Construction of $S_g$-bundles with $e_i \neq 0$

\[ \begin{array}{cccccc}
\tilde{E}^* & \rightarrow & E_3 & \rightarrow & E_2 & \rightarrow & (E_1)^* \\
\downarrow S_g & \downarrow S_g & \downarrow S_g & \rightarrow & S_g & \rightarrow & \Delta \\
E & \rightarrow & \tilde{E} & \rightarrow & E_2 & \rightarrow & E_1 & \rightarrow & E \\
\text{fiberwise cyclic branched cover over} & \end{array} \]

Induction: Start with $e_i(E) \neq 0$. \(\text{dim}(M) = 2i\)

1. diagonal construction:
   - bump up dim by 2,
   - new bundle has diagonal section $\Delta$.

2. take cover over base so pullback bundle has $m$-fold fiberwise cover
3. now can take cover of base so preimage of $\Delta$ is divis. by $m$

\[ \begin{aligned}
\text{Prop } e_i(\tilde{E}^*) &= -dm^2 \left( 1 - m^{-i+1} \right) e_i(E) \\
d &= \text{degree of } \tilde{E} \rightarrow E
\end{aligned} \]

The proof is analogous to that of: $e_i(\tilde{M}) = 2e_i(M) - i(\tilde{D}, \tilde{D})$ above.

So $e_i(E) \neq 0 \Rightarrow e_i(\tilde{E}^*) \neq 0$. Morita calls this the $m$-construction on $E \rightarrow M$.
Atiyah's construction is $\tilde{E}$-construction on $S_g \rightarrow \text{pt.}$
Higher Dimensional Surface Bundles

Goal. $e_i \neq 0 \ \forall i$.

Iterated surface bundles. $C_0 = \{ * \}$

$C_{i+1} = \{ \text{finite covers of } S_g \text{-bundles over elts of } C_i, \ g \geq 2 \}$

e.g. $C_1 = \{ S_g : \ g \geq 2 \}$

Choose $E \in C_i$ surf. bundle with $e_i(E) \neq 0$. Note: $e_0$ always $\neq 0$, which is why you can use the trivial bundle in Atiyah's construction. Will use to construct $\tilde{E} \in C_{i+1}$ with $e_{i+1}(\tilde{E}) \neq 0$.

Step 1. $C_i \to C_{i+1}$

Given $S_g$-bundle, $\pi: E \to M$

$\Rightarrow E^* = \pi^*(E) = \{ (u, u') \in E \times E : \pi(u) = \pi(u') \}$

Bundle structure: $\pi' : E^* \to E$

$(u, u') \mapsto u$

Have a bundle map:

$$
\begin{array}{ccc}
E^* & \xrightarrow{\phi} & E \\
\downarrow \pi' & & \downarrow \pi \\
E & \xrightarrow{\pi} & M
\end{array}
$$

$E^*$ comes with a section $\Delta = \{ (u, u) \}$, which intersects each fiber in one point.

Write $V$ for $\Delta^* \in H^2(E^*; \mathbb{Z})$

$V_m \in H^2(E^*; \mathbb{Z}_m)$ the mod $m$ reduction
Example. \( E = S^g, M = \ast. \)
\[ \implies E^* = S^g \times S^g, \quad \Delta = \text{usual diagonal.} \]

Step 2. Given an \( S^g \)-bundle \( E \to M \)

\[ \exists \text{ finite cover } M_1 \to M \]
\[ \text{s.t. } p^*(E) \text{ admits } m\text{-fold (unbranched) cover along fibers.} \]

Note. Step 2 not needed in \( \ast \) case since \( S^g \times S^g \to S^g \) admits
\[ m\text{-fold cover over fibers for any } m. \]

Pf. Pick any \( m\)-fold \( \tilde{S}^g \to S^g \)

Denote \( h : M \to \text{MCG}(S^g) \) the monodromy.

