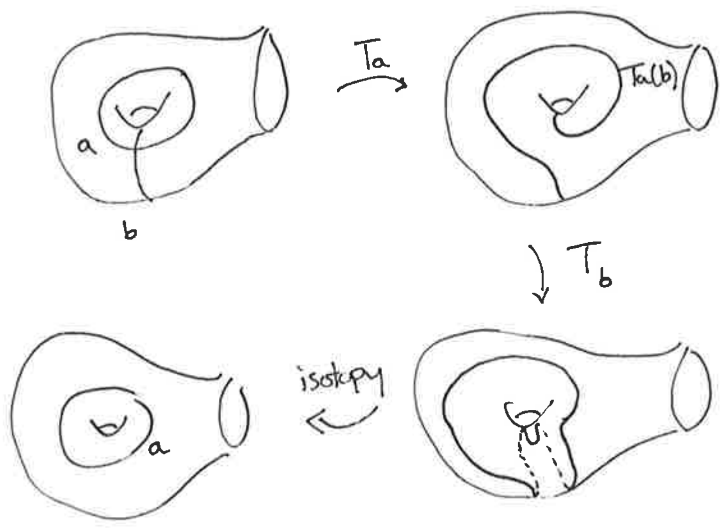
$\Rightarrow$ $MCG(S_g)$ gen. by nonsep twists if $MCG(S_g - c)$ is.

But $MCG(S_g - c) \cong MCG(S_{g-1}, 2)$

② ~~& induction~~ (applied twice) $\Rightarrow$ $MCG(S_{g-1}, 2)$ is gen by nonsep twists if $MCG(S_{g-1})$ is.

Done by induction. Base case is $MCG(S_1) \cong SL_2\mathbb{Z}$ gen by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$.

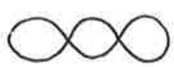
* Use the relation $T_b T_a(b) = a$ for $i(a,b) = 1$.



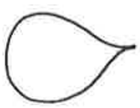
Connectivity of $C(S_g)$

Take two vertices of $C(S_g)$, represent them by s.c.c. a, b in S_g.
Choose smooth fns f_0, f_1 s.t. a is a level set of f_0, b of f_1.
Connect f_0 to f_1 by a path $f_t \in C^\infty(S_g, \mathbb{R})$.

Cerf Lemma. Any path $f_t \in C^\infty(S_g, \mathbb{R})$ can be approx. by $g_t \in C^\infty(S_g, \mathbb{R})$ so each g_t is in one of following classes:

① Morse functions with at most 2 coincident critical values ← crit. values passing each other



② functions with distinct crit vals and exact one degen. crit pt of the form $x^3 \pm y^2 + c$ ← crit vals merging/splitting

Claim. Each g_t has a level set rep. a vertex of $C(S_g)$.

Nearby curves are isotopic $\Rightarrow$ $\{t : v \in C(S_g)$ is rep by a level set of $g_t\}$ is open in $\mathbb{R}$

Also, level sets of the same g_t are disjoint.
Result follows from compactness of $[0,1]$.

Remains to prove claim. Take nbd of crit set: 

If two circles bound disks, modify the function to get rid of this crit pt.
Look at another crit pt.

Or: Given $f : S_g \to \mathbb{R} \rightsquigarrow$ graph Γ_f by crushing conn. comp. of level sets.

easy Euler char. count. → $\text{rk}(\Gamma_f) = g$. except in case ② above where $\text{rk}(\Gamma_f) = g-1$. ↙ this is where $g \geq 2$ used!

Any nontrivial cocycle ($=$ pt) in Γ_f corresponds to a nontrivial level set in S_g. (this shows $N(S_g)$ connected!)