The Euler Class

$e \in H^n(\tilde{G}_n; \mathbb{Z})$

$\rightsquigarrow$ e is n-dim class for oriented $\mathbb{R}^n$-bundles

idea: given n-chain, put it in gen. pos wrt 0-section,
count intersection points with sign. ~~It really think of this as n-cochain dual to these pts.~~

The Euler class satisfies:

(1) $e(f^*(E)) = f^* e(E)$

(2) $e(\overline{E}) = -e(E)$

(3) $e(E_1 \oplus E_2) = e(E_1) \cup e(E_2)$

(4) $e(E) = -e(E)$ n odd (i.e. $e(E)$ is 2-torsion)

(5) $e(E) = 0$ if E has nonvan section

(6) $\langle e(M), [M] \rangle = \chi(M)$

Instability. Unlike w_i, c_i the class e is unstable:

$$e(E \oplus \text{trivial}) = 0 \qquad \text{(nonvan section)}$$

The construction of e requires one tool.

Let $E' = E - 0\text{-sec}$.

We'll show $\exists\, c \in H^n(E, E')$ restricting in each fiber to a gen for $H^n(\mathbb{R}^n, \mathbb{R}^n \setminus 0)$. c = Thom class

Define e = restriction of c to 0-section: $H^*(E, E') \to H^*(E) \to H^*(B)$

This does just what we want:

To compute, perturb intersections to lie in fibers.

Thom Isomorphism

Orientability. $\mathbb{R}^n \to E \to B \rightsquigarrow$ disk bundle $D^n \to D(E) \to B$ and
sphere bundle $S^{n-1} \to S(E) \to B$

Say $E, D(E)$ orientable if $S(E)$ is

$S(E)$ orientable if the map $H^{n-1}(S^{n-1};\mathbb{Z}) \circlearrowleft$ induced by any loop in B is id.

e.g. T^2 is orientable S^1 bundle over S^1, K.B. nonorientable.

Thom class. A Thom class is a $c \in H^n(D(E), S(E);\mathbb{Z})$ restricting to gen for $H^n(D^n, S^{n-1};\mathbb{Z})$ in each fiber.

<u>Thm</u> E orientable $\Rightarrow c$ exists.

Thom isomorphism. The map
$$H^i(B;\mathbb{Z}) \to H^{i+n}(D(E), S(E);\mathbb{Z})$$
$$b \mapsto p^*(b) \cup c$$
is isom. $\forall\, i \geq 0$, and $H^i(D(E), S(E);\mathbb{Z}) = 0 \quad i < n$.

Thom space. $T(E) = D(E)/S(E)$ disk fibers $\rightsquigarrow$ spheres S^n in $T(E)$, all spheres meet at basept.

Thom class $\leftrightarrow$ elt of $H^n(T(E), x_0;\mathbb{Z}) \cong H^n(T(E);\mathbb{Z})$.
restricting to gen of $H^n(S^n;\mathbb{Z})$ in each "fiber"

Thom isom $\rightsquigarrow H^i(B;\mathbb{Z}) \cong \tilde{H}^{n+i}(T(E);\mathbb{Z})$

$T(E)$ central to Thom's work on cobordism.

Thom Class

* all coeffs = $\mathbb{Z}$

Thm. Every orientable bundle $E \to B$ has a Thom class

Pf. Assume B = connected CW complex. (finite dim)

Claim. $H^i(D(E), S(E)) \xrightarrow{\cong} H^i(D^n, S^{n-1})$ $\forall$ fibers.

Say B is k-dim, assume true for smaller dim complexes.
For concreteness $i = n$. Other cases easier.
Set U = nbd of B^{k-1}, $V = \coprod$ open k-cells
Mayer-Vietoris:

$$0 \longrightarrow H^n(D(E), S(E)) \longrightarrow H^n(D(E)_U, S(E)_U) \oplus H^n(D(E)_V, S(E)_V) \xrightarrow[\text{diff map}]{\psi} H^n(D(E)_{U\cap V}, S(E)_{U\cap V})$$

$$H^n(D(E)_U, S(E)_U) \cong H^n(D^n, S^{n-1}), \qquad H^n(D(E)_V, S(E)_V) \cong \bigoplus_{|V|} H^n(D^n, S^{n-1})$$

induction

$\uparrow$ by induction & $U \cap V \simeq \coprod S^{k-1}$ & $A \hookrightarrow B$ weak h.e. $\Rightarrow E_A \hookrightarrow E$ weak h.e.

