Vector Bundles

Fix a vector space V

$$\begin{array}{ccc} V & \to & E \\ & & \downarrow p \\ & & B \end{array}$$

① Fibers $p^{-1}(b)$ have structure of V.

② B covered by $\mathcal{U}$ s.t. $\exists$

$p^{-1}(U) \longrightarrow U \times V$ homeo resp. structure on fibers.

local trivialization

Examples

① Trivial bundle $E = B \times V$.

② Möbius bundle over S^1.

③ Tangent bundle to a smooth manifold M

$TM = \{(x,v) : v \in T_x M\}$

$p(x,v) = x$

V.S. structure: $k_1(x,v_1) + k_2(x,v_2) = (x, k_1 v_1 + k_2 v_2)$

By defn, M locally diffeo to $U \subseteq \mathbb{R}^n$ open.
So suffices to show TU locally trivial. easy

④ Normal bundle to $M \hookrightarrow N$

Locally: $\mathbb{R}^n \hookrightarrow \mathbb{R}^{n+k}$ (Tubular nbhd thm).

⑤ Canonical bundle over $\mathbb{R}P^n$

$\mathbb{R}P^n$ = space of lines in $\mathbb{R}^{n+1}$ $\cong S^n/\text{antipode}$
Canonical line bundle: $\{(\ell, v) : v \in \ell\}$
Local trivialization near ℓ: orthog. proj. to ℓ in $\mathbb{R}^{n+1}$.
e.g. $(\ell', v) \mapsto (\ell', \mathrm{proj}_\ell(v)) \in U \times \ell$.

Allow $n = \infty$.

⑥ Orthogonal complement to ⑤

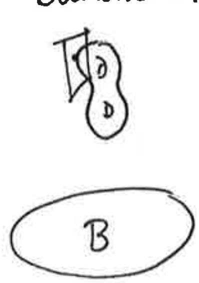
$E^\perp = \{(\ell, v) \in \mathbb{R}P^n \times \mathbb{R}^{n+1} : v \perp \ell\}$
Again, orthog proj gives local trivialization.

Q. $E^\perp \cong T\mathbb{R}P^n$?

⑦ Grassmann manifold

G_n = space of n-planes in $\mathbb{R}^\infty$ thru 0.
$E_n = \{(P, v) \in G_n \times \mathbb{R}^\infty : v \in P\}$
& $E_n^\perp = \{(P, v) \in G_n \times \mathbb{R}^\infty : v \perp P\}$

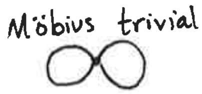
⑧ Vertical bundle of surface bundle

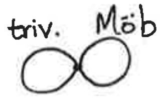


Char. classes for surface bundles defined in terms of char. classes for these vector bundles.

Isomorphism

$p_1 : E_1 \to B$ is isomorphic to $p_2 : E_2 \to B$
if $\exists$ homeo $h : E_1 \to E_2$ s.t. $h|_{p_1^{-1}(b)}$ is a v.s. $\cong$ to $p_2^{-1}(b)$.

N.B. 

$\ncong$ 

& bundles over different spaces can't be isomorphic(!)

Examples

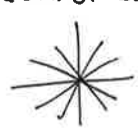
① $NS^n \cong S^n \times \mathbb{R}$
via $(x, tx) \mapsto (x, t)$

② $TS^1 \cong S^1 \times \mathbb{R}$
via $(z, izt) \mapsto (z, t)$

We say S^1 is <u>parallelizable</u> 

Q. Which manifolds are parallelizable? S^2?
(All 3-manifolds!)

③ Cancn. line bundle over $\mathbb{RP}^1$ $\cong$ Möbius bundle over $\mathbb{RP}^1$
after traveling around base, fibers get flipped:



Q. Is $T\mathbb{RP}^n \cong E^\perp$?

SECTIONS

A section of $p: E \to B$ is $s: B \to E$ s.t. $p \circ s = id$.

e.g. 0-section

Some bundles have non~~zero~~ vanishing sections, some do not.

For example: A section of TM is a vector field on M.

We showed nonvan vect field $\Rightarrow \chi(M) = 0$.

So $\chi(M) \neq 0 \Rightarrow TM$ has no nonvan. sec.

e.g. $\chi(S^n) = 2$ n even.

Can show S^n has nonvan. vect field n odd.

FACT: An n-dim bundle is trivial $\Leftrightarrow$ it has n sections s_i that are lin. ind. over each point of B.

$\Rightarrow$ obvious

$\Leftarrow$ there is a contin. map

$$B \times \mathbb{R}^n \longrightarrow E$$

$$(b, t_1, \ldots, t_n) \longmapsto \sum t_i s_i(b)$$

clearly isom. on fibers

need to show inverse is continuous

follows from: inversion of matrices is continuous.

Spheres: TS^1 trivial by $s(z) = iz$

TS^3 trivial by $s_1(z) = iz$, $s_2(z) = jz$, $s_3(z) = kz$

TS^7 trivial by similar construction w/ octonians.

(all other TS^n nontrivial!)

