Stiefel-Whitney and Chern Classes

First, we will show that characteristic classes exist by defining specific ones, the SW classes w_i and the Chern classes c_i. Then we will show these are all char. classes (in the $\mathbb{R}, \mathbb{Z}_2$ & $\mathbb{C}, \mathbb{Z}$ cases, resp.) by computing $H^*(G_n; \mathbb{Z}_2)$ and $H^*(G_n(\mathbb{C}); \mathbb{Z})$.

<u>Thm.</u> $\exists!$ seq. of fns $w_1, w_2, \ldots$ assigning to each real v.b. $E \to B$ a class $w_i(E) \in H^i(B; \mathbb{Z}_2)$ s.t.

(i) $w_i(f^*(E)) = f^*(w_i(E))$

(ii) $w(E_1 \oplus E_2) = w(E_1) \cup w(E_2)$ $\qquad w = 1 + w_1 + w_2 + \cdots$

(iii) $w_i(E) = 0 \quad i > \dim E$

(iv) $w_1(\text{canon. bundle} \to \mathbb{R}P^\infty)$ is gen. of $H^1(\mathbb{R}P^\infty; \mathbb{Z}_2)$.

w = total SW class. $\qquad$ (iii) $\Rightarrow$ it is a finite sum.

(ii) is Whitney sum formula.

(iv) $\Rightarrow$ the w_i are not all zero!

(i) $\Rightarrow w_i(B \times \mathbb{R}^n) = 0 \quad i > 0$. (ii) $\Rightarrow w_i$ stable $\qquad$ <u>Cor</u>: $w_i(TS^n) = 0$ $\quad i > 0$. or: $w(TS^n) = 1$.

For complex bundles, have $c_i \in H^{2i}(B; \mathbb{Z})$. Thm is same except:

(iv) $c_1(\text{canon} \to \mathbb{C}P^\infty)$ gen. $H^2(\mathbb{C}P^\infty; \mathbb{Z})$.

Proof requires one tool from alg. top...

THE LERAY-HIRSCH THEOREM

When does $H^*(E)$ look like $H^*(F\times B)$? First, recall:

<u>KÜNNETH FORMULA</u>. $$H^*(X;R)\otimes_R H^*(Y;R) \xrightarrow{\cong} H^*(X\times Y;R)$$
$$a\otimes b \longmapsto p_1^*(a)\cup p_2^*(b)$$

For a fiber bundle, $H^*(E)\to H^*(F)$ not. nec. surj, so don't always have a map the other way. To get a Künneth-like formula, must add this to the assumptions.

General theme in bundle theory: try to extend an object related to the fiber (inner prod, cohom. class) to whole bundle.

<u>L-H</u> Theorem. Let $F\to E\to B$ be a fiber bundle, R a ring s.t.

(i) $H^n(F;R)$ is a free f.g. R-module $\forall n$.

(ii) $\exists c_j\in H^{k_j}(E;R)$ s.t. the $i^*(c_j)$ form a basis for $H^*(F;R)$

Then: $$H^*(B;R)\otimes_R H^*(F;R) \xrightarrow{\cong} H^*(E,R)$$
$$\sum b_i\otimes i^*(c_j) \longmapsto p^*(b_i)\cup c_j$$

In other words: $H^*(E;R)$ a free $H^*(B;R)$ module w/ basis c_j. Module structure given by $\cup$.

- The c_i do exist for product bundles: pull back via projection.
- The c_i do not exist for $S^1\to S^3\to S^2$ as $H^1(S^3)=1$.

Pf. of LH (a few words) Using long ex. seq. for a pair, plus excision, you reduce to understanding

$$p^{-1}(B^{n-1}) \to B^{n-1} \qquad (n\text{-skeleton})$$

$$p^{-1}(n\text{-cell}) \to n\text{-cell}$$

Former works by induction, latter by local triviality. $\blacksquare$

Pf of SW Thm. $\pi: E \to B$

$\rightsquigarrow P(\pi): P(E) \to B$ $\qquad$ $P(E)$ = space of lines, fibers $\mathbb{RP}^{n-1}$

To use L-H, need $x_i \in H^i(P(E); \mathbb{Z}_2)$ restricting to gens for $H^i(\mathbb{RP}^{n-1}; \mathbb{Z}_2)$.

$(E \to B) \rightsquigarrow g: E \to \mathbb{R}^\infty$ lin. inj on fibers.