Orientability $\Rightarrow$ can choose the gens for the $\oplus$ in the middle consistently

$\Rightarrow \ker \psi \cong \mathbb{Z} = \{(a, (a, \ldots, a))\}$

$\Rightarrow H^n(D(E), S(E)) \cong \mathbb{Z}$.

for mod 2 version skip this step.

$\boxed{\text{Can rewrite everything with } (E, E - (0\text{-sec})) \text{ \& } (\mathbb{R}^n, \mathbb{R}^n - 0)}$

Moreover the isom is given by restriction to fibers as

$$H^n(D(E), S(E)) \xrightarrow{\cong} \ker \psi \underset{\cong}{\xrightarrow[\text{any factor}]{\text{proj to}}} H^n(D^n, S^{n-1})$$

this map is restriction to fibers.

∎

Relative LH $\Rightarrow H^*(D(E), S(E))$ = free $H^*(B)$-module ~~over~~ w/ basis c

$\cong H^*(B)$

This is the Thom isomorphism.

Properties of the Euler Class

(1) Naturality. A pullback $f^*(E)$ comes with a map $f^*(E) \xrightarrow{\tilde{f}} E$ that is a lin. isom. on fibers. Thus $\tilde{f}$ pulls back the Thom class to a Thom class: $\tilde{f}^*(c(E)) = c(f^*(E))$
$\tilde{f}|_B = f$ so when we pass through

$$H^*(E, E') \longrightarrow H^*(E) \longrightarrow H^*(B)$$

we get the result.

(2) Negation. Basically obvious – negating the orientation of E negates all signs of intersection.

(3) Whitney sum. ~~Reduces~~ Consider $p_i : E_1 \oplus E_2 \longrightarrow E_i$. (linear on fibers)
Say $c(E_1) \in H^m(E_1, E_1')$ $c(E_2) \in H^n(E_2, E_2')$
Want: $p_1^*(c(E_1)) \smile p_2^*(c(E_2)) = c(E_1 \oplus E_2)$
Reduces to showing

$$H^m(\mathbb{R}^{m+n}, \mathbb{R}^{m+n} - \mathbb{R}^n) \times H^n(\mathbb{R}^{m+n}, \mathbb{R}^{m+n} - \mathbb{R}^m) \longrightarrow H^{m+n}(\mathbb{R}^{m+n}, \mathbb{R}^{m+n} \setminus 0)$$

takes (gen, gen) $\longmapsto$ gen.

(4) Odd dimensions. Use (2) plus the fact that negation is an orientation reversing automorphism

(5) Nonvanishing sections. Basically obvious – in the presence of a nonvan. section, any n-chain in B can be pushed completely off of B.

(6) Euler characteristic

We know $\langle e(M), M\rangle$ = self-int of M in TM

Step 1. $\langle e(M), M\rangle$ = self-int of Δ in $M\times M$.

Step 2. Latter = sum of indices of Lefschetz fixed pts of an $f: M\to M$

Step 3. Choose an f and compute

Step 1. Self-int of M in any $2n$-dim man. U equals $\langle e(N_U M), M\rangle$

Remains to show: $N_{M\times M}\Delta \cong TM$

A vector $(u,v)\in T_xM\times T_xM \cong T_{(x,x)} M\times M$

is tangent to $\Delta \iff u=v$

hence normal to $\Delta \iff u=-v$

The isomorphism $TM \longrightarrow N_{M\times M}\Delta$ is

$$(x,v)\longmapsto ((x,x),(v,-v)).$$

Step 2. $f: M\to M$ is Lefschetz if $Df - I$ invertible at each pt

The index of f at a fixed pt is $+1$ if $\det(Df-I)>0$, -1 o.w.

