Pontryagin Classes

Complexification. $E \to B \rightsquigarrow E^{\mathbb{C}} \to B$

$E^{\mathbb{C}} = E \otimes \mathbb{C}$ or $E \oplus E$ with $i(x,y) = (-y,x)$.

Pontryagin classes. $p_i(E) = (-1)^i c_{2i}(E^{\mathbb{C}}) \in H^{4i}(B;\mathbb{Z})$

Why only even c_i? The $c_{2i+1}(E^{\mathbb{C}})$ are determined by the w_i:

$$c_{2i+1}(E^{\mathbb{C}}) = \beta(w_{2i}(E)\, w_{2i+1}(E))$$

$\uparrow$ Bockstein: $H^*(G_n;\mathbb{Z}_2) \circlearrowleft$

Relations to other classes. (1) $p_i(E) \mapsto w_{2i}(E)^2$ via $H^{4i}(B;\mathbb{Z}) \to H^{4i}(B;\mathbb{Z}_2)$

(2) $p_n(E) = e(E)^2$ E = orient. $\mathbb{R}^{2n}$-bundle.

Pf. Whitney sum, $c_{2i} \mapsto w_{4i}$, $c_{2n} = e$.

Later: $p_1(M^4) = \sigma(M^4)$

We can now describe all $\mathbb{Z}$ char classes for real (oriented) bundles.

<u>Thm</u>. (1) $H^*(G_n;\mathbb{Z})/\text{torsion} \cong \mathbb{Z}[p_1, \dots, p_{\lfloor n/2 \rfloor}]$

(2) $H^*(\tilde{G}_n;\mathbb{Z})/\text{torsion} \cong \begin{cases} \mathbb{Z}[\tilde{p}_1, \dots, \tilde{p}_{\lfloor n/2 \rfloor}] & n = 2k+1 \\ \mathbb{Z}[\tilde{p}_1, \dots, \tilde{p}_{\frac{n}{2}-1}, e] & n = 2k \end{cases}$

where $p_i = p_i(E_n)$, $\tilde{p}_i = p_i(\tilde{E}_n)$, $e = e(\tilde{E}_n)$.

All torsion is 2-torsion, so lies in $H^*(G_n;\mathbb{Z}_2)$. It is the image of the Bockstein homomorphism $\beta: H^*(G_n;\mathbb{Z}_2) \circlearrowleft$

Quick idea: Start with $0 \to \mathbb{Z}_2 \to \mathbb{Z}_4 \to \mathbb{Z}_2 \to 0$

Apply $\mathrm{Hom}(C_n(G_n), -) \rightsquigarrow$ LES in H^*

Get $\beta: H^n(G_n;\mathbb{Z}_2) \to H^{n+1}(G_n;\mathbb{Z}_2)$

(notice $\deg c_{2i+1} = \deg w_{2i} w_{2i+1} + 1$).

Gysin Sequence

The computation of $H^*(G_n;\mathbb{Z})$ needs one final tool:

$$\cdots \to H^{i-n}(B) \xrightarrow{\cup e} H^i(B) \xrightarrow{p^*} H^i(S(E)) \to H^{i-n+1}(B) \to \cdots$$

This sequence is the LES for $(D(E), S(E))$ in disguise:

$$\begin{tikzcd}
\cdots \arrow[r] & H^i(D(E),S(E)) \arrow[r, "j^*"] & H^i(D(E)) \arrow[r] & H^i(S(E)) \arrow[r] & H^{i+1}(D(E),S(E)) \arrow[r] & \cdots \\
\cdots \arrow[r] & H^{i-n}(B) \arrow[r, "\cup e"] \arrow[u, "\cong", "\Phi = \text{Thom}"'] & H^i(B) \arrow[r, "p^*"] \arrow[u, "\cong", "p^*"'] & H^i(S(E)) \arrow[r] \arrow[u, "="] & H^{i-n+1}(B) \arrow[r] \arrow[u, "\cong", "\Phi = \text{Thom}"'] & \cdots
\end{tikzcd}$$

Commutativity of first square.
$$\begin{aligned} j^*\Phi(b) &= j^*(p^*(b)\cup c) \\ &= p^*(b) \cup j^*(c) \\ &= p^*(b) \cup p^*(e) \\ &= p^*(b\cup e). \end{aligned}$$

The map $H^i(S(E)) \to H^{i-n+1}(B)$ is called the Gysin map.
It is defined s.t. the third square commutes.
For B a k-manifold, it can also be defined by:

$$H^i(S(E)) \xrightarrow{P.D.} H_{k+(n-1)-i}(S(E)) \xrightarrow{p_*} H_{k+(n-1)-i}(B) \xrightarrow{PD} H^{i-n+1}(B).$$

Or: given an i-cochain φ on $S(E)$ we evaluate on an $(i-n+1)$-chain τ in B by taking the pullback S^{n-1} bundle over τ and applying φ to this.

Computing with Gysin

The computation of $H^*(G_n;\mathbb{Z})$ is modeled on the following argument for $H^*(G_n;\mathbb{Z}_2)$.

$E_n \xrightarrow{\pi} G_n$ universal bundle

$S(E_n) = \{(v,\ell)\}$ ℓ = n-plane in $\mathbb{R}^\infty$, $v \in \ell$ unit.

Define $p : S(E_n) \longrightarrow G_{n-1}$

$(v,\ell) \longmapsto v^\perp \subseteq \ell$

This is a fiber bundle, with fiber S^∞ = unit vectors in $\mathbb{R}^\infty$ $\perp$ to given $(n-1)$-plane.

S^∞ contractible $\Rightarrow p^*$ is $\cong$ on H^*.

Gysin: $\cdots \longrightarrow H^i(G_n) \xrightarrow{\cup e} H^{i+n}(G_n) \xrightarrow{\eta} H^{i+n}(G_{n-1}) \longrightarrow H^{i+1}(G_n) \longrightarrow \cdots$

Key step. $\eta(w_j(E_n)) = w_j(E_{n-1})$.

By defn η is the composition $H^*(G_n) \xrightarrow{\pi^*} H^*(S(E_n)) \xleftarrow[\cong]{p^*} H^*(G_{n-1})$

induced by $G_{n-1} \xleftarrow{p} S(E_n) \xrightarrow{\pi} G_n$

Take pullback $\pi^*(E_n) = \{(v,w,\ell) : \ell \in G_n,\ v,w \in \ell,\ v \text{ unit}\}$

$\cong L \oplus p^*(E_{n-1})$

where L is subbundle with $w \in \operatorname{span}(v)$.

$p^*(E_{n-1})$ is subbundle with $w \perp v$.

But L is trivial: it has section (v,v,ℓ)

So: $\pi^* w_j(E_n) = w_j \pi^*(E_n) = w_j(L \oplus p^*(E_{n-1}))$

$= w_j\, p^*(E_{n-1}) = p^* w_j(E_{n-1})$ as desired.

Thus η surjective. Now induct on n!