Cohomology of Grassmannians

We showed $w_i\left((E_1)^n \to (G_1)^n\right) \neq 0 \quad 0 \le i \le n.$
Naturality $\Rightarrow$ $w_i(E_n) \neq 0 \quad 0 \le i \le n.$

Let $f: (\mathbb{RP}^\infty)^n \longrightarrow G_n$ be classifying map for $(E_1)^n$.
& $w_i = w_i(E_n)$.
Then:

$$\mathbb{Z}_2[w_1, \ldots, w_n] \longrightarrow H^*(G_n; \mathbb{Z}_2) \xrightarrow{f^*} H^*((\mathbb{RP}^\infty)^n; \mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha_1, \ldots, \alpha_n]$$

sends w_i to i^{th} symm. poly. σ_i in the α_j.

Fact. The σ_i are alg. indep.

$\Rightarrow$ above map is inj
$\Rightarrow$ $\mathbb{Z}_2[w_1, \ldots, w_n] \hookrightarrow H^*(G_n; \mathbb{Z}_2)$.

<u>Thm</u> $H^*(G_n; \mathbb{Z}_2) = \mathbb{Z}_2[w_1, \ldots, w_n]$

also: $H^*(G_n(\mathbb{C}); \mathbb{Z}) = \mathbb{Z}[c_1, \ldots, c_n]$

<u>Pf</u>. We showed $\operatorname{im} f^*$ contains $\mathbb{Z}_2[\sigma_1, \ldots, \sigma_n]$
Also $\operatorname{im} f^*$ contained in $\mathbb{Z}_2[\sigma_1, \ldots, \sigma_n]$ since permuting the $\mathbb{RP}^\infty$ factors gives same bundle with α_i's permuted.
So:

$$\mathbb{Z}_2[w_1, \ldots, w_n] \longrightarrow H^*(G_n; \mathbb{Z}_2) \xrightarrow{f^*} \mathbb{Z}_2[\sigma_1, \ldots, \sigma_n] \cong \mathbb{Z}_2[w_1, \ldots, w_n]$$

f^* surjective. To show f^* injective.

Focus on r-grading:

$$(\mathbb{Z}_2[w_1,\dots,w_n])_r \longrightarrow H^r(G_n;\mathbb{Z}_2) \longrightarrow (\mathbb{Z}_2[w_1,\dots,w_n])_r$$

Since composition surj, suffices to show $\dim H^r(G_n;\mathbb{Z}_2) \le \dim(\mathbb{Z}_2[w_1,\dots,w_n])_r$.

Let $p(r,n)$ = #partitions of r into n nonneg integers.

Step 1. $\dim(\mathbb{Z}_2[w_1,\dots,w_n])_r = p(r,n)$.

$w_1^{r_1} w_2^{r_2} \cdots w_n^{r_n} \in (\mathbb{Z}_2[w_1,\dots,w_n])_r$ means

$r_1 + 2r_2 + \cdots + nr_n = r$ (since $w_i \in H^i$)

$\rightsquigarrow$ partition of r: $r_n \le r_n + r_{n-1} \le \cdots \le r_n + \cdots + r_1$

Step 2. $\dim H^r(G_n;\mathbb{Z}_2) \le$ # Schubert cells of dim r.

General fact about cell complexes

Step 3. # Schubert cells in G_n of dim r $= p(r,n)$.

A partition $a_1 \le a_2 \le \cdots \le a_n$

$\rightsquigarrow$ Schubert symbol $(a_1+1, a_2+2, \dots, a_n+n)$.

Example. $r=10$, $n=6$.

partition: $0,0,1,1,3,5$

Schubert cell: $(1,2,4,5,8,11)$

monomial: $w_1^2 w_2^2 w_4$

$\boxtimes$