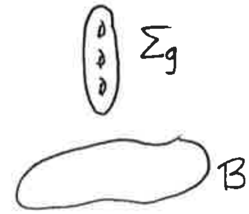
Characteristic Classes for Surface Bundles: An Overview

Surface bundles. These are smooth fiber bundles

$$\begin{array}{c} \Sigma_g \to E \\ \quad\ \downarrow \\ \quad\ B \end{array}$$



i.e. B covered by U s.t. $p^{-1}(U) \cong U \times \Sigma_g$ (restriction to fibers smooth)

Examples. $B \times \Sigma_g$

$M\varphi$ = mapping torus of $\varphi : \Sigma_g \to \Sigma_g$. $B = S^1$

$M\varphi \times S^1 \to T^2$

Isomorphism. As before, a homeo $E \xrightarrow{p} B$ to $E' \xrightarrow{p'} B$ taking $p^{-1}(b)$ to $(p')^{-1}(b)$ by diffeo.

Pullback. As before, given $f : A \to B$, we set

$$f^*(E) = \{(a,x) : f(a) = p(x)\}$$

Characteristic classes. Fix g, R. A char class is a fn

$$\chi : \{\Sigma_g\text{-bundles}\}/\cong \;\longrightarrow H^*(\text{Base}; R)$$

that is natural:

$$\chi(f^*(E)) = f^*\chi(E).$$

Why? Surface bundles are basic fiber bundles / manifolds.

Want invariants.

There are other applications to mapping class groups.

We study surface bundles in analogy with vector bundles.

- A Grassmannian for surface bundles

$C(\Sigma_g, \mathbb{R}^\infty)$ = space of smooth (oriented) submanifolds of $\mathbb{R}^\infty$ diffeo to Σ_g.

$\mathcal{E}(\Sigma_g, \mathbb{R}^\infty) = \{(x, S) \in \mathbb{R}^\infty \times C(\Sigma_g, \mathbb{R}^\infty) : x \in S\}$

$\mathcal{E}(\Sigma_g, \mathbb{R}^\infty) \longrightarrow C(\Sigma_g, \mathbb{R}^\infty)$ is a Σ_g-bundle.

We will show:

$$\{\Sigma_g\text{-bundles over } B\}/_{\cong} \longleftrightarrow [B, C(\Sigma_g, \mathbb{R}^\infty)]$$

and so (fixing g, $\mathbb{R}$):

$$\{\text{char. classes for } \Sigma_g\text{-bundles}\} \longleftrightarrow H^* C(\Sigma_g, \mathbb{R}^\infty).$$

- The mapping class group

In vector bundle case, can reduce structure group to $O(n)$ i.e. transition maps can be taken to be isometries on fibers. Have an analogous reduction here.

$$\begin{aligned} MCG(\Sigma_g) &= \pi_0 \mathrm{Diff}^+(\Sigma_g) \\ &= \mathrm{Diff}^+(\Sigma_g) / \text{isotopy} \end{aligned}$$

We'll show: $\mathrm{Diff}(\Sigma_g)$ has contractible components, i.e.

$$\mathrm{Diff}^+(\Sigma_g) \simeq MCG(\Sigma_g)$$

From this we can deduce:

$$\left\{\begin{matrix} \Sigma_g\text{-bundles} \\ \text{over } B \end{matrix}\right\} \longleftrightarrow [B, K(MCG(\Sigma_g), 1)]$$

$$\longleftrightarrow \mathrm{Hom}(\pi_1(B), MCG(\Sigma_g)) \Big/ \underbrace{MCG(\Sigma_g)}_{\text{conj.}}$$

and so:

$$\left\{\begin{matrix} \text{Char. classes} \\ \text{for } \Sigma_g\text{-bundles} \end{matrix}\right\} \longleftrightarrow H^* MCG(\Sigma_g).$$

- *Morita-Mumford-Miller classes.*

Here Gysin means:

$$H^{2i+2}(E) \xrightarrow{PD} H_{n-2i}(E) \xrightarrow{\mathrm{proj}_*} H_{n-2i}(B) \xrightarrow{PD} H^{2i}(B)$$

Given $\Sigma_g \to E \to M$ = smooth manifold

Let V = vertical 2-plane bundle on E

Define $e_i(E) = \mathrm{Gysin}(e^{i+1}) \in H^{2i}(M)$.

We'll see: e_1 is proportional to: signature, WP form, 1^{st} Pontryagin class.

<u>Thm</u> $\lim\limits_{g\to\infty} H^*(MCG(\Sigma_g^1); \mathbb{Q}) \cong \mathbb{Q}[e_1, e_2, \dots]$

i.e. the e_i exactly describe the stable rational char. classes.

- Unstable classes

We know $\chi(MCG(\Sigma_g)) = \zeta(1-2g)/2-2g$. So there are lots of other char. classes. Almost nothing is known.

COHOMOLOGY OF MAPPING CLASS GROUPS

coeff $= \mathbb{Q}$

THM. $\mathrm{vcd}(MCG(\Sigma_g)) = 4g-5$ $\Rightarrow$ $H^i(MCG(\Sigma_g)) = 0$ $i > 4g-5$ (any coeff)
(although $H^{4g-5}(MCG(\Sigma_g)) = 0$).

Low dim's:
$H^1(MCG(\Sigma_g)) = 0$ $g \geq 0$.
$H^2(MCG(\Sigma_g)) = \mathbb{Q}$ $g \geq 4$
$H^3(MCG(\Sigma_g)) = 0$ $g \geq 6$
$H^4(MCG(\Sigma_g)) = \mathbb{Q}^2$ $g \geq 10$.

Low genus:
$H^*(MCG(T^2)) = 0$.
$H^*(MCG(\Sigma_2)) = 0$
$H^*(MCG(\Sigma_3)) = \mathbb{Q}[\overset{c_2,}{c_6}]$ c_5, c_6 unstable.
$H^*(MCG(\Sigma_4)) = \mathbb{Q}[\overset{c_2,}{c_4}, c_5]$

Stability. $H^i(MCG(\Sigma_g^1))$ indep of g, $g \geq 3i/2 + 1$.

Also, $H^i(MCG(\Sigma_g^1)) \cong H^i(MCG(\Sigma_g))$ in this case.

Mumford Conjecture. $H^i(MCG(\Sigma_\infty^1)) = \mathbb{Q}[e_1, e_2, \ldots]$ $e_i \in H^{2i}$ i^{th} MMM class

Euler char. $\chi(MCG(\Sigma_g)) = \zeta(1-2g)/2-2g \sim (-1)^g \, (2g-1)!/2^{2g-1} \pi^{2g}$
$\Rightarrow$ $> 2^g$ unstable classes. use: $p(n) \sim \frac{1}{n} e^{\pi\sqrt{2n/3}}$

Applications. ① $\mathrm{Diff}^+(\Sigma_g) \xrightarrow{\pi} MCG(\Sigma_g)$ has no section
pf: $\pi^*(e_3) = 0$.

② Odd e_i are geometric, cobordism invar, vanish on handlebody group.