Vector Bundles

Fix a vector space \( V \)

\[
V \rightarrow E \\
p \\
B
\]

1. Fibers \( p^{-1}(b) \) have structure of \( V \).
2. \( B \) covered by \( U \) s.t. \( \exists \)

\[
p^{-1}(U) \rightarrow U \times V \quad \text{homeo resp. structure on fibers.}
\]

\[\text{local trivialization}\]

Examples

1. Trivial bundle \( E = B \times V \).
2. Möbius bundle over \( S^1 \).
3. Tangent bundle to a smooth manifold \( M \)

\[
TM = \{(x, v) : v \in T_xM\}
\]

\[
p(x, v) = x
\]

v.s. structure: \( k_1(x, v_1) + k_2(x, v_2) = (x, k_1 v_1 + k_2 v_2) \)

By defn, \( M \) locally diffeo to \( U \subseteq \mathbb{R}^n \) open.
So suffices to show \( TU \) locally trivial. easy

4. Normal bundle to \( M \subseteq N \)

Locally: \( \mathbb{R}^n \hookrightarrow \mathbb{R}^{n+k} \quad \text{(Tubular nbhd thm)} \).
5. Canonical bundle over $\mathbb{RP}^n$

$\mathbb{RP}^n = \text{space of lines in } \mathbb{R}^{n+1} \cong S^n/\text{antipode.}$

Canonical line bundle: $\{(l, v) : v \in l\}$

Local trivialization near $l$: orthog. proj. to $l$ in $\mathbb{R}^{n+1}$

\[ (l', v) \mapsto (l', \text{proj}_l(v)) \in U \times l. \]

Allow $n = \infty$.

6. Orthogonal complement to 5

$E^\perp = \{(l, v) \in \mathbb{RP}^n \times \mathbb{R}^{n+1} : v \perp l\}$

Again, orthog proj gives local trivialization.

Q. $E^\perp \cong T\mathbb{RP}^n$?

7. Grassmann manifold

$G_n = \text{space of } n\text{-planes in } \mathbb{R}^\infty \text{ thru } 0$.

$E_n = \{(P, v) \in G_n \times \mathbb{R}^\infty : v \in P\}$

$E^\perp_n = \{(P, v) \in G_n \times \mathbb{R}^\infty : v \perp P\}$

8. Vertical bundle of surface bundle

Char. classes for surface bundles defined in terms of char. classes for these vector bundles.
ISOMORPHISM

\( p_1 : E_1 \to B \) is isomorphic to \( p_2 : E_2 \to B \) if \( \exists \) homeo \( h : E_1 \to E_2 \) s.t. \( h|_{p_1^{-1}(b)} \) is a vs. \( \cong \) to \( p_2^{-1}(b) \).

\( \text{N.B. } \infty \neq \text{triv. Möb} \)

\( \& \) bundles over different spaces can't be isomorphic(!)

EXAMPLES

1. \( \mathbb{N}\mathcal{S}^n \cong S^n \times \mathbb{R} \)
   via \((x, tx) \mapsto (x, t)\)

2. \( TS' \cong S^1 \times \mathbb{R} \)
   via \((z, izt) \mapsto (z, t)\)

We say \( S' \) is parallelizable. 

Q. Which manifolds are parallelizable? \( S^2 ? \)
   (All 3-manifolds!)

3. Can\( \mathcal{C} \) line bundle over \( T\mathbb{R}^n \cong \) Möbius bundle over \( T\mathbb{R}^2 \)
   after traveling around base, fibers get flipped:

Q. Is \( T\mathbb{R}^n \cong E^2 ? \)
A section of $p: E \rightarrow B$ is $s: B \rightarrow E$ s.t. $p \circ s = \text{id}$.

e.g. $O$-section

Some bundles have non-vanishing sections, some do not.

For example: A section of $TM$ is a vector field on $M$.

We showed nonvan vect field $\Rightarrow \chi(M) = 0$.

So $\chi(M) \neq 0 \Rightarrow TM$ has no nonvan. sec.

e.g. $\chi(S^n) = 2$ n even.

Can show $S^n$ has nonvan. vect field n odd.

**Fact**: An $n$-dim bundle is trivial $\iff$ it has $n$ sections $s_i$ that are lin. ind. over each point of $B$.

$\Rightarrow$ obvious

$\Leftarrow$ there is a contin. map

$$B \times \mathbb{R}^n \rightarrow E$$

$$(b, t_1, \ldots, t_n) \mapsto \sum t_i s_i(b)$$

Clearly isom. on fibers

Need to show inverse is continuous

Follows from: inversion of matrices is continuous.

**Spheres**: $T S^1$ trivial by $s_i(z) = iz$

$T S^3$ trivial by $s_1(z) = iz$, $s_2(z) = jz$, $s_3(z) = kz$

$T S^7$ trivial by similar construction w/ octonians.

(all other $T S^n$ nontrivial!)
**Direct Sum**

\[
p_1 : E_1 \rightarrow B, \quad p_2 : E_2 \rightarrow B
\]

\[
E_1 \oplus E_2 = \{(v_1, v_2) \in E_1 \times E_2 : p_1(v_1) = p_2(v_2)\}
\]

\[
p : E_1 \oplus E_2 \rightarrow B
\]

\[
(v_1, v_2) \mapsto p(v_1)
\]

$E_1 \oplus E_2$ a vector bundle because:

1. products of vb's are vb's
2. restrictions of vb's are vb's.