This number equals the sign of intersection of

Δ with graph $\Gamma(f)$ at $(x, f(x))$

Idea: Check sign of $\underbrace{(v_1,v_1),\dots,(v_n,v_n)}_{T\Delta},\ \underbrace{(v_1,Df(v_1)),\dots,(v_n,Df(v_n))}_{T\Gamma(f)}$

$\xrightarrow{\text{Gauss}}$ $(v_1,v_1),\dots,(v_n,v_n),\ (0,(Df-I)(v_1)),\dots,(0,(Df-I)(v_n))$ ← last n span $0\times T\Delta$

$\xrightarrow{\text{Gauss}}$ $(v_1,0),\dots,(v_n,0),\ (0,(Df-I)(v_1)),\dots,(0,(Df-I)(v_n))$

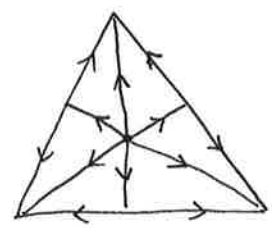
Claim follows.

or $\begin{vmatrix} I & I \\ I & Df \end{vmatrix} = \begin{vmatrix} I & I \\ 0 & Df-I \end{vmatrix} = |Df-I|$

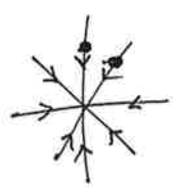
But $\Delta\cap\Gamma(f) = \Delta\cap\Delta$, so done.

Step 3. Find a nice Lefschetz function.

Choose a vector field, say one pointing from barycenters of higher dim. simplices to barycenters of lower dim simplices (actually, gradient flow for any Morse fn will work).

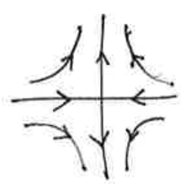


At a vertex:



face

edge:



Then f is time ε flow.

In the 3 cases, Df is $\begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$, $\begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}$, $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

So $\det(Df - I)$ is $+$ $\quad -$ $\quad +$

as desired.

Thom Isomorphism

The Thom Isom. reduces to a rel version of Leray-Hirsch.

Fiber bundle pairs. $F \to E \xrightarrow{p} B$ with $E' \subseteq E$ s.t. $E' \xrightarrow{p} B$
a bundle with fibers $F' \subseteq F$, compatible trivializations $\rightsquigarrow (E,E') \to B$
e.g. $S(E) \subseteq D(E)$

THM (Relative Leray-Hirsch). Say $(F,F') \to (E,E') \xrightarrow{p} B$ a f.b. pair
s.t. $H^*(F,F')$ f.g. free R-mod in each dim.
If $\exists\ c_j \in H^*(E,E')$ whose restrictions form a basis for $H^*(F,F')$
in each fiber then $H^*(E,E')$ = free $H^*(B)$-module w/ basis $\{c_j\}$.

Pf Main idea: Construct a related bundle $\hat{E}$, apply absolute LH to $\hat{E}$.

Construction of $\hat{E}$. Let M = mapping cyl of $p: E' \to B$ note $E' \subseteq M$

$$\hat{E} = M \amalg_{E'} E$$

$$\hat{F} = \text{cone on } \hat{E} = \text{mapping cyl. of const. map}$$

Key isomorphism. $H^*(\hat{E}) \cong H^*(\hat{E}, B) \oplus H^*(B)$ as $H^*(B)$ modules

$H^*(\hat{E},B) \cong H^*(E,E')$ ← killing E' in E same as killing M in $\hat{E}$, same as killing B in M in $\hat{E}$.

* splitting from retraction $p: \hat{E} \to B$.

Let $\hat{c}_j$ correspond to $(c_j, 0)$. The c_j & 1 restrict to basis's
for $H^*(\hat{F}) \cong H^*(F,F')$

LH $\Rightarrow$ $H^*(\hat{E})$ free $H^*(B)$-modules, basis $\{1, \hat{c}_j\}$
$\Rightarrow$ c_j free basis for $H^*(E,E')$.

Euler Class via Poincaré Duality

Fix some oriented $\mathbb{R}^n \longrightarrow E \longrightarrow B$ = smooth, oriented, k-manifold.

Let D = disk bundle of E.

D is an oriented $(n+k)$-manifold with ∂, so it has Poincaré duality

$$\begin{aligned} H^i(M, \partial M) &\xrightarrow{\cong} H_{n+k-i}(M) \\ \alpha &\longmapsto [M] \frown \alpha = \alpha^* \end{aligned}$$

relative fundamental class

Regard the fundamental class $[B]$ as elt of $H_k(D)$

via the map on H_* induced by $B \hookrightarrow D$.

<u>Prop.</u> $[B] = c^*$ in $H_k(D)$.

