Construction of S_g-Bundles with $e_i \neq 0$

$$
\begin{tikzcd}
\tilde{E}^* \arrow[r] \arrow[d, "S_{g''}"] & E_3^* \arrow[r] \arrow[d, "S_{g'}"] & E_2^* \arrow[r] \arrow[d, "S_{g'}"] & (E_1^*)' \arrow[r] \arrow[d, "S_{g'}"] & E_1^* \arrow[r] \arrow[d, "S_{g^*}"] & E^* \arrow[d, "S_g"] & E \arrow[d, "S_g"] \\
\tilde{E} \arrow[r] & \tilde{E} \arrow[r] & E_2 \arrow[r] & E_1 \arrow[r] & E_1 \arrow[r] & E & M
\end{tikzcd}
$$

$D \subset E_3^*$, $\Delta \subset E^*$

Induction: Start with $e_i(E) \neq 0$. $\dim(M) = 2i$

diagonal construction: bump up dim by 2, new bundle has diagonal section Δ. (1)

take cover over base so pullback bund. has m-fold fiberwise cover (2)

take the m-fold fiberwise cover

Kill action of $\pi_1(\text{base})$ on $H^1(\text{fiber}; \mathbb{Z}_m)$ and ensure $H^1(E_1) \to H^1(E_2)$ is 0 mod m.

now can take cover of base so preimage of Δ is divis. by m (3)

fiberwise cyclic branched cover over D

The steps labeled (1), (2), (3) are the new ones.

Morita calls this the m-construction on $E \to M$.
Atiyah's construction is 2-construction on $S_g \to pt$.

Prop $e_{i+1}(\tilde{E}^*) = -dm^2(1 - m^{-(i+2)})\, e_i(E)$ $\qquad d = \text{degree of } \tilde{E} \to E$

The proof is analogous to that of: $e_1(\tilde{M}) = 2e_1(M) - i(\tilde{D}, \tilde{D})$ above.

So $e_i(E) \neq 0 \Rightarrow e_{i+1}(\tilde{E}^*) \neq 0$.

Higher Dimensional Surface Bundles

Goal. $e_i \neq 0 \ \forall i$.

Iterated surface bundles. $C_0 = \{*\}$

$C_{i+1} = \{$finite covers of S_g-bundles over elts of C_i, $g \geq 2\}$

e.g. $C_1 = \{S_g : g \geq 2\}$

Choose $E \in C_i$ surf. bundle with $e_i(E) \neq 0$.
Will use to construct $\tilde{E} \in C_{i+1}$ with $e_{i+1}(\tilde{E}) \neq 0$.

note: e_0 always $\neq 0$, which is why you can use the trivial bundle in Atiyah's construction.

Step 1. $C_i \to C_{i+1}$

Given S_g-bundle $\pi : E \to M$

$\rightsquigarrow E^* = \pi^*(E) = \{(u,u') \in E \times E : \pi(u) = \pi(u')\}$

Bundle structure: $\pi' : E^* \to E$

$(u,u') \mapsto u$

Have a bundle map:

$$\begin{tikzcd} E^* \arrow[r, "q"] \arrow[d, "\pi'"'] & E \arrow[d, "\pi"] \\ E \arrow[r, "\pi"] & M \end{tikzcd} \qquad q(u,u') = u'$$

E^* comes with a section $\Delta = \{(u,u)\}$, which intersects each fiber in one point.

Write ν for $\Delta^* \in H^2(E^*;\mathbb{Z})$

$\nu_m \in H^2(E^*;\mathbb{Z}_m)$ the mod m reduction example. $E = S_g$, $M = *$.

$\rightsquigarrow E^* = S_g \times S_g$, Δ = usual diagonal.

Step 2. Given an S_g-bundle $E \to M$

$\exists$ finite cover $M_1 \xrightarrow{p} M$

s.t. $p^*(E)$ admits m-fold (unbranched) cover along fibers.

Note. Step 2 not needed in e_1 case since $S_g \times S_g \to S_g$ admits m-fold cover over fibers for any m.