Direct Sum

$p_1 : E_1 \to B,\ p_2 : E_2 \to B \rightsquigarrow$

$$E_1 \oplus E_2 = \{(v_1, v_2) \in E_1 \times E_2 : p_1(v_1) = p_2(v_2)\}$$

$$\begin{aligned} p : E_1 \oplus E_2 &\to B \\ (v_1, v_2) &\to p(v_1) \end{aligned}$$

$E_1 \oplus E_2$ a vector bundle because ① products of vb's are vb's
② restrictions of vb's are vb's.

$E_1 \oplus E_2$ is restriction of $E_1 \times E_2$ to diagonal $B \subseteq B \times B$.

Trivial $\oplus$ trivial = trivial <u>but</u>
Nontrivial $\oplus$ trivial can be trivial!

e.g. $\underbrace{TS^n \oplus NS^n}$ trivial. Say TS^n <u>stably</u> trivial.
$\quad\quad NS^n$: trivial

also: $E \oplus E^\perp \to \mathbb{R}P^n$ trivial via $(\ell, v, w) \mapsto (\ell, v+w)$
$n=1$ case: Möbius $\oplus$ Möbius = trivial

A useful exercise related to last example: Show there are exactly two $\mathbb{R}^n$ bundles over S^1. Similarly, exactly two S^1-bundles over S^1.

EXAMPLE. $T\mathbb{RP}^n$ stably isom. to $\bigoplus E$ ← canon. line bundle.

Start with $TS^n \oplus NS^n \cong S^n \times \mathbb{R}^{n+1}$

Quotient by $(x,v) \sim (-x,-v)$ on both sides.

$TS^n/\sim \;\cong T\mathbb{RP}^n$ since $(x,v) \mapsto (-x,-v)$ is map on TS^n induced by $x \mapsto -x$.

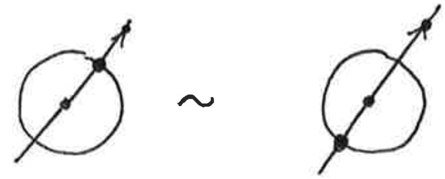
$NS^n/\sim \;\cong \mathbb{RP}^n \times \mathbb{R}$ via the section $x \mapsto (x,x)$

Claim: $(S^n \times \mathbb{R}^{n+1})/\sim \;\cong \bigoplus_{i=1}^{n+1} E$

First, $\sim$ preserves factors, so

$$(S^n \times \mathbb{R}^{n+1})/\sim \;\cong \bigoplus_{i=1}^{n+1} (S^n \times \mathbb{R})/\sim$$

But $(S^n \times \mathbb{R})/\sim \;\cong E$, as



Using quaternions, $T\mathbb{RP}^3 \cong \mathbb{RP}^3 \times \mathbb{R}^3$

~~As above~~ So $T\mathbb{RP}^3 \oplus$ trivial line bundle $\cong \mathbb{RP}^3 \times \mathbb{R}^4$

As above $T\mathbb{RP}^3 \oplus$ trivial line bundle $\cong \bigoplus_{i=1}^{4} E$

$$\Rightarrow \bigoplus_{i=1}^{4} E \cong \mathbb{RP}^3 \times \mathbb{R}^4.$$

NEXT GOAL

Prop. B = compact Hausdorff
$\forall\ E \to B\ \exists\ E' \to B$ s.t. $E \oplus E'$ trivial.

Step 1. Inner Products

Inner product on V: pos. def. symm. bilinear form.
Inner product on E: map $E \oplus E \to \mathbb{R}$ restricting to inner prod. on each fiber.

Paracompact: Hausdorff + every open cover admits a part. of unity.
Compact Hausdorff, CW complex, metric space $\Rightarrow$ paracompact

Prop. B paracompact $\Rightarrow$ $E \to B$ has an inner product.
Pf: Exercise.

Step 2. Orthogonal complements

Prop. B paracompact, $E_0 \to B$ subbundle of $E \to B$.
$\exists\ E_0^\perp$ s.t. $E_0 \oplus E_0^\perp \cong E$.

Pf. Choose inner product, $E_0^\perp$ = orthog. comp. in each fiber.
Need to check local triviality
Over $U \subseteq B$ choose m sections s_i for E_0, $n-m$ for E.
Apply Gram-Schmidt — continuous.
New sections trivialize E_0 & $E_0^\perp$ simultaneously. $\square$

Note: $E_0 \oplus E_0^\perp \cong E$ via FACT above.

To prove that any E has E' with $E \oplus E'$ trivial, it now suffices to show:

Prop. B = compact Hausdorff
Any $\mathbb{R}^n$-bundle $E \to B$ is a subbundle of $B \times \mathbb{R}^N$.

Pf. Choose: $U_1, \dots, U_k$ s.t. $p^{-1}(U)$ trivial
$h_i : U_i \longrightarrow U_i \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$
φ_i = part of unity subord to U_i

Define: $g_i : E \to \mathbb{R}^n$ — linear inj. on each fiber
$v \mapsto \varphi_i(p(v)) h_i(v)$ — with $\varphi_i \neq 0$.

$g : E \to \mathbb{R}^{nk}$ — linear inj. on all fibers.
$v \mapsto (g_1(v), \dots, g_k(v))$

$f : E \to B \times \mathbb{R}^{nk}$
$v \mapsto (p(v), g(v))$.

$\mathrm{Im}(f)$ is a subbundle. Project in 2^{nd} coord to get local triv. over U_i. ∎