Madsen-Weiss Theorem

We know $\mathbb{Q}[e_1, e_2, \ldots] \hookrightarrow H^*(MCG(S_\infty^1))$
Want to show this is surjective
Will do this by relating $H^*(MCG(S_\infty^1))$ to a "familiar" space.

$S_\infty =$  $\cdots$
$\mathcal{G}_{S_g^1} =$ space of subsurfaces of $(0,g] \times \mathbb{R}^\infty$ diffeo to S_g^1 and that agree on ∂S_g^1 with a fixed embedding of S_∞.
$= K(MCG(S_g^1), 1)$
$\mathcal{G}_{S_g^1} \hookrightarrow \mathcal{G}_{S_{g+1}^1} \rightsquigarrow \mathcal{G}_{S_\infty} = \bigcup \mathcal{G}_{S_g^1}$
Harer stability $\Rightarrow H_i(\mathcal{G}_{S_\infty}) = \lim_g H_i(\mathcal{G}_{S_g^1}) = \lim_g H_i(MCG(S_g^1))$

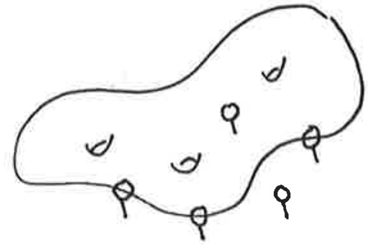
$AG_{n,d}$ = affine Grassmannian of d-planes in $\mathbb{R}^n$
$\cong G_{n,d}^\perp$ since affine plane determined by plane thru 0 & $\perp$ vector
$AG_{n,d}^+$ = 1-pt comp
$\cong$ Thom space for $G_{n,d}^\perp$ when $n < \infty$.

Theorem. $H_*(\mathcal{G}_{S_\infty}) \cong H_*(\Omega_0^\infty AG_{\infty,2}^+)$ basept @ ∞

In general, the $\mathbb{Q}$-cohomology of a loop space is a tensor product of a polynomial algebra on even-dim gens and an exterior alg. on odd-dim gens (assuming the loop space is path conn and has f.g. $\mathbb{Z}$-homology in each dim).

Scanning Map

Take some point in G_{n,S_g} :



With a small lens we either see an almost-flat 2-plane or $\emptyset$.
If we identify the lens with $\mathbb{R}^n$, get a pt in $AG^+_{n,2}$ (slope is same as in lens but position of plane given by lens $\to \mathbb{R}^n$).
Near ∞, lens sees $\emptyset$ $\rightsquigarrow$

$$S^n = \mathbb{R}^n \cup \{\infty\} \longrightarrow AG^+_{n,2}$$

i.e. a point in $\Omega^n AG^+_{n,2}$

As we move in G_{n,S_g} can vary the size of the lens continuously.
As we let n increase, have :

$$\begin{tikzcd} G_{n,S_g} \arrow[r, hook] \arrow[d] & G_{n+1,S_g} \arrow[d] \\ \Omega^n AG^+_{n,2} \arrow[r] & \Omega^{n+1} AG^+_{n+1,2} \end{tikzcd}$$

where bottom row obtained by applying Ω^n to the map $\quad AG^+_{n,2} \longrightarrow \Omega AG^+_{n+1,2}$
obtained by translating a plane from $-\infty$ to ∞ in $n+1^{st}$ coord.
Taking limit over n:

$$G_{S_g} \longrightarrow \Omega^\infty AG^+_{\infty,2} \qquad \text{"Scanning map"}$$

Note that the target does not depend on g, which is why we should expect to consider some limit over g in order to get an isomorphism.

A First Outline

Fix d (for us $d=2$)

C^n = space of all smooth, oriented d-dim submanifolds of $\mathbb{R}^n$ that are properly embedded (maybe disconn, open, empty).

Topology: pts are close if they are close in C^∞ top. on a large ball

Note C^n is path conn: radial expansion from a pt not on the manifold gives a path to the empty manifold.

Prop. $C^n \simeq AG^+_{n,d}$

Pf. Want to rescale from 0, but this is not continuous since we can push a manifold off 0, changing image from nonempty plane to empty plane.

Fix: For $M \in C^n$ choose tub. nbd $N = N(M)$ continuously.

If $0 \notin N$, rescale as above.

If $0 \in N$, rescale in tangent dir from $1 \longrightarrow \infty$ as before

in normal dir $1 \longrightarrow \lambda$ where

$\lambda = 1$ near 0-sec, $\lambda = \infty$ near frontier.

