Cup product and intersections

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Abstract

This is a handout for an algebraic topology course. The goal is to explain a geometric interpretation of the cup product. Namely, if X is a closed oriented smooth manifold, if A and B are oriented submanifolds of X, and if A and B intersect transversely, then the Poincaré dual of $A \cap B$ is the cup product of the Poincaré duals of A and B. As an application, we prove the Lefschetz fixed point formula on a manifold. As a byproduct of the proof, we explain why the Euler class of a smooth oriented vector bundle is Poincaré dual to the zero set of a generic section.

1 Statement of the result

A question frequently asked by algebraic topology students is: "What does cup product *mean*?" Theorem 1.1 below gives a partial answer to this question. The theorem says roughly that on a manifold, cup product is Poincaré dual to intersection of submanifolds. This is arguably the most important thing to know about cup product.

To state the theorem precisely, let X be a closed oriented smooth manifold of dimension n. Let A and B be oriented smooth submanifolds of X of dimensions $n-i$ and $n-j$ respectively. Assume that A and B intersect transversely. This means that for every $p \in A \cap B$, the map $T_pA \oplus T_pB \to T_pX$ induced by the inclusions is surjective. Then $A \cap B$ is a submanifold of dimension $n-(i+j)$, and there is a short exact sequence

$$0 \longrightarrow T_p(A \cap B) \longrightarrow T_pA \oplus T_pB \longrightarrow T_pX \longrightarrow 0.$$

This exact sequence determines an orientation of $A \cap B$. We will adopt the following convention. We can choose an oriented basis

$$u_1, \ldots, u_{n-i-j}, v_1, \ldots, v_j, w_1, \ldots, w_i$$

for T_pX such that $u_1, \ldots, u_{n-i-j}, v_1, \ldots, v_j$ is an oriented basis for T_pA and $u_1, \ldots, u_{n-i-j}, w_1, \ldots, w_i$ is an oriented basis for T_pB. We then declare that $u_1, \ldots, u_{n-i-j}$ is an oriented basis for $T_p(A \cap B)$.

The most important case is when A and B have complementary dimension, i.e. $i + j = n$, so that $A \cap B$ is a finite set of points. In this case an intersection p is positively oriented if and only if the isomorphism $T_pA \oplus T_pB \simeq T_pX$ is orientation preserving.

Now recall that there is a Poincaré duality isomorphism

$$\begin{aligned} H^i(M;\mathbb{Z}) &\xrightarrow{\simeq} H_{n-i}(M), \\ \alpha &\longmapsto [M] \frown \alpha. \end{aligned}$$

The images of the fundamental classes of A, B, and $A \cap B$ under the inclusions into X define homology classes $[A] \in H_{n-i}(X)$, $[B] \in H_{n-j}(X)$, and $[A \cap B] \in H_{n-i-j}(X)$. We denote their Poincaré duals by $[A]^* \in H^i(X;\mathbb{Z})$, $[B]^* \in H^j(X;\mathbb{Z})$, and $[A \cap B]^* \in H^{i+j}(X;\mathbb{Z})$. We now have:

Theorem 1.1. *Cup product is Poincaré dual to intersection:*

$$[A]^* \smile [B]^* = [A \cap B]^* \in H^{i+j}(X;\mathbb{Z}).$$

This theorem only partially answers the question of what cup product means, because it only works in a smooth manifold, and moreover not every homology class in a smooth manifold can be represented by a submanifold (although counterexamples to this last statement are somewhat hard to come by). In any case, Theorem 1.1 is very useful. Before proving it, we consider some examples.

