Algebraic Independence of the MMMs

Thm. Fix n. $\exists\, g$ s.t.

$$\mathbb{Q}[e_1, e_2, \ldots] \longrightarrow H^*(MCG(S_g^1); \mathbb{Q})$$

is injective up to degree $2n$ (in fact $g = 3n$).

i.e. $\mathbb{Q}[e_1, e_2, \ldots] \hookrightarrow H^*(MCG(S_\infty^1))$

Pf. Choose $g_1, \ldots, g_n$ s.t. $e_i \in MCG(S_{g_i}^1)$ is nonzero $i = 1, \ldots, n$.
(i.e. do our bundle construction for surfaces with boundary)
Choose d_j s.t. $j d_j \geq n$, set $g = \sum d_j g_j$

$$\rightsquigarrow \iota : MCG(S_{g_1}^1)^{d_1} \times \cdots \times MCG(S_{g_n}^1)^{d_n} \hookrightarrow MCG(S_g^1)$$

Fact: $\iota^*(e_i) = \sum_{j=1}^{n} p_j^*(e_i)$ $\qquad p_j$ = proj to jth factor

(the point is that the euler classes live in separate subbundles).

Now just apply the Künneth formula. The image of any polynomial of deg $\leq 2n$ will have one term in the direct sum of the form

$$e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_N} \otimes 1 \otimes \cdots \otimes 1$$

which is $\neq 0$ by construction.

Computing H_2.

- First show e_1 generates a $\mathbb{Z}$ in $H^2(MCG(S_g))$ $g \geq 3$.
- Then use Hopf formula, to show $H^2(MCG(S_g))$ is a quotient of $\mathbb{Z}$ for $g \geq 4$ and of $\mathbb{Z} \oplus \mathbb{Z}_2$ for $g = 3$.
- Remains to show $H^2(MCG(S_3)) = \mathbb{Z} \oplus \mathbb{Z}_2$.

There is: $1 \to \mathcal{I}(S_3) \to MCG(S_3) \to Sp_6(\mathbb{Z}) \to 1$

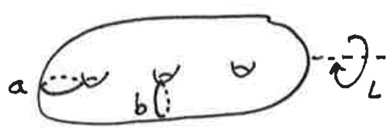
$\rightsquigarrow$ 5-term sequence:

$$H_2(MCG(S_3)) \to H_2(Sp_6(\mathbb{Z})) \to H_1(\mathcal{I}(S_3))_{Sp_6(\mathbb{Z})} \to H_1(MCG(S_3)) \to H_1(Sp_6(\mathbb{Z}))$$

But: $H_1(MCG(S_3)) = 0$.

$H_2(Sp_6(\mathbb{Z})) = \mathbb{Z} \oplus \mathbb{Z}_2$ Stein '75.

Remains: $H_1(\mathcal{I}(S_3))_{Sp_6(\mathbb{Z})} \cong \mathcal{I}(S_3)/[MCG(S_3), \mathcal{I}(S_3)] \cong 1.$ Johnson '79

Pf. 

and choose $h \in MCG(S_3)$ s.t. $h(b) = a$.

In $\mathcal{I}/[MCG, \mathcal{I}]$:

$$\begin{aligned} [T_b, \iota] &= h[T_b, \iota]h^{-1} \quad \text{since } [T_b, \iota] \in \mathcal{I}(S_3) \\ &= [hT_bh^{-1}, h\iota h^{-1}] \\ &= [T_a, \iota[\iota^{-1}, h]] \\ &= [T_a, \iota]\, \iota [T_a, [\iota^{-1}, h]] \iota^{-1} \\ &= 1 \quad \text{since } T_a \leftrightarrow \iota \\ &\qquad \text{and } \iota \leftrightarrow h \text{ in } Sp. \\ &\qquad (\text{so } [\iota^{-1}, h] \in \mathcal{I}). \end{aligned}$$

Benson-Cohen: $H_2(MCG(S_2))$ consists of 2,3,5-torsion only.