

THE GROUP OF LINE BUNDLES

We'll first show: $\text{Vect}^1(X)$ is a group under \otimes .

and then: $\text{Vect}^1(X) \cong H^1(X; \mathbb{Z}_2)$. The isom. is w_1 !

Gluing construction of vector bundles. Given $p: E \rightarrow B$, $\{U_\alpha\}$,

$h_\alpha: p^{-1}(U_\alpha) \rightarrow U_\alpha \times \mathbb{R}^n$, can recover

$$E = (\coprod U_\alpha \times \mathbb{R}^n) / \sim$$

where $(x, v) \in U_\alpha \times \mathbb{R}^n \sim h_\beta h_\alpha^{-1}(x, v) \in U_\beta \times \mathbb{R}^n$ $x \in U_\alpha \cap U_\beta$.

Write $g_{\beta\alpha}$ for the gluing func. $h_\beta h_\alpha^{-1}: U_\alpha \cap U_\beta \rightarrow GL_n(\mathbb{R})$.

\rightarrow cocycle condition: $g_{\gamma\beta} g_{\beta\alpha} = g_{\gamma\alpha}$ on $U_\alpha \cap U_\beta \cap U_\gamma$

Conversely: any collection of gluing functions satisfying cocycle cond gives rise to a vector bundle.

The gluing functions for $E_1 \otimes E_2$ are the tensor products of the gluing functions for E_1, E_2 .

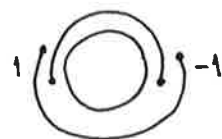
In general, \otimes on $\text{Vect}^n(X)$ is comm, assoc, and has identity = trivial line bundle.

For $n=1$, also have inverses. In fact, each elt is its own inverse.

Example. Möbius $\rightarrow S^1$ has gluing fns 1, -1

$$1 \otimes 1 = 1 \quad -1 \otimes -1 = 1$$

\Rightarrow Möbius \otimes Möbius $\rightarrow S^1$ is trivial.



For general line bundles, we obtain inverse by replacing gluing matrices by their inverses, as $t \otimes t^{-1} = 1$.

Cocycle condition still works since 1×1 matrices commute.

Endow E w/ inner product \rightsquigarrow rescale all h_{α} with isometries

\Rightarrow all gluing fns ± 1 . \Rightarrow gluing fns for $E \otimes E$ all 1.

$\Rightarrow E \otimes E$ trivial.

We have: $\text{Vect}^1(X) = [X, G_1] \cong H^1(X; \mathbb{Z}_2)$
 \uparrow isom. of sets $\quad \uparrow$ since $G_1 = \mathbb{R}P^\infty$ is $K(\mathbb{Z}_2, 1)$.

Prop. $w_1: \text{Vect}^1(X) \xrightarrow{\cong} H^1(X; \mathbb{Z}_2)$ $X = \text{CW-complex}$.

Pf. First show w_1 a homomorphism.

Step 1. $w_1(L_1 \otimes L_2) = w_1(L_1) + w_1(L_2)$
 for $L_i \rightarrow G_1 \times G_1$ the pullback of $E_i \rightarrow G_1$
 via $\pi_i: G_1 \times G_1 \rightarrow G_1$.

Have $H^*(G_1 \times G_1) \cong \mathbb{Z}_2[\alpha_1] \otimes \mathbb{Z}_2[\alpha_2] \cong \mathbb{Z}_2[\alpha_1, \alpha_2]$

$H^*(G_1 \vee G_2) \cong \mathbb{Z}_2[\alpha_1] \oplus \mathbb{Z}_2[\alpha_2]$

This is an isom. on $H^1: \mathbb{Z}_2 \oplus \mathbb{Z}_2 = \{0, \alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$

So suffices to compute $w_1(L_1 \otimes L_2 \rightarrow G_1 \vee G_1)$

Over $G_1 \vee *$, L_2 trivial $\Rightarrow L_1 \otimes L_2 \cong L_1 \otimes 1 \cong L_1$

Similar for $* \vee G_1$

$\Rightarrow w_1(L_1 \otimes L_2) = \alpha_1 + \alpha_2 = w_1(L_1) + w_1(L_2)$.

\uparrow use naturality of pullback via $G_1 \rightarrow G_1 \vee G_1$

Step 2. (Naturality) E_1, E_2 arbitrary ^{line} bundles
 $E_i = f_i^*(E_1) \quad f_i: X \rightarrow G_1$
 Let $F = (f_1, f_2): X \rightarrow G_1 \times G_1$
 $F^*(L_i) = f_i^*(E_1) = E_i$

follow your nose...

$$\begin{aligned} w_1(E_1 \otimes E_2) &= w_1(F^*(L_1) \otimes F^*(L_2)) = w_1(F^*(L_1 \otimes L_2)) \\ &= F^*(w_1(L_1 \otimes L_2)) = F^*(w_1(L_1) + w_1(L_2)) \\ &= F^*(w_1(L_1)) + F^*(w_1(L_2)) \\ &= w_1(F^*(L_1)) + w_1(F^*(L_2)) \\ &= w_1(E_1) + w_1(E_2). \end{aligned}$$

The isomorphism $[X, G_1] \rightarrow H^1(X; \mathbb{Z}_2)$
 is $[f] \mapsto f^*(\alpha)$

It factors as $[X, G_1] \rightarrow \text{Vect}^1(X) \rightarrow H^1(X; \mathbb{Z}_2)$

$$[f] \mapsto f^*(E_1) \mapsto w_1(f^*(E_1)) = f^*(w_1(E_1)) = f^*(\alpha)$$

First map is bij, comp is isom \Rightarrow 2nd map bij. \square

We can unravel the last step. Want to define

$$H^1(X; \mathbb{Z}_2) \rightarrow \text{Vect}^1(X)$$

inverse to w_1 . Given $c \in H^1$, define an \mathbb{R} -bundle skeleton by skeleton. On 1-skeleton, use c to decide between Möbius & trivial bundle. As c is a cocycle, it is trivial on any loop bounding a 2-cell, so can extend over 2-skeleton and higher.