The Group of Line Bundles

We'll first show: $\mathrm{Vect}^1(X)$ is a group under $\otimes$.

and then: $\mathrm{Vect}^1(X) \cong H^1(X;\mathbb{Z}_2)$. The isom. is w_1!

Gluing construction of vector bundles. Given $p: E \to B$, $\{U_\alpha\}$,

$h_\alpha : p^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{R}^n$, can recover

$$E = \left(\coprod U_\alpha \times \mathbb{R}^n\right)/\sim$$

where $(x,v) \in U_\alpha \times \mathbb{R}^n \sim h_\beta h_\alpha^{-1}(x,v) \in U_\beta \times \mathbb{R}^n$ $\quad x \in U_\alpha \cap U_\beta$.

Write $g_{\beta\alpha}$ for the gluing func. $h_\beta h_\alpha^{-1} : U_\alpha \cap U_\beta \longrightarrow GL_n(\mathbb{R})$.

$\rightsquigarrow$ cocycle condition: $g_{\gamma\beta} g_{\beta\alpha} = g_{\gamma\alpha}$ on $U_\alpha \cap U_\beta \cap U_\gamma$

Conversely: any collection of gluing functions satisfying cocycle cond gives rise to a vector bundle.

The gluing functions for $E_1 \otimes E_2$ are the tensor products of the gluing functions for E_1, E_2.

In general, $\otimes$ on $\mathrm{Vect}^n(X)$ is comm, assoc, and has identity = trivial line bundle.

For $n=1$, also have inverses. In fact, each elt is its own inverse.

Example. Möbius $\to S^1$ has gluing fns $1, -1$

$1 \otimes 1 = 1 \quad -1 \otimes -1 = 1$

$\Rightarrow$ Möbius $\otimes$ Möbius $\longrightarrow S^1$ is trivial.

For general line bundles, we obtain inverse by replacing gluing matrices by their inverses, as $t \otimes t^{-1} = 1$.

Cocycle condition still works since 1×1 matrices commute.

Endow E w/ inner product $\rightsquigarrow$ rescale all h_α with isometries

$\Rightarrow$ all gluing fns ± 1. $\Rightarrow$ gluing fns for $E \otimes E$ all 1.

$\Rightarrow$ $E \otimes E$ trivial.

We have: $\underbrace{\mathrm{Vect}^1(X) = [X, G_1]}_{\text{isom. of sets}} \cong H^1(X; \mathbb{Z}_2)$ since $G_1 = \mathbb{RP}^\infty$ is $K(\mathbb{Z}_2, 1)$.

Prop. $w_1 : \mathrm{Vect}^1(X) \xrightarrow{\cong} H^1(X; \mathbb{Z}_2)$ $\qquad X$ = CW-complex.

Pf. First show w_1 a homomorphism.

Step 1. $w_1(L_1 \otimes L_2) = w_1(L_1) + w_1(L_2)$

for $L_i \to G_1 \times G_1$ the pullback of $E_1 \to G_1$

via $\pi_i : G_1 \times G_1 \to G_1$.

Have $H^*(G_1 \times G_1) \cong \mathbb{Z}_2[\alpha_1] \otimes \mathbb{Z}_2[\alpha_2] \cong \mathbb{Z}_2[\alpha_1, \alpha_2]$

$H^*(G_1 \vee G_2) \cong \mathbb{Z}_2[\alpha_1] \oplus \mathbb{Z}_2[\alpha_2]$

This is an isom. on H^1 : $\mathbb{Z}_2 \oplus \mathbb{Z}_2 = \{0, \alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$

So suffices to compute $w_1(L_1 \otimes L_2 \to G_1 \vee G_1)$

Over $G_1 \vee *$, L_2 trivial $\Rightarrow L_1 \otimes L_2 \cong L_1 \otimes 1 \cong L_1$

Similar for $* \vee G_1$

$\Rightarrow w_1(L_1 \otimes L_2) = \alpha_1 + \alpha_2 = w_1(L_1) + w_1(L_2)$.

use naturality of pullback via $G_1 \to G_1 \vee G_1$

Step 2. (Naturality) E_1, E_2 arbitrary line bundles

$$E_i = f_i^*(E_1) \quad f_i : X \to G_1.$$

$$\text{Let } F = (f_1, f_2) : X \to G_1 \times G_1$$

$$F^*(L_i) = f_i^*(E_1) = E_i$$

follow your nose...

$$\begin{aligned} w_1(E_1 \otimes E_2) &= w_1(F^*(L_1) \otimes F^*(L_2)) = w_1(F^*(L_1 \otimes L_2)) \\ &= F^*(w_1(L_1 \otimes L_2)) = F^*(w_1(L_1) + w_1(L_2)) \\ &= F^*(w_1(L_1)) + F^*(w_1(L_2)) \\ &= w_1(F^*(L_1)) + w_1(F^*(L_2)) \\ &= w_1(E_1) + w_1(E_2). \end{aligned}$$

The isomorphism is

$$[X, G_1] \longrightarrow H^1(X; \mathbb{Z}_2)$$
$$[f] \longmapsto f^*(\alpha)$$

It factors as

$$[X, G_1] \longrightarrow \mathrm{Vect}^1(X) \longrightarrow H^1(X; \mathbb{Z}_2)$$
$$[f] \longmapsto f^*(E_1) \longmapsto w_1(f^*(E_1)) = f^*(w_1(E_1)) = f^*(\alpha)$$

First map is bij, comp is isom $\Rightarrow$ 2^{nd} map bij. ▨

We can unravel the last step. Want to define

$$H^1(X; \mathbb{Z}_2) \longrightarrow \mathrm{Vect}^1(X)$$

inverse to w_1. Given $\varphi \in H^1$, define an $\mathbb{R}$-bundle skeleton by skeleton. On 1-skeleton, use φ to decide between Möbius & trivial bundle. As φ is a cocycle, it is trivial on any loop bounding a 2-cell, so can extend over 2-skeleton and higher.