The Grassmann Manifold.

We just showed

$$[B, G_n] \longrightarrow \{\mathbb{R}^n\text{-bundles over } B\}$$
$$f \longmapsto f^*(E_n)$$

is well defined.

Want to show it is a bijection. First, let's discuss the topology of G_n & E_n.

G_n = set of all n-dim subspaces of $\mathbb{R}^\infty$.
V_n = Stiefel manifold
= space of orthonormal n-frames in $\mathbb{R}^\infty$.

V_n has a natural topology as a subspace of S^∞, and there is a quotient

direct limit topology.

$$V_n \longrightarrow G_n.$$

Endow G_n with quotient topology.

Define $E_n = \{(\ell, v) \in G_n \times \mathbb{R}^\infty : v \in \ell\}$, $p(\ell, v) = \ell$.

<u>Lemma</u>. $E_n \xrightarrow{p} G_n$ is a vector bundle.

Pf. Let $\ell \in G_n$, $\pi_\ell : \mathbb{R}^\infty \to \ell$ orthog. proj.

$$U_\ell = \{\ell' \in G_n : \pi_\ell(\ell') \text{ has dim } n\}.$$

Steps: ① U_ℓ open (check preim in V_n open).

② $h : p^{-1}(U_\ell) \longrightarrow U_\ell \times \ell$ is a local triv.
$(\ell', v) \longrightarrow (\ell', \pi_\ell(v))$

h clearly a bij, lin. iso on each fiber.

Need: h, h^{-1} continuous (lin alg).

THEOREM. X paracompact. The map $[X, G_n] \to \mathrm{Vect}^n(X)$, $f \mapsto f^*(E_n)$ is a bijection.

Example. $M \subseteq \mathbb{R}^N$ submanifold. Define $f: M \to G_n$ by $x \mapsto T_x M$. Then $TM \cong f^*(E_n)$.

Pf. Key observation: For $E \to X$ an $\mathbb{R}^n$-bundle, an iso $E \cong f^*(E_n)$ is equivalent to a map $E \to \mathbb{R}^\infty$ that is a lin inj. on each fiber.

Indeed, given $f: X \to G_n$ and $E \xrightarrow{\cong} f^*(E_n)$ have:

$$\begin{tikzcd} E \arrow[r, "\cong"] \arrow[dr, "p"'] & f^*(E_n) \arrow[r] \arrow[d] & E_n \arrow[r] \arrow[d] & \mathbb{R}^\infty \\ & X \arrow[r, "f"] & G_n & \end{tikzcd}$$

Top row is the desired map.

Conversely, given $g: E \to \mathbb{R}^\infty$ lin inj. on each fiber, define $f: X \to G_n$ by $x \mapsto g(p^{-1}(x))$.
$\tilde{f}: E \to E_n$ by $v \mapsto g(v)$.
This gives diagram as above, by univ. prop. of pullbacks.

Surjectivity. Let $p: E \to X$ be an $\mathbb{R}^n$-bundle
(for simplicity, X = compact Hausdorff)

Choose cover $U_1, \dots, U_N$ s.t. E trivial over U_i.
& partition of unity $\varphi_1, \dots, \varphi_N$.

Define $g_i: p^{-1}(U_i) \longrightarrow U_i \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$

& $g: E \longrightarrow \mathbb{R}^n \times \cdots \times \mathbb{R}^n \subseteq \mathbb{R}^\infty$

$v \longmapsto (\varphi_1 g_1(v), \dots, \varphi_N g_N(v))$

φ_i means $\varphi_i \circ p$ = scalar

Check g a lin. inj. on each fiber.

Injectivity. Say $E \cong f_0^*(E_n), f_1^*(E_n)$
for $f_0, f_1 : X \to G_n$.

$\rightsquigarrow g_0, g_1 : E \to \mathbb{R}^\infty$ lin inj on each fiber.

To show $g_0 \sim g_1$ via maps that are lin inj on each fiber.

$\Rightarrow f_0 \sim f_1$ via $f_t(x) = g_t(p^{-1}(x))$.

Use:

$$\begin{tikzcd}
g_0 \arrow[d, "①"] & & g_1 \arrow[d, "②"] \\
\text{odd coords} \arrow[rr, "\text{straight line}", "③"'] & & \text{even coords}
\end{tikzcd}$$

N.B. ③ only makes sense b/c g_0, g_1 are both maps from a fixed space E to $\mathbb{R}^\infty$.

e.g. $g_0 \longrightarrow$ odd coords via $(x_1, x_2, \dots) \longmapsto (1-t)(x_1, x_2, \dots) + t(x_1, 0, x_2, \dots)$

At each stage, lin. inj. on fibers. ▨

The Thm has an immediate corollary: v.b.'s over paracompact bases have inner products. Pull back obvious one on $\mathbb{R}^\infty$.