This takes $AG^+_{n,d}$ to itself. ▨

Filter C^n by $C^{n,0} \subseteq C^{n,1} \subseteq \cdots \subseteq C^{n,n} = C^n$

where $C^{n,k}$ = subspace of C^n consisting of manifolds lying in $\mathbb{R}^k \times (0,1)^{n-k}$

i.e. manifolds that extend to ∞ in only k directions.

There is: $C^{n,k} \longrightarrow \Omega C^{n,k+1}$ by translating from $-\infty$ to ∞ in $(k+1)$st coord.

Putting these together:

$$\mathcal{C}^{n,0} \longrightarrow \Omega\mathcal{C}^{n,1} \longrightarrow \Omega^2\mathcal{C}^{n,2} \longrightarrow \cdots \longrightarrow \Omega^n\mathcal{C}^n$$

The composition takes a compact manifold and translates it to ∞ in all directions. (can think of this as scanning with an ∞'ly large lens; shrinking the lens gives a homotopy to the original scanning map).

Would like: $\mathcal{C}^{n,k} \longrightarrow \Omega\mathcal{C}^{n,k+1}$ is a homotopy equivalence.

Easier: $k>0$ case. works for any $d \geq 0$.

Harder: $k=0$ case. when $d=2$, works after passing to limits where $n,g \to \infty$. uses group completion theorem. only get a homology equivalence:

$$H_*(\mathcal{C}_\infty) \cong H_*(\Omega_0\mathcal{C}^{\infty,1}).$$

So the main thread for the MW Thm is:

$$\begin{aligned} H_*(\mathcal{C}_\infty) &\cong H_*(\Omega_0\mathcal{C}^{\infty,1}) && \text{harder delooping} \\ &\cong \lim H_*(\Omega_0\mathcal{C}^{n,1}) \\ &\cong \lim H_*(\Omega_0^n\mathcal{C}^n) && \text{easier delooping} \\ &\cong \lim H_*(\Omega_0^n AG_{n,2}^+) && \text{above Prop.} \\ &\cong H_*(\Omega_0^\infty AG_{\infty,2}^+) \end{aligned}$$

DELOOPING – THE EASIER CASE

Want: $C^{n,k} \simeq \Omega C^{n,k+1}$ $\quad k > 0$.

Road map: $C^{n,k} \simeq M^{n,k} \simeq \Omega B M^{n,k} \simeq \Omega C_0^{n,k+1}$

Step 1. $M^{n,k} = \{ (M,a) \in C^n \times [0,\infty) : M \subseteq \mathbb{R}^k \times (0,a) \times (0,1)^{n-k-1} \}$

This is a monoid version of $C^{n,k}$, analogous to the Moore loopspace, which is a monoid version of ΩX.

The map
$$\begin{aligned} C^{n,k} &\longrightarrow M^{n,k} \\ M &\longmapsto (M,1) \end{aligned}$$
is a homotopy equivalence.

Step 2. $M^{n,k} \simeq \Omega B M^{n,k}$

A topological monoid M has a classifying space BM

Construction is analogous to group case: p-simplices $\longleftrightarrow (m_1, \ldots, m_p)$

faces obtained by dropping m_1, m_p

& multiplying $m_i m_{i+1}$

There is a space of p-simplices with topology from $\coprod_p \Delta^p \times M^p$ and face identifications.

There is a map
$$\begin{aligned} M &\longrightarrow \Omega BM \\ m &\longmapsto (m) \end{aligned}$$

General fact: This is a hom. eq. when $\pi_0 M$ is a group with mult. coming from mult. in M.

So we want: $\pi_0 M^{n,k}$ is a group.

Prop. $\pi_0 C^{n,k} = \begin{cases} 0 & k > d \\ \Omega^{SO}_{d-k,\,n-k} & k \le d \end{cases}$

$\uparrow$ Cobordism group of closed, oriented $(d-k)$-manifolds in $\mathbb{R}^{n-k}$.

Pf. A point of $C^{n,k}$ is a d-mnfld $M \subseteq \mathbb{R}^n$ with $p : M \longrightarrow \mathbb{R}^k$ proper.

Can perturb M s.t. p is transverse to $0 \in \mathbb{R}^k$.

$k > d$: $p(M)$ misses 0. Expand radially from 0 in $\mathbb{R}^k$ to get path to empty manifold.