$\rightsquigarrow P(g): P(E) \to \mathbb{RP}^\infty$

Let α = gen for $H^1(\mathbb{RP}^\infty; \mathbb{Z}_2)$

$x = P(g)^*(\alpha)$ $\quad \leftarrow$ easy to see this generates H^1(fiber). also indep. of g

$x_i = x^i$.

i.e. $x \in \mathrm{Hom}(H_1(P(E)), \mathbb{Z}_2)$ records whether a line comes back w/ same or. after the loop.

L-H $\Rightarrow$ $H^*(P(E))$ a free $H^*(B)$-module with basis $1, x, \dots, x^{n-1}$

$\Rightarrow x^n$ = unique linear combo:

$$x^n + w_1(E)x^{n-1} + \dots + w_n(E)\cdot 1 = 0.$$

for some $w_i(E) \in H^*(B; \mathbb{Z}_2)$.

Also set $w_i(E) = 0$ for $i > n$

$w_0(E) = 1$.

These are the SW classes. Need to check properties (i)–(iv), uniqueness.

(i) Naturality

Say
$$\begin{tikzcd} E' \arrow[r, "\tilde{f}"] \arrow[d] & E \arrow[r, "g"] \arrow[d] & \mathbb{R}^\infty \\ B' \arrow[r, "f"] & B & \end{tikzcd}$$

$\rightsquigarrow \quad P(\tilde{f})^* \, x(E) = x(E')$

$\Rightarrow P(\tilde{f})^* \, x_i(E) = x_i(E')$

Commutativity $\Rightarrow$ module structure pulls back

i.e. $\quad x^n + w_1(E)x^{n-1} + \cdots + w_n(E)\cdot 1 = 0$

$\rightsquigarrow \quad x^n + f^*(w_1(E))\, x^{n-1} + \cdots + f^*(w_n(E))\cdot 1 = 0$

But this defines $w_i(E')$ so $w_i(E') = f^*(w_i(E)) \ \forall i$.

(ii) Whitney sum - similar flavor

(iii) $w_i(E) = 0 \quad i > n$ by definition.

(iv) $w_1(CB \to \mathbb{RP}^\infty) \neq 0$.

Almost by definition: x(loop in $P(E)$) measures whether or not a line comes back to where it started with same or different orientation.

$$x + w_1(CB)\cdot 1 = 0.$$
$$\Rightarrow w_1(CB) = x.$$

For uniqueness of w_i, need a tool.

Splitting Principle. Given $E \to B$ $\exists$ $f: A \to B$ s.t.

(i) $f^*(E)$ splits as a sum of line bundles

(ii) $f^*: H^*(B) \to H^*(A)$ injective

Now, the w_i are unique because:

(iv) determines $w_1(CB \to \mathbb{R}P^\infty)$

(iii) determines $w_i(CB \to \mathbb{R}P^\infty)$ $i > 1$.

(i) determines w_i (line bundles)

(ii) determines w_i (sum of line bundles)

SP + (i) determines w_i (any bundle).

Pf of SP. $A = F(E) =$ flag bundle of E

$=$ space of orthog. splittings $l_1 \oplus \cdots \oplus l_n$ of E into lines

$f: A \to B$ projection

$f^*(E) = \{$(splitting of fiber over b, vector in fiber over b)$\}$

This has n obvious linear subbundles, which give the splitting.

For (ii) use Leray-Hirsch $\Longrightarrow$ $H^*(B) \cdot 1$ a summand of $H^*(A)$.

IMPORTANT EXAMPLE.

$(E_1)^n \longrightarrow (G_1)^n$ $\qquad E_1$ = Canon. line bundle

$(E_1)^n \cong \oplus \pi_i^*(E_1)$ $\qquad \pi_i : (G_1)^n \longrightarrow G_1$ $\qquad$ true for any $E^n \to B^n$

$\Rightarrow \ w((E_1)^n) = \prod (1+\alpha_i) \in \mathbb{Z}_2[\alpha_1, \dots, \alpha_n] \cong H^*((\mathbb{RP}^\infty)^n; \mathbb{Z}_2)$

$\rightarrow \ w_i((E_1)^n) =$ ith symm. poly σ_i in the α_j

e.g. for $n=3$:

$$\sigma_1 = \alpha_1 + \alpha_2 + \alpha_3$$
$$\sigma_2 = \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3$$
$$\sigma_3 = \alpha_1\alpha_2\alpha_3$$

So all w_i nonzero $i \leq n$.

Next: We'll use this to show

$$\mathbb{Z}_2[w_1, \dots, w_n] \hookrightarrow H^*(G_n; \mathbb{Z}_2)$$