Goal: Construct a cover \( \tilde{M} \to M \) and a monodromy
\[ \tilde{h} : \tilde{M} \to \text{MCG}(\tilde{S}^g) \]
\[ \text{s.t. } \tilde{h}(x) \text{ is a lift of } h(x) \quad \forall x \in \pi_1(\tilde{M}). \]

Then check: the combination of the two covering maps
(of base and fiber) give a covering map of bundles.*

Need two facts about \( \text{MCG} : \)
\[ 1. \ \text{Out}(\pi_1(S^g)) = \text{MCG}^+(S^g) \]
\[ 2. \ \text{MCG}(S^g) \text{ has torsion-free subgroup} \]
\[ \text{of finite index, e.g. } \ker (\text{MCG}(S^g) \to \text{Sp}(2g, \mathbb{Z})) \]

monodromy of \( f^* \)

* In general, \( \pi_1 \text{pullback} \) is given by composition of \( f_* \) (on \( \pi_1 \)) with
original monodromy.

Cover along fibers given by lifting monodromy to \( \text{MCG} \) of cover.
Choose \( \tilde{\Gamma} \leq \text{Aut } \pi_1(S_g) \) finite index, preserves \( \pi_1(S_g) \)

\[ \Rightarrow r : \tilde{\Gamma} \rightarrow \text{Aut } \pi_1(S_g) \rightarrow \text{MCG}(\tilde{S}_g) \]

Note: \( r(\tilde{\Gamma}, \text{Inn } \pi_1(S_g)) \) consists of torsion since any \( x \in \pi_1(S_g) \) has a power in \( \pi_1(S_g) \), which then is an inner aut of \( \pi_1(S_g) \).

\[ \Rightarrow \exists \; \tilde{\Gamma}_2 < \tilde{\Gamma}_1 \text{ finite index s.t. } \tilde{\Gamma}_2 \cap \text{Inn } \pi_1(S_g) = 1. \]

(using \( \boxtimes \) above).

\[ \Rightarrow \tilde{\Gamma}_2 = \pi_1(S_g) \text{ finite index in } \text{MCG}(S_g) \]

\[ \Rightarrow \tilde{\Gamma}_2 < \text{MCG}(S_g) \text{ finite index (intersect all conjugates of } \tilde{\Gamma}_2 \}) \]

and \( \tilde{\Gamma}_2 \rightarrow \text{MCG}(\tilde{S}_g) \) is well defined.

Let \( \tilde{M} \rightarrow M \) be the cover given by

\[ \pi_1(M) \rightarrow \text{MCG}(S_g) / \tilde{\Gamma}_2 \]

Then \( \tilde{h} : \tilde{M} \rightarrow \text{MCG}(\tilde{S}_g) \) given by

\[ \pi_1(\tilde{M}) \rightarrow \tilde{\Gamma}_2 \rightarrow \text{MCG}(\tilde{S}_g). \]

In other words, we showed: Given \( S_g \rightarrow S_g \), \( \exists \) finite index \( \Gamma \leq \text{MCG}(S_g) \) and a \( \Gamma \rightarrow \text{MCG}(\tilde{S}_g) \) where each \( f \in \Gamma \) maps to a lift of \( f \).

Then if the original bundle \( E \) has monodromy \( g : \pi_1(M) \rightarrow \text{MCG}(S_g) \)

the monodromy cover of \( M \) is the one corresponding to \( g\tilde{\Gamma}(\tilde{M}) \) and the monodromy after taking the fiberwise cover is \( g\tilde{\Gamma}(\tilde{M}) \rightarrow \pi_1(M) \rightarrow \tilde{\Gamma} \rightarrow \text{MCG}(\tilde{S}_g). \)
Step 3. \( E \in C_n, \quad \Delta \in H^2(E) \quad \text{all coeff} = \mathbb{Z}/m\mathbb{Z} \)

Then \( \exists \) finite cover \( \widetilde{E} \to E \) s.t. \( p^*(\Delta) = 0 \).

Induct on \( n \).

Reduce to case \( E = S_g \)-bundle by taking pullbacks.

Apply Step 2, then take \( m \)-fold fitewise cover.