$k \le d$: $p^{-1}(0) = M \cap (\{0\} \times \mathbb{R}^{n-k}) = M_0 \rightsquigarrow [M_0] \in \Omega^{SO}_{d-k,n-k}$

$\rightsquigarrow \varphi : \pi_0 C^{n,k} \longrightarrow \Omega^{SO}_{d-k,n-k}$

$[M] \longmapsto [M_0]$

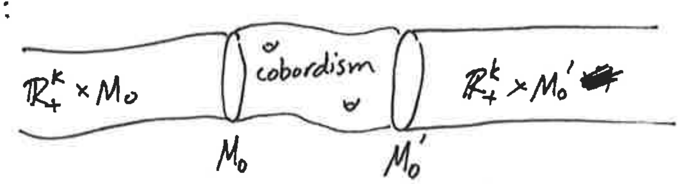
This is a homom since both group ops are disj. union.
and surjective since $[\mathbb{R}^k \times M_0] \longmapsto [M_0]$

Remains: φ injective.

First we claim any M is path conn to $\mathbb{R}^k \times M_0$ (first make M agree with $\mathbb{R}^k \times M_0$ on a nbd of M_0, then expand radially)

Now if $\varphi([M]) = [M_0]$ equals $\varphi([M']) = [M_0']$
can assume $M = \mathbb{R}^k \times M_0$, $M' = \mathbb{R}^k \times M_0'$ and $M_0 \sim M_0'$

Build a manifold:



Translating right gives path to $\mathbb{R}^k \times M_0$,
and left gives path to $\mathbb{R}^k \times M_0'$ so $[M] = [M']$ in $\pi_0 C^{n,k}$. ∎

STEP 3. $BM^{n,k} \simeq C_0^{n,k+1}$

We will define a natural map $\nabla : BM^{n,k} \longrightarrow C_0^{n,k+1}$

A point in $BM^{n,k}$ is given by $(m_1, \ldots, m_p) \in (M^{n,k})^p$, $(w_0, \ldots, w_p)$

A stupid map (ignoring the w_i) is:

$(m_1, \ldots, m_p) \longmapsto m_1 m_2 \cdots m_p = \bigcup M_i$ where M_i is a manifold with $(k+1)^{st}$ coord in $[a_{i-1}, a_i]$

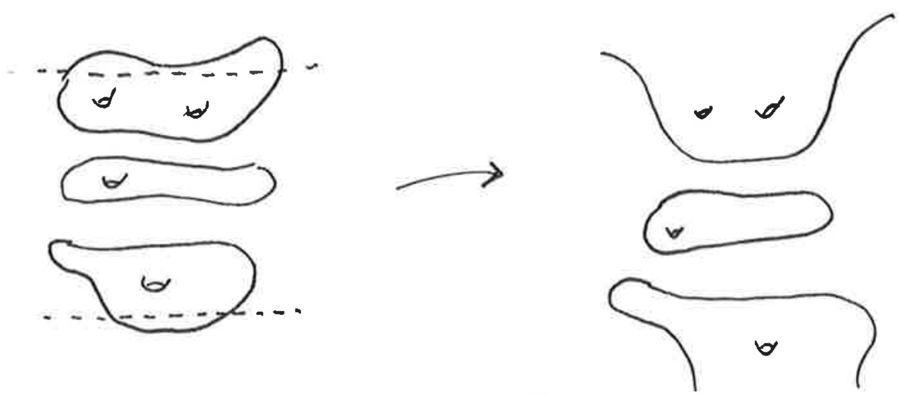
This map is not continuous upon passage to faces:

① When w_0 or $w_p \to 0$, M_1 or M_p suddenly deleted.

② When $w_0 \longrightarrow 0$ $m_2 \cdots m_p$ suddenly shifts by $a_1 - a_0$ in $(k+1)^{st}$ coord

Can easily address ②: translate in $(k+1)^{st}$ coord so barycenter $b = \sum w_i a_i$ equals 0.

Idea for ①: truncate M_1, M_p a little at a time.



precisely: $a_i^+ = \max\{a_i, b\}$ $b^+ = \sum w_i a_i^+$ "upper & lower barycenters"

$a_i^- = \min\{a_i, b\}$ $b^- = \sum w_i a_i^-$

$\nabla(m_1 \cdots m_p)$ obtained by stretching $\mathbb{R}^k \times (b^-, b^+) \times \mathbb{R}^{n-k-1}$ to $\mathbb{R}^k \times \mathbb{R} \times \mathbb{R}^{n-k-1}$

Need to check ∇ is $\cong$ on π_q $\forall q$.