Example 1.2. Consider the complex projective space $X = \mathbb{C}P^n$. Recall that $\mathbb{C}P^n$ has a cell decomposition with one cell in each of the dimensions $0, 2, \ldots, 2n$. Thus $H^*(\mathbb{C}P^n;\mathbb{Z})$ is isomorphic to $\mathbb{Z}$ in degrees $0, 2, \ldots, 2n$, and 0 in all other degrees. Moreover, $H^{2i}(\mathbb{C}P^n;\mathbb{Z})$ has a canonical generator α_i, which is the Poincaré dual of a complex $(n-i)$-plane in $\mathbb{C}P^n$, with the complex orientation. Now a generic $(n-i)$-plane intersects a generic $(n-j)$-plane transversely in an $(n-i-j)$-plane with the complex orientation (or the empty set when $i + j > n$). So by Theorem 1.1,

$$\alpha_i \smile \alpha_j = \begin{cases} \alpha_{i+j}, & i + j \leq n, \\ 0, & i + j > n. \end{cases}$$

Note that the nondegeneracy of the cup product pairing in Poincaré duality implies that $\alpha_i \smile \alpha_j = \pm\alpha_{i+j}$, but the above calculation determines the signs (or lack thereof).

Example 1.3. Let $X = T^2 = \mathbb{R}^2/\mathbb{Z}^2$. Then $H_1(T^2) \simeq \mathbb{Z}^2$, and we choose generators $[A]$ and $[B]$ where A and B are circles in the x and y directions of $\mathbb{R}^2$, respectively. Also $H_0(T^2) = \mathbb{Z}$ has a canonical generator $[p]$, which is the class of a point p. Let α, β, μ denote the Poincaré duals of $[A], [B], [p]$. Now $A \cap B$ is a positively oriented point, while $B \cap A$ is a negatively oriented point. Thus

$$\alpha \smile \beta = \mu, \qquad \beta \smile \alpha = -\mu.$$

On the other hand,

$$\alpha \smile \alpha = \beta \smile \beta = 0.$$

This follows from the sign-commutativity of the cup product. In terms of Theorem 1.1, to compute $\alpha \cup \alpha$ we need to calculate the signed intersection number of two transversely intersecting submanifolds A_1, A_2 representing the class $[A]$. We cannot take $A_1 = A_2$, but we can take A_1 and A_2 to be parallel, in which case they do not intersect.

Note that the basis $\{[A], [B]\}$ of $H_1(X)$ has a dual basis $\{[A]', [B]'\}$ of $\mathrm{Hom}(H_1(X), \mathbb{Z}) = H^1(X; \mathbb{Z})$ with $\langle [A], [A]' \rangle = \langle [B], [B]' \rangle = 1$ and $\langle [A], [B]' \rangle = \langle [B], [A]' \rangle = 0$. This dual basis does *not* consist of the Poincaré duals of $[A]$ and $[B]$. Rather, we have $[A]' = \beta$ and $[B]' = -\alpha$. We can verify these equations by computing their cup products with α and β. For example, to check that $\alpha \cup [A]' = \mu$, we have

$$[X] \frown (\alpha \cup [A]') = ([X] \frown \alpha) \frown [A]' = [A] \frown [A]' = 1 = [X] \frown \mu.$$

In general, if X is a closed oriented manifold of dimension n, we define the *intersection pairing*

$$\cdot : H_{n-i}(X) \otimes H_{n-j}(X) \longrightarrow H_{n-i-j}(X)$$

by applying Poincaré duality, taking the cup product, and then applying Poincaré duality again:

$$\alpha \cdot \beta := [X] \frown (\alpha^* \smile \beta^*).$$

Theorem 1.1 then says that if A and B are transversely intersecting oriented submanifolds representing α and β, then

$$\alpha \cdot \beta = [A \cap B].$$

In particular, if X is connected and if $\dim(A) + \dim(B) = \dim(X)$, then $\alpha \cdot \beta \in H_0(X) = \mathbb{Z}$ is simply the signed number of intersection points

$\#(A \cap B)$. When X is not connected, we usually interpret $\alpha \cdot \beta$ to be the image of this element of $H_0(X)$ under the augmentation map $H_0(X) \to \mathbb{Z}$, i.e. the total signed number of intersection points.