Take another pullback to kill action on \( H^i(\text{fiber}) \)

and kill \( H^i(\text{base}) \)

\[
\begin{array}{cccccc}
E_2^* & \to & (E_1^*)' & \to & E_1^* & \to & E \\
\downarrow \pi & & \downarrow S_g & & \downarrow S_g & & \downarrow S_g \\
E_2 & \to & E_1 & \to & E_1 & \to & M \\
\end{array}
\]

Denote \( E_2^* \to E \) by \( p_0^* \)

Claim: \( \exists \nu \in H^2(E_2) \) s.t. \( p_0^*(\Delta) = \pi^*(\nu) \)

Pf: Sere spectral seq (below)

By induction, \( \exists \) finite cover \( \widetilde{E} \to E_2 \) s.t. \( \nu_1 \to 0 \)

in \( H^2(\widetilde{E}) \):

\[
\begin{array}{c}
E_3^* \to E_2^* \\
\downarrow & \downarrow \\
\widetilde{E} & \to E_2
\end{array}
\]

By commutativity, the result follows.
SERRE SPECTRAL SEQUENCE

Want to prove claim. Write $F \to E \to B$ for $Sg \to E_2^* \to E_2$

Page 2 of Serre SS:

By construction, all $\mathbb{Z}/m$ coeffs are trivial.

$H^0(B; H^0(F)) \to H^1(B; H^0(F)) \to H^2(B; H^2(F))$

$H^0(B; H^1(F)) \to H^1(B; H^1(F)) \to H^2(B; H^1(F))$

$H^0(B; H^0(F)) \to H^1(B; H^0(F)) \to H^2(B; H^0(F))$

The Serre SS package gives two things

1. There is a filtration $F_2 \subseteq F_1 \subseteq F_0 = H^2(E)$ s.t.

   $F_i / F_{i+1} \cong E_{0, 2-i}^{i, i}$

2. The map

   $H^2(E) \to E_{0, 2}^{0, 2} \to E_{2}^{0, 2} = H^2(F)$

   is the one induced by $F \to E$.

   (The map $H^2(E) \to E_{0, 2}^{0, 2}$ comes from 1, the other map comes from the SS)

What are the $F_i$?

- $F_2 / F_3 = F_2 = \frac{1}{2} E_{0, 2}^{2, 0}$
- $F_1 / E_{0, 2}^{2, 0} = E_{0, 1}^{1, 1}$
- $H^2(E) / F_1 = E_{0, 2}^{0, 2}$

Still need to determine $F_1$. Have:

$1 \to F_1 \to H^2(E) \to E_{0, 2}^{0, 2} \to 1$

The term $E_{0, 2}^{0, 2}$ is a subgap of $E_{2}^{0, 2}$ (it is the kernel of the differential shown above). So by 2,

$F_1 = K = \ker (H^2(E) \to H^2(F))$
In other words, we have two short exact seqs:

\[ 1 \to K \to H^2(E) \to E^{0,2}_\infty \to 1 \]

\[ 1 \to E^{2,0}_\infty \to K \to E^{1,1}_\infty \to 1 \quad \leftarrow \text{typo in Morita!} \]

Recall, we have \( p^*_o(\Delta) \in H^2(E) \), we want to show it lives in \( E^{2,0}_\infty = H^2(B) \).

**Step 1.** Image of \( p^*_o(\Delta) \) in \( E^{0,2}_\infty \) is 0, i.e. \( p^*_o(\Delta) \in K \).

Recall we took an m-fold fiberwise cover

\[
\begin{array}{ccc}
S_g & \to & S_g \\
\downarrow & & \downarrow \\
E^*_2 & \to & E^* \\
\end{array}
\]

The map \( H^2(S_g) \to H^2(S_g') \) is zero.

The map \( H^2(E^*_2) \to E^{0,2}_\infty \) is the map \( H^2(E^*_2) \to H^2(S_g') \)

Use commutativity.

**Step 2.** Image of \( p^*_o(\Delta) \) in \( E^{1,1}_\infty \) is 0, i.e. \( p^*_o(\Delta) \in E^{2,0}_\infty = H^2(B) \)

Recall we arranged \( \beta \) s.t. \( H'(E) \to H'(E_2) \) is zero.