We now know $\quad [B, G_n] \longleftrightarrow \{\text{vector bundles over } B\}$

so $\quad$ char. classes $\longleftrightarrow H^*(G_n)$

Cell Structure on G_n.

First recall cell structure on $G_1 = \mathbb{RP}^\infty$

one i-cell e_i $\forall i$.

e_i glued to e_{i-1} by degree 2 map

$e_i \leftrightarrow \{\ell \in \mathbb{RP}^\infty : \ell \subseteq \mathbb{R}^{i+1}\}$

Will generalize this.

A Schubert symbol $\sigma = (\sigma_1, \ldots, \sigma_n)$ is a seq of integers

s.t. $\quad 1 \leq \sigma_1 < \sigma_2 < \cdots < \sigma_n$

Let $e(\sigma) = \{\ell \in G_n : \dim(\ell \cap \mathbb{R}^{\sigma_i}) - \dim(\ell \cap \mathbb{R}^{\sigma_i - 1}) = 1 \;\forall i\}$

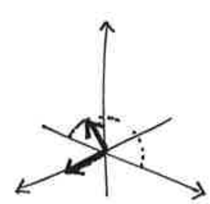
<u>Prop.</u> The $e(\sigma)$ are the cells of a CW structure on G_n.

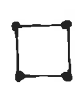
$$\dim e(\sigma) = \sum_{i=1}^{n} (\sigma_i - i)$$

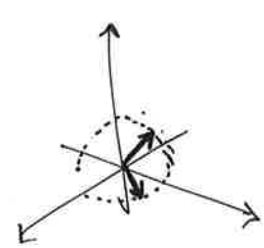
Examples. Consider in G_2:

$e(1,2) = \bullet$

$e(1,3) =$



$e(2,3) =$ 



Proof of Prop. Let H_i = hemisphere in $S^{\sigma_i - 1} \subseteq \mathbb{R}^{\sigma_i}$ s.t. σ_i-coord non-neg.

$$e(\sigma) \longleftrightarrow \{(b_1, \dots, b_n) \in V_n : b_i \in \operatorname{int} H_i\}$$

Let $E(\sigma) = \{(b_1, \dots, b_n) \in V_n : b_i \in H_i\}$

Main step: $E(\sigma)$ a closed ball of dim $\Sigma(\sigma_i - i)$

$n=1$ case: $E(\sigma) = H_1$ ✓

$n>1$ case: Define
$$\pi : E(\sigma) \to H_1$$
$$(b_1, \dots, b_n) \mapsto b_1$$
$$p : E(\sigma) \to \pi^{-1}(e_{\sigma_1})$$
rotate fiber over b_1 to $\pi^{-1}(e_{\sigma_1})$ by rotating b_1 to e_{σ_1}, fixing orthog. comp. of $\langle b_1, e_{\sigma_1} \rangle$

Then $\pi \times p : E(\sigma) \to H_1 \times \pi^{-1}(e_{\sigma_1})$ is a contin. bij $\Rightarrow$ homeo.
(exercise: Hausdorff)

Remains to check $\pi^{-1}(e_{\sigma_1})$ a ball.
Induct on n. $\pi^{-1}(e_{\sigma_1}) \longleftrightarrow E(\sigma_2 - 1, \dots, \sigma_n - 1)$

Span takes int $E(\sigma)$ to $e(\sigma)$ bijectively.
Since G_n has quotient top. from $V_n \rightsquigarrow$ homeo.

Need to check that the CW complex obtained from the $E(\sigma)$ give right topology. Induct on skeleta. ∎

Other versions: $\quad \mathrm{Vect}^n_{\mathbb{C}}(X) \longleftrightarrow [X, G_n(\mathbb{C})]$

$$\mathrm{Vect}^n_+(X) \longleftrightarrow [X, \tilde{G}_n]$$

Note $\mathrm{Vect}^n_+(S^1)$ trivial $\Longrightarrow [S^1, \tilde{G}_n]$ trivial

$\Longrightarrow \pi_1(\tilde{G}_n) = 1.$

$\Longrightarrow \tilde{G}_n =$ univ. cover of G_n.

For $f : X \longrightarrow G_n$, $f^*(E)$ orientable iff f lifts to $\tilde{G}_n$ & in this case, orientations correspond to choices of lifts.

<u>Prop.</u> G_n is a manifold.

<u>Pf.</u> ~~First show~~ Clear for interior of a top-dim. cell. But G_n is homogeneous: $\exists$ homeo taking any pt to any other pt, ie the one induced by a linear map.