Pf. Pick any m-fold $\tilde{S}_g \to S_g$

Denote $h : M \to MCG(S_g)$ the monodromy.

Goal: Construct a cover $\tilde{M} \to M$ and a monodromy

$\tilde{h} : \tilde{M} \to MCG(\tilde{S}_g)$ s.t.

$\tilde{h}(\alpha)$ is a lift of $h(\alpha)$ $\forall\ \alpha \in \pi_1(\tilde{M})$.

Then check: the combination of the two covering maps (of base and fiber) give a covering map of bundles.*

Need two facts about MCG: ① $\mathrm{Out}\,\pi_1(S_g) = MCG^{\pm}(S_g)$

② $MCG(S_g)$ has torsion free subgp of finite index, eg.

$\ker(MCG(S_g) \longrightarrow Sp(2g, \mathbb{Z}_3))$

* In general, monodromy of f^* pullback is given by composition of f_* (on π_1) with original monodromy.

Cover along fibers given by lifting monodromy to MCG of cover.

Choose $\tilde{\Gamma}_1 \leq \mathrm{Aut}\,\pi_1(S_g)$ finite index, preserves $\pi_1(\tilde{S}_g)$

$\rightsquigarrow$ $r: \tilde{\Gamma}_1 \longrightarrow \mathrm{Aut}\,\pi_1(\tilde{S}_g) \longrightarrow MCG(\tilde{S}_g)$

note: $r(\tilde{\Gamma}_1 \cap \mathrm{Inn}\,\pi_1(S_g))$ consists of torsion since any $\alpha \in \pi_1(S_g)$ has a power in $\pi_1(\tilde{S}_g)$, which then is an inner aut of $\pi_1(\tilde{S}_g)$.

$\Rightarrow \exists\ \tilde{\Gamma}_2 < \tilde{\Gamma}_1$ finite index s.t. $\tilde{\Gamma}_2 \cap \mathrm{Inn}\,\pi_1(S_g) = 1$.

(using ② above).

$\Rightarrow \Gamma_2 = \pi(\tilde{\Gamma}_2)$ finite index in $MCG(S_g)$

$\rightsquigarrow \Gamma_3 \lhd MCG(S_g)$ finite index (intersect all conjugates of Γ_2)

and $\Gamma_3 \longrightarrow MCG(\tilde{S}_g)$ is well defined.

Γ_3 not needed unless we want a reg. cover.

Let $\tilde{M} \to M$ be the cover given by

$$\pi_1(M) \longrightarrow MCG(S_g)/\Gamma_3$$

Then $\tilde{h}: \tilde{M} \longrightarrow MCG(\tilde{S}_g)$ given by

$$\pi_1(\tilde{M}) \longrightarrow \Gamma_3 \longrightarrow MCG(\tilde{S}_g).$$

∎

In other words, we showed: Given $\tilde{S}_g \to S_g$, $\exists$ finite index $\Gamma < MCG(S_g)$ and a $\Gamma \to MCG(\tilde{S}_g)$ where each $f \in \Gamma$ maps to a lift of f.

Then if the original bundle E has monodromy $\rho: \pi_1(M) \longrightarrow MCG(S_g)$ the ~~monodromy~~ cover of M is the one corresponding to $\rho^{-1}(\Gamma)$ and the monodromy after taking the fiberwise cover is $\rho^{-1}(\Gamma) \hookrightarrow \pi_1(M) \longrightarrow \Gamma \longrightarrow MCG(\tilde{S}_g)$.

Step 3. $E \in C_n$, $\Delta \in H^2(E)$ all coeff $= \mathbb{Z}/m\mathbb{Z}$

Then $\exists$ finite cover $\tilde{E} \xrightarrow{p} E$ s.t. $p^*(\Delta) = 0$.

Induct on n.

Reduce to case $E = S_g$-bundle by taking pullbacks.

Apply Step 2, then take m-fold fiberwise cover.