$E_1 \oplus E_2$ is restriction of $E_1 \times E_2$ to diagonal $B = B \times B$.

**Trivial $\oplus$ trivial = trivial but**

Nontrivial $\oplus$ trivial can be trivial!

\[e.g. \quad TS^n \oplus NS^n \text{ trivial. Say } TS^n \text{ stably trivial.}\]

\[\text{trivial}\]

also: \[E \oplus E^\perp \rightarrow \mathbb{R}P^n \text{ trivial via } (l, v, w) \mapsto (l, v + w)\]

\[n=1 \text{ case: } \text{Möbius } \oplus \text{Möbius } = \text{trivial}\]

A useful exercise, related to last example: Show there are exactly two $\mathbb{R}^n$ bundles over $S^1$. Similarly, exactly two $S^1$-bundles over $S^1$.
EXAMPLE. \( T\mathbb{R}P^n \) stably isom. to \( \bigoplus E \) line bundle.

Start with \( TS^n \oplus NS^n \cong S^n \times \mathbb{R}^{n+1} \).
Quotient by \( (x,v) \sim (-x,-v) \) on both sides.

\( TS^n/\sim \cong T\mathbb{R}P^n \) since \( (x,v) \mapsto (-x,-v) \) is map on \( TS^n \)
induced by \( x \mapsto -x \).

\( NS^n/\sim \cong \mathbb{R}P^n \times \mathbb{R} \) via the section \( x \mapsto (x,x) \)

Claim: \( (S^n \times \mathbb{R}^{n+1})/\sim \cong \bigoplus_{i=1}^{n+1} E \)

First, \( \sim \) preserves factors, so
\( (S^n \times \mathbb{R}^{n+1})/\sim \cong \bigoplus_{i=1}^{n+1} (S^n \times \mathbb{R})/\sim \)

But \( (S^n \times \mathbb{R})/\sim \cong E \), as

Using quaternions, \( T\mathbb{R}P^3 \cong T\mathbb{R}P^3 \times \mathbb{R}^3 \)
As above, \( T\mathbb{R}P^3 \oplus \text{trivial line bundle} \cong \mathbb{R}P^3 \times \mathbb{R}^4 \)
As above, \( T\mathbb{R}P^3 \oplus \text{trivial line bundle} \cong \bigoplus_{i=1}^4 E \)

\[ \Rightarrow \bigoplus_{i=1}^4 E \cong \mathbb{R}P^3 \times \mathbb{R}^4. \]
Next Goal

Prop. $B = \text{compact Hausdorff}$

$\forall E \to B \exists E' \to B \text{ s.t. } E \oplus E' \text{ trivial.}$

Step 1. Inner Products

Inner product on $V$: pos. def. symm. bilinear form.
Inner product on $E$: map $E \oplus E \to \mathbb{R}$ restricting to inner prod. on each fiber.

Paracompact: Hausdorff + every open cover admits a part. of unity.
Compact Hausdorff, CW complex, metric space $\Rightarrow$ paracompact

Prop. $B$ paracompact $\Rightarrow E \to B$ has an inner product.
$\blacksquare$: Exercise.

Step 2. Orthogonal complements

Prop. $B$ paracompact, $E_0 \to B$ subbundle of $E \to B$.

$\exists E^\perp$ s.t. $E_0 \oplus E^\perp \cong E$.

$\blacksquare$. Choose inner product, $E_0^\perp = \text{orthog. comp. in each fiber}$.
Need to check local triviality
Over $U \subseteq B$ choose m sections $s_i$ for $E_0$, $n-m$ for $E$.
Apply Gram-Schmidt $-$ continuous.
New sections trivialize $E_0$ & $E_0^\perp$ simultaneously.

Note: $E_0 \oplus E^\perp \cong E$ via FACT above.
To prove that any $E$ has $E'$ with $E \oplus E'$ trivial, it now suffices to show:

**Prop.** $B = \text{compact Hausdorff}$

Any $\mathbb{R}^n$-bundle $E \to B$ is a subbundle of $B \times \mathbb{R}^n$.

**Pf.** Choose: $U_i, \ldots, U_k$ s.t. $p^{-1}(U)$ trivial

$h_i : U_i \to U_i \times \mathbb{R}^n \to \mathbb{R}^n$

$q_i = \text{part of unity subord to } U_i$

Define: $g_i : E \to \mathbb{R}^n$

$v \mapsto (q_i(p(v)))h_i(v)$

linear inj. on each fiber with $q_i \neq 0$.

$g : E \to \mathbb{R}^{nk}$

$v \mapsto (g_1(v), \ldots, g_k(v))$

linear inj. on all fibers.

$f : E \to B \times \mathbb{R}^{nk}$

$v \mapsto (p(v), g(v))$.

$\text{Im}(f)$ is a subbundle. Project in 2nd coord to get local triv. over $U_i$. ❑