Thom class

So: An explicit cochain $\{2\text{-cells of } B\} \longrightarrow \mathbb{Z}$ representing u is given by counting intersections of a section with 2-cells of B (assuming gen. pos.). Actually, can replace the section with any subspace homotopic/homologous to B.

Pf. Apply three isomorphisms (WLOG B connected):

sphere bundle

$$\begin{array}{ccccccc} \mathbb{Z} = H^0(B) & \xrightarrow{\text{Thom}} & H^n(D, S) & \xrightarrow{\text{P.D.}} & H_k(D) & \longrightarrow & H_k(B) = \mathbb{Z} \\ & & \| & & & & \\ & & H^n(D, \partial D) & & & & \\ 1 & \longmapsto & c & \longmapsto & c^* & & \end{array}$$

Since the composition $\mathbb{Z} \to \mathbb{Z}$ is an iso, $c^* = \pm[B]$.

(Must work harder to get the sign.) ▨

Circle Bundles and the Euler Class

There are correspondences:

$$\mathbb{C}^1\text{-bundles} \longleftrightarrow \text{oriented } \mathbb{R}^2\text{-bundles} \longleftrightarrow \text{oriented } S^1\text{-bundles}$$

Both $\rightarrow$ are easy.

First $\leftarrow$ via Euc. metric. $\mathbb{C}$-structure is rotation by π.

Second $\leftarrow$ uses $\mathrm{Diff}^+(S^1) \simeq \mathrm{Isom}^+(S^1) \cong S^1$.

This implies we can modify the local trivializations so they remember distance on S^1. Then build $\mathbb{R}^2$-fibers by coning off S^1-fibers.

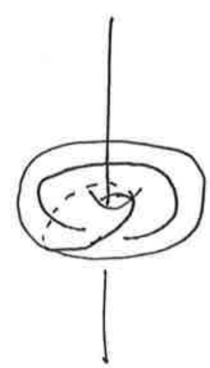
Key example. (Hopf bundle $S^1 \to S^3 \to S^2$) $\longleftrightarrow$ (CLB $\to \mathbb{CP}^1$)

$\mathbb{C}$-description $\quad S^3 = \{(z,w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\}$

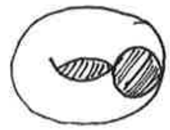
$(z,w) \longmapsto w/z \in \hat{\mathbb{C}}$

or $(z,w) \longmapsto$ line spanned by $(z,w) \in \mathbb{CP}^1$

Topological description



There are two $D^2 \times S^1$



The bundles over the two ∂D^2 are equal as sets

$\rightsquigarrow$ a map $S^3 \to S^2$

Euler class via sections of S^1-bundles

A bundle $S^1 \to E \to X$ is trivial iff it has a section.
For X = CW complex, can try to build a section inductively over skeleta
Say s_i = section over $X^{(i)}$
s_i extends over D^{i+1} iff $S^i \cong \partial D^{i+1} \xrightarrow{\text{attach}} X^{(i)} \xrightarrow{s_i} S^1$ is homot. trivial (exercise)
But we know: $\pi_i(S^1) = \begin{cases} \mathbb{Z} & i=1 \\ 0 & \text{o.w.} \end{cases}$

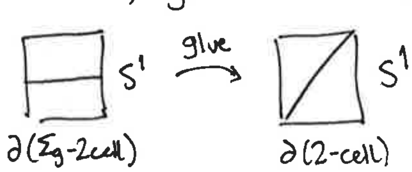
So only obstruction is over 2-skeleton.

Can use this idea to build a cochain $\{2\text{-cells of } X\} \to \mathbb{Z}$.
Step 1. Choose any section s_1 over $X^{(1)}$
Step 2. Take degrees of maps $\partial D^2 \to S^1$ as above.
Can check directly this is a cocycle. It vanishes $\iff$ trivial bundle.
(see Candel-Conlon).
It turns out this is the Euler class. See below.

We will show:
c_1 for $\mathbb{C}^1$-bundles $\longleftrightarrow$ e for or. $\mathbb{R}^2$-bundles $\longleftrightarrow$ e for or. S^1-bundles

We already showed: $c_1 : \mathrm{Vect}^1_{\mathbb{C}}(X) \xrightarrow{\cong} H^2(X;\mathbb{Z})$

For $X = \Sigma_g$ can build explicitly the unique E_k s.t. $e(E_k) = k \in \mathbb{Z} \cong H^2(\Sigma_g;\mathbb{Z})$.
Idea: Remove a 2-cell. Take trivial bundle over complement, trivial over 2-cell, glue with a twist on $\partial = T^2$



Dehn surgery on $\Sigma_g \times S^1$
Dehn twist in fiber direction.

use Dehn surgery description.