Take another pullback to kill action on H^1(fiber)

and kill H^1(base)

$$
\begin{tikzcd}
 & & & \Delta \arrow[d, phantom, "\rotatebox{-90}{$\in$}"] \\
E_2^* \arrow[r] \arrow[d, "\pi"', "S_{g'}"] & (E_1^*)' \arrow[r] \arrow[d, "S_{g'}"] & E_1^* \arrow[r] \arrow[d, "S_g"] & E \arrow[d, "S_g"] \\
v \quad E_2 \arrow[r] & \tilde{E}_1 \arrow[r] & E_1 \arrow[r] & M
\end{tikzcd}
$$

Denote $E_2^* \to E$ by p_0^*

Claim: $\exists$ $v \in H^2(E_2)$ s.t. $p_0^*(\Delta) = \pi^*(v)$

Pf: Serre spectral seq (below)

By induction, $\exists$ finite cover $\tilde{E} \to E_2$ s.t. $v \mapsto 0$ in $H^2(\tilde{E})$:

$$
\begin{tikzcd}
E_3^* \arrow[r] \arrow[d] & E_2^* \arrow[d] \\
\tilde{E} \arrow[r] & E_2
\end{tikzcd}
$$

By commutativity, the result follows.

Serre Spectral Sequence

Want to prove claim. Write $F \to E \to B$ for $S_g \to E_2^* \to E_2$

Page 2 of Serre SS:

By construction, all $\mathbb{Z}/m$ coeffs are trivial.

$$\begin{array}{|lll}
H^0(B;H^2(F)) & H^1(B;H^2(F)) & H^2(B;H^2(F)) \\
H^0(B;H^1(F)) & H^1(B;H^1(F)) \to & H^2(B;H^1(F)) \\
H^0(B;H^0(F)) & H^1(B;H^0(F)) \to & H^2(B;H^0(F)) \\
\hline
\end{array}$$

The Serre SS package gives ~~three~~ two things

① There is a filtration $F_2 \subseteq F_1 \subseteq F_0 = H^2(E)$ s.t.

$$F_i/F_{i+1} \cong E_\infty^{i,2-i}$$

② The map

$$H^2(E) \to E_\infty^{0,2} \to E_2^{0,2} = H^2(F)$$

is the one induced by $F \hookrightarrow E$.

(the map $H^2(E) \to E_\infty^{0,2}$ comes from ①, the other map comes from the SS)

What are the F_i?

- $F_2/F_3 = F_2 = E_\infty^{2,0}$
- $F_1/E_\infty^{2,0} = E_\infty^{1,1}$
- $H^2(E)/F_1 = E_\infty^{0,2}$

Still need to determine F_1. Have:

$$1 \to F_1 \to H^2(E) \to E_\infty^{0,2} \to 1$$

The term $E_\infty^{0,2}$ is a subgp of $E_2^{0,2}$ (it is the kernel of the differential shown above). So by ②,

$$F_1 = K = \ker\left(H^2(E) \to H^2(F)\right)$$

In other words, we have two short exact seqs:

$$1 \to K \to H^2(E) \to E_\infty^{0,2} \to 1$$

$$1 \to E_\infty^{2,0} \to K \to E_\infty^{1,1} \to 1 \qquad \leftarrow \text{typo in Morita!}$$

Recall, we have $p_0^*(\Delta) \in H^2(E)$, we want to show it lives in $E_\infty^{2,0} = H^2(B)$.

Step 1. Image of $p_0^*(\Delta)$ in $E_\infty^{0,2}$ is 0, i.e. $p_0^*(\Delta) \in K$.

Recall we took an m-fold fiberwise cover

$$\begin{tikzcd}
S_{g'} \arrow[r] \arrow[d] & S_g \arrow[d, "\Delta"] \\
E_2^* \arrow[r] & E^*
\end{tikzcd}$$

The map $H^2(S_g) \to H^2(S_{g'})$ is zero.

The map $H^2(E_2^*) \to E_\infty^{0,2}$ is the map $H^2(E_2^*) \to H^2(S_{g'})$

Use commutativity.

Step 2. Image of $p_0^*(\Delta)$ in $E_\infty^{1,1}$ is 0, i.e. $p_0^*(\Delta) \in E_\infty^{2,0} = H^2(B)$

Recall we arranged s.t. $H^1(E) \to H^1(E_2)$ is zero.