The Group of Line Bundles

We'll first show: \( \text{Vect}^1(X) \) is a group under \( \otimes \).
and then: \( \text{Vect}^1(X) \cong H^1(X; \mathbb{Z}) \). The isom. is \( w_1 \)!

Gluing construction of vector bundles. Given \( p : E \to B, \{ U_\alpha \}, \)
\( h_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n \), can recover
\( E = ( \bigsqcup U_\alpha \times \mathbb{R}^n ) / \sim \)
where \((x,v) \in U_\alpha \times \mathbb{R}^n \sim h_\beta h_\alpha^{-1}(x,v) \in U_\beta \times \mathbb{R}^n \) \( x \in U_\alpha \cap U_\beta \).
Write \( g_\alpha x \) for the gluing func. \( h_\beta h_\alpha^{-1} : U_\alpha \cap U_\beta \to GL_n(\mathbb{R}) \).
\( \sim \) cocycle condition: \( g_\beta g_\alpha = g_{\alpha \beta} \) on \( U_\alpha \cap U_\beta \cap U_\gamma \).
Conversely: any collection of gluing functions satisfying
\( \sim \) cocycle cond. gives rise to a vector bundle.

The gluing functions for \( E_1 \otimes E_2 \) are the tensor products of
the gluing functions for \( E_1, E_2 \).

In general, \( \otimes \) on \( \text{Vect}^n(X) \) is comm, assoc., and has
identity = trivial line bundle.

For \( n=1 \), also have inverses. In fact, each elt is its
own inverse.

Example. Möbius \( \to S^1 \) has gluing fns 1, -1

\[ 1 \otimes 1 = 1 \quad -1 \otimes -1 = 1 \]
\[ \Rightarrow \text{Möbius} \otimes \text{Möbius} \to S^1 \text{ is trivial.} \]
For general line bundles, we obtain inverse by replacing gluing matrices by their inverses, as $t \otimes t^{-1} = 1$.

Cocycle condition still works since $1 \times 1$ matrices commute.

Endow $E$ w/inner product $\Rightarrow$ rescale all $h_x$ with isometries
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\Rightarrow$ all gluing $f$s $\pm 1$.
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\Rightarrow$ gluing $f$s for $E \otimes E$ all $1$.

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\Rightarrow E \otimes E$ trivial.

We have: $\text{Vect}^1(X) = \begin{bmatrix} X, G_1 \end{bmatrix} \cong H^1(X; \mathbb{Z}_2)$

$\uparrow$

$\leftarrow$ since $G_1 = \text{RP}^\infty$ is $K(\mathbb{Z}_2, 1)$.

**Prop.** $\text{W}_1 : \text{Vect}^1(X) \xrightarrow{\cong} H^1(X; \mathbb{Z}_2)$

$X = \text{CW}$-complex.

**If.** First show $\text{W}_1$ a homomorphism.

**Step 1.** $\text{W}_1(L_1 \otimes L_2) = \text{W}_1(L_1) + \text{W}_1(L_2)$

for $L_i \rightarrow G_1 \times G_1$ the pullback of $E_1 \rightarrow G_1$

via $\pi_i : G_1 \times G_1 \rightarrow G_1$.

Have $H^*(G_1 \times G_1) \cong \mathbb{Z}_2[x_1] \otimes \mathbb{Z}_2[x_2] \cong \mathbb{Z}_2[x_1, x_2]$

$H^*(G_1 \times G_1) \cong \mathbb{Z}_2[x_1] \otimes \mathbb{Z}_2[x_2]$

This is an isom. on $H^1 : \mathbb{Z}_2 \otimes \mathbb{Z}_2 = \{0, x_1, x_2, x_1 + x_2\}$

So suffices to compute $\text{W}_1(L_1 \otimes L_2 \rightarrow G_1 \times G_1)$

Over $G_1 \times *$, $L_2$ trivial $\Rightarrow L_1 \otimes L_2 \cong L_1 \otimes 1 \cong L_1$

Similar for $\star \times G_1$

$\Rightarrow \text{W}_1(L_1 \otimes L_2) = \alpha_1 + \alpha_2 = \text{W}_1(L_1) + \text{W}_1(L_2)$.

$\uparrow$ use naturality of pullback via $G_1 \rightarrow G_1 \times G_1$. 
Step 2. (Naturality) \( E_1, E_2 \) arbitrary bundles
\[ E_i = f_i^*(E_i) \quad f_i : X \to G_i. \]
Let \( F = (f_1, f_2) : X \to G_1 \times G_1 \)
\[ F^*(L_i) = f_i^*(E_i) = E_i. \]

follow your nose...
\[ W_1(E_1 \otimes E_2) = W_1(F^*(L_1) \otimes F^*(L_2)) = W_1(F^*(L_1 \otimes L_2)) = F^*(W_1(L_1 \otimes L_2)) = F^*(W_1(L_1)) + F^*(W_1(L_2)) = W_1(F^*(L_1)) + W_1(F^*(L_2)) = W_1(E_1) + W_1(E_2). \]

The isomorphism \( [X, G_i] \to H^i(X; \mathbb{Z}_2) \)
is \( [f] \mapsto f^*(\alpha) \)
It factors as \( [X, G_i] \to \text{Vect}^i(X) \to H^i(X; \mathbb{Z}_2) \)
\[ [f] \mapsto f^*(E_i) \mapsto W_1(f^*(E_i)) = F^*(W_1(E_i)) = f^*\alpha \]
First map is bij, comp is isom \(\Rightarrow\) 2nd map bij. \(\square\)

We can unravel the last step. Want to define
\[ H^i(X; \mathbb{Z}_2) \to \text{Vect}^i(X) \]
inverse to \( W_1 \). Given \( \alpha \in H^i \), define an \( \mathbb{R} \)-bundle skeleton by skeleton. On 1-skeleton, use \( \alpha \) to decide between Moebius & trivial bundle. As \( \alpha \) is a cocycle, it is trivial on any loop bounding a 2-cell, so can extend over 2-skeleton and higher.