Exercise. $g=0$ $E_k = L(k,1)$ $L(0,1) = S^2 \times S^1$

note $\underset{\underset{\mathbb{RP}^3}{\|}}{L(2,1)} = UTS^2$ since have same Euler class.

$g=1$ $E_k = M_{\left[\begin{smallmatrix}1 & k \\ 0 & 1\end{smallmatrix}\right]}$

<u>Prop.</u> For $\mathbb{C} \to E \to X$, $c_1 = e = e$.

<u>Pf.</u> First compare e for S^1-bundles with c_1.

If we believe e is a char class, then we know it is a deg 1 poly in the c_i $\Rightarrow$ it is a multiple of c_1.

So suffices to check on $CLB \to \mathbb{CP}^1$.

By defn $c_1(CLB) = \alpha = 1 \in \mathbb{Z} \cong H^2(\mathbb{CP}^1)$.

We choose trivializations of the circle bundle $S^1 \to S^3 \to S^2$ over Δ, Δ^c and show corresponding sections over $S^1 = \partial\Delta$ intersect in one pt. This means (up to sign) $e=1$.

Over Δ: $\alpha \mapsto (\alpha, 1)/\text{norm}$

Δ^c: $\alpha \mapsto (1, \alpha)/\text{norm}$ $(\infty \mapsto (1,0))$

On $\partial\Delta$ these equal only for $\alpha = 1$.

exercise: check e for top. description.

We'll also show the two e's are same for X a manifold. ~~image in~~ ~~Hom(H₂(X;Z),Z)~~).

Idea: Suppose have a section of E over ∂D^2 of degree 1.

i.e. $(1,\theta) \mapsto \theta$.

Can try to extend to a section of assoc. $\mathbb{R}^2$-bundle.

$(r,\theta) \mapsto (r,\theta)$

There is one zero, at origin. So the cocycle we constructed for S^1-bundles counts intersection pts (with sign) of elts of $H^2(X;\mathbb{Z})$ with themselves.

Using this, and axioms for c_i can again show $e = c_1$.

Milnor-Wood Inequality

Thm. If $E \to \Sigma_g$ is oriented S^1-bundle with $g \geq 1$ and has a foliation transverse to the fibers, then
$$|e(E)| \leq |\chi(\Sigma_g)|.$$

Will show: $UT(\Sigma_g)$ realizes this bound.

There is a correspondence:
$$\left\{\begin{array}{l}\text{oriented } S^1\text{-bundles} \\ \text{over } M \text{ with} \\ \text{transverse foliation}\end{array}\right\} \longleftrightarrow \{\pi_1(M) \longrightarrow \mathrm{Homeo}^+(S^1)\}$$

$\to$ is monodromy (the foliation identifies pts of fibers).

$\leftarrow$ is: $\tilde{M} \times S^1 / \pi_1(M)$ by diag action gives the bundle, foliation by $\tilde{M} \times pt$ descends.

Unit tangent bundle of Σ_g. We already know $e(UT(\Sigma_g)) = \chi(\Sigma_g)$.

Need to find foliation.

Setup: $\tilde{\Sigma}_g = \mathbb{H}^2$ $\quad UT(\mathbb{H}^2) \cong \mathbb{H}^2 \times S^1$ (triv. given by proj. to $\partial_\infty \mathbb{H}^2 = S^1$)

$\pi_1(\Sigma_g) \longrightarrow \mathrm{Isom}^+(\mathbb{H}^2)$ via deck trans.

induces action on $UT(\mathbb{H}^2)$.

Quotient is precisely $UT(\Sigma_g)$, as desired.

$\downarrow$ So leaves are unit vectors with asymptotic rays.

Above theorem due to Wood. Milnor showed if the bundle admits a flat connection (curvature $=0$) then $|e(E)| \leq |\chi(\Sigma_g)|/2$. (This is a strictly stronger assumption.)

Later we'll use this to prove $\mathrm{Diff}^+(\Sigma_{g,1}) \longrightarrow \mathrm{MCG}(\Sigma_{g,1})$ has no section.