There is also an obvious analogue of Theorem 1.1 for unoriented manifolds using $\mathbb{Z}/2$ coefficients.

2 The Lefschetz fixed point theorem

A more interesting application of Theorem 1.1 is given by the following version of the Lefschetz fixed point theorem.

Let X be a closed smooth manifold and let $f : X \to X$ be a smooth map. A *fixed point* of f is a point $p \in X$ such that $f(p) = p$. The fixed point p is *nondegenerate* if

$$1 - df_p : T_pX \to T_pX$$

is invertible. If p is nondegenerate, we define the *Lefschetz sign* $\epsilon(p) \in \{\pm 1\}$ to be the sign of $\det(1 - df_p)$. It is a fact, which we will not prove here, that if f is "generic", then all the fixed points are nondegenerate, in which case there are only finitely many of them. In this situation we define the signed count of fixed points

$$\# \operatorname{Fix}(f) := \sum_{f(p)=p} \epsilon(p) \in \mathbb{Z}.$$

The Lefschetz theorem then says:

Theorem 2.1. *Let X be a closed smooth manifold and let $f : X \to X$ be a smooth map with all fixed points nondegenerate. Then*

$$\# \operatorname{Fix}(f) = \sum_i (-1)^i \operatorname{Tr}\left(f_* : H_i(X;\mathbb{Q}) \to H_i(X;\mathbb{Q})\right).$$

Note that it follows from the universal coefficient theorem that the above traces are integers.

Example 2.2. The Poincaré-Hopf index theorem asserts that if X is a closed smooth manifold and if V is a vector field on X with isolated zeroes, then

$$\sum_{V(p)=0} \deg(V,p) = \chi(X). \tag{1}$$

Here $\deg(V,p)$ is an integer defined as follows. Use local coordinates around p to regard V as a map from $\mathbb{R}^n$ to $\mathbb{R}^n$ with $V(0)=0$. Restricting V to a small sphere around 0 gives a map $S^{n-1} \to \mathbb{R}^n \setminus \{0\} \approx S^{n-1}$, and $\deg(V,p)$ is the degree of this map. This does not depend on the choice of local coordinates.

We say that the zero p of V is *nondegenerate* if the derivative $\nabla V_p : T_pX \to T_pX$ is invertible. In this case

$$\deg(V,p) = \operatorname{sign}(\det(\nabla V_p)) \in \{\pm 1\}.$$

If all zeroes of V are nondegenerate, then $f = \exp(tV)$ for $t > 0$ small is a diffeomorphism of X with nondegenerate fixed points corresponding to the zeroes of V. One can check that if p is a fixed point, then the Lefschetz sign $\epsilon(p) = \deg(V,p)$. Consequently,

$$\sum_{V(p)=0} \deg(V,p) = \#\operatorname{Fix}(f) = \sum_i \dim H_i(X;\mathbb{Q}) = \chi(X).$$

Here the second equality follows from the Lefschetz fixed point theorem because f is homotopic to the identity. This proves the Poincaré-Hopf index theorem in the nondegenerate case. (One can deduce the general case of the Poincaré-Hopf index theorem by showing that the left hand side of (1) does not depend on V.)

To prove the Lefschetz theorem, we will do intersection theory in $X \times X$. We will assume that X is orientable, although this assumption can be removed, see below. Define the *diagonal*

$$\Delta := \{(x,x) \mid x \in X\} \subset X \times X.$$

Also, define the *graph*

$$\Gamma(f) := \{(x, f(x)) \mid x \in X\} \subset X \times X.$$

There is an obvious bijection between fixed points of f and intersections of the graph and the diagonal. More precisely, we have:

Lemma 2.3. *f has nondegenerate fixed points if and only if $\Gamma(f)$ and Δ intersect transversely in $X \times X$. In that case, for each fixed point p, the Lefschetz sign $\epsilon(p)$ agrees with the sign of the intersection of $\Gamma(f)$ and Δ at (p,p).*

Proof. Exercise. □

It follows from Lemma 2.3 and Theorem 1.1 that if f has nondegenerate fixed points, then

$$\# \operatorname{Fix}(f) = [\Gamma(f)] \cdot [\Delta].$$

To complete the proof of the Lefschetz theorem, we now compute the intersection number $[\Gamma(f)] \cdot [\Delta]$. This requires the following lemmas about intersection theory in $X \times X$.

Recall first that for any topological spaces X and Y there is a homology cross product

$$\times : H_i(X) \otimes H_j(Y) \to H_{i+j}(X \times Y).$$

If X and Y are smooth manifolds and if A and B are closed oriented submanifolds of X and Y respectively, then $[A] \times [B] = [A \times B]$.

Returning to the proof of the Lefschetz theorem, write $n = \dim(X)$, and if $\alpha \in H_*(X)$ has pure degree, denote this degree by $|\alpha|$.

Lemma 2.4. *Let $\alpha, \beta, \gamma, \delta \in H_*(X)$ with $|\alpha| + |\beta| = |\gamma| + |\delta| = n$. Then*

$$(\alpha \times \beta) \cdot (\gamma \times \delta) = \begin{cases} (-1)^{|\beta|}(\alpha \cdot \gamma)(\beta \cdot \delta) & \text{if } |\beta| = |\gamma|, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 2.5. *If $\alpha, \beta \in H_*(X)$ with $|\alpha| + |\beta| = n$, then*

$$[\Gamma(f)] \cdot (\alpha \times \beta) = (-1)^{|\alpha|} f_* \alpha \cdot \beta.$$

Note that if $\alpha, \beta, \delta, \gamma$ can be represented by submanifolds, and if one arranges these submanifolds to intersect transversely, then the above two lemmas correspond under Theorem 1.1 (up to checking signs) to obvious set-theoretic facts. In general, these lemmas follow from basic properties of cap and cross products, and we leave the details as an exercise.

Now let $\{e_k\}$ be a basis for the vector space $H_*(X;\mathbb{Q})$, consisting of elements of pure degree. Let $\{e'_k\}$ be the dual basis of $H_*(X;\mathbb{Q})$, with respect to the intersection pairing. That is, if $e_k \in H_m(X;\mathbb{Q})$, then $e'_k \in H_{n-m}(X;\mathbb{Q})$ satisfies $e_i \cdot e'_j = \delta_{i,j}$. This dual basis exists and is unique since the intersection pairing is a perfect pairing.

Recall that by the Künneth theorem, $H_*(X \times X;\mathbb{Q}) = H_*(X;\mathbb{Q}) \otimes H_*(X;\mathbb{Q})$, with the isomorphism given by the homology cross product. Then $\{e_i \times e'_j\}$ is a basis for $H_*(X \times X;\mathbb{Q})$, and so is $\{e'_i \times e_j\}$.

Lemma 2.6. $[\Delta] = \sum_k e_k \times e'_k$.

Proof. Since the intersection pairing is perfect, it is enough to check that both sides of the equation have the same intersection pairing with $e_i' \times e_j$ when $|e_i'| + |e_j| = n$. By Lemma 2.4, and Lemma 2.5 with $f = \mathrm{id}_X$, we have

$$\begin{aligned}\left(\sum_k e_k \times e_k'\right) \cdot (e_i' \times e_j) &= \sum_{\{k : |e_k'| = |e_i'|\}} (-1)^{|e_i'|} (e_k \cdot e_i')(e_k' \cdot e_j) \\ &= \sum_k (-1)^{|e_i'|} e_i' \cdot e_j \\ &= [\Delta] \cdot (e_i' \times e_j).\end{aligned}$$

□

Proof of Theorem 2.1. By Lemmas 2.5 and 2.6, we have

$$\begin{aligned}[\Gamma(f)] \cdot [\Delta] &= [\Gamma(f)] \cdot \sum_k e_k \times e_k' \\ &= \sum_k (-1)^{|e_k|} f_* e_k \cdot e_k' \\ &= \sum_i (-1)^i \operatorname{Tr}\left(f_* : H_i(X;\mathbb{Q}) \to H_i(X;\mathbb{Q})\right).\end{aligned}$$

□

Exercise 2.7. How can one remove the assumption that X is orientable? (Hint: see handout on homology with local coefficients.)

3 The Thom isomorphism theorem

We now assume some familiarity with vector bundles. Let E be a real vector bundle over B of rank n. Regard B as a subset of E via the zero section. The proof of Theorem 1.1 will use part (a) of the following theorem, which is a version of the Thom isomorphism theorem.

Theorem 3.1. *Let $\pi : E \to B$ be an oriented rank n real vector bundle. Then:*

(a) There is a unique cohomology class $u \in H^n(E, E \setminus B; \mathbb{Z})$ such that for every $x \in B$, the restriction of u to $H^n(E_x, E_x \setminus \{0\}; \mathbb{Z}) \simeq \mathbb{Z}$ is the prefered generator determined by the orientation.

(b) The map

$$\begin{aligned} H^p(B;\mathbb{Z}) &\longrightarrow H^{p+n}(E, E\setminus B;\mathbb{Z}),\\ \alpha &\longmapsto \pi^*\alpha \smile u \end{aligned}$$

is an isomorphism.

The class u is called the *Thom class* of E.

Proof. There is a relative version of the Leray-Serre spectral sequence[1] with

$$E_2^{p,q} = H^p(B; \{H^q(E_x, E_x \setminus \{0\}; \mathbb{Z})\})$$

which converges to $H^*(E, E \setminus B; \mathbb{Z})$. The orientation of E identifies the twisted coefficient system $\{\{H^q(E_x, E_x \setminus \{0\}; \mathbb{Z})\}$ with $\mathbb{Z}$ for $q = n$ and with 0 for $q \neq n$. Part (a) follows instantly, as well as the fact that $H^p(B;\mathbb{Z}) \simeq H^{p+n}(E, E \setminus B; \mathbb{Z})$. The fact that the latter isomorphism is given by the cup product with u requires a bit more work which we omit.

A longer proof of the Thom isomorphism which does not use spectral sequences may be found in [3]. □

Remark 3.2. In the Thom isomorphism theorem, one can replace the coefficient ring $\mathbb{Z}$ by any commutative ring R with unit. If $R = \mathbb{Z}/2$, an orientation of E is not needed.

When E has a metric, one can give a slightly more intuitive description of the Thom class as follows. Let D be the unit disk bundle over B, consisting of vectors of length ≤ 1. Define S to be the unit sphere bundle over B, consisting of vectors of length equal to 1. By excision there is a canonical isomorphism $H^*(E, E \setminus B; \mathbb{Z}) \simeq H^*(D, D \setminus B; \mathbb{Z})$, and the latter is isomorphic to $H^*(D, S; \mathbb{Z})$, by the long exact sequence of the triple $(D, D \setminus B, S)$. Thus we have a canonical isomorphism

$$H^*(E, E \setminus B; \mathbb{Z}) = H^*(D, S; \mathbb{Z}).$$

The Thom class can then be described as the unique element of $H^n(D, S; \mathbb{Z})$ which for each $x \in B$ restricts to the generator of $H^n(D_x, S_x; \mathbb{Z})$ determined by the orientation.

Intuitively, the Thom class u, evaluated on an n-chain α, counts the intersections of α with the zero-section. Lemma 4.1 below is a precise version of this intuition, as will eventually become clear.

[1] At this point in the course we have not yet introduced spectral sequences, but we present this short proof here to demonstrate why they are worth learning.

4 Proof of the main theorem

We now prove Theorem 1.1. The proof here is based on material in [2, 3], with some modifications. Below, if A is a subspace of X, let $\imath_A^X : A \to X$ denote the inclusion.

Recall that if M is a closed oriented n-manifold with boundary, then there is a relative fundamental class $[M] \in H_n(M, \partial M)$, and a Poincaré duality isomorphism

$$\begin{aligned} H^i(M, \partial M; \mathbb{Z}) &\xrightarrow{\simeq} H_{n-i}(M; \mathbb{Z}), \\ \alpha &\longmapsto [M] \frown \alpha. \end{aligned}$$

The following lemma says that in the smooth case, the Thom class is just the Poincaré dual of the zero section.

Lemma 4.1. *Let B be a closed smooth oriented k-manifold, and let E be a smooth rank n oriented real vector bundle over B, with unit disk bundle D. Then*

$$\left(\imath_B^D\right)_* [B] = [D] \frown u \in H_k(D).$$

Here E is given any metric; we know that a metric exists. Also B is regarded as a submanifold of D via the zero section. Finally, the orientation on D is determined by the orientations of the fibers and the base, in that order. (It would perhaps be more usual to use the other order, but then we would have more minus signs in our formulas.)

Proof. Without loss of generality, B is connected. We now have isomorphisms

$$\mathbb{Z} = H^0(B; \mathbb{Z}) \xrightarrow{\pi^*(\cdot) \smile u} H^n(D, \partial D; \mathbb{Z}) \xrightarrow{[D] \frown} H_k(D) \xrightarrow{\pi_*} H_k(B) = \mathbb{Z}.$$

The generator $1 \in H^0(B; \mathbb{Z})$ maps to $[D] \frown u \in H_k(D)$, and this must map to a generator of $H_k(B)$. Since π_* is an isomorphism on homology, it follows that $[D] \frown u = \pm \left(\imath_B^D\right)_* [B]$.

Unfortunately, I do not know a very satisfactory way to nail down the sign from this point of view, since the details of the definition of the cap/cup product have been abstracted away. One approach is to pass to real coefficients and use de Rham cohomology, where the cup product is just wedge product of differential forms, and so the sign is easily understood in terms of the orientations. See for instance [1]. $\square$

Returning to the setting of the main theorem, let $N_A^X = N$ be a tubular neighborhood of A, which we can regard as an oriented rank i vector bundle over A, using the orientation convention in Lemma 4.1. The Thom class of A can be regarded as an element

$$u_A \in H^i(N, N \setminus A; \mathbb{Z}).$$

Let $u_A^{X,X\setminus A}$ denote the inverse of u_A under the excision isomorphism

$$H^i(X, X \setminus A; \mathbb{Z}) \to H^i(N, N \setminus A; \mathbb{Z}).$$

Let u_A^X denote the image of $u_A^{X,X\setminus A}$ under the "restriction"

$$H^i(X, X \setminus A; Z) \longrightarrow H^i(X; \mathbb{Z}).$$

Lemma 4.2. *$[A]^* = u_A^X \in H^i(X; \mathbb{Z})$.*

Proof. Since Poincaré duality is an isomorphism, we can equivalently prove

$$\left(\imath_A^X\right)_* [A] = [X] \frown u_A^X \in H_{n-i}(X). \tag{2}$$

To do so, consider the commutative diagram

$$\begin{tikzcd}
H_n(X) \otimes H^i(X, X \setminus A) \arrow[r] \arrow[d] & H_n(X, X \setminus A) \otimes H^i(X, X \setminus A) \arrow[r] \arrow[d, "\frown"] & H_n(N, N \setminus A) \otimes H^i(N, N \setminus A) \arrow[d, "\frown"] \\
H_n(X) \otimes H^i(X) \arrow[r, "\frown"] & H_{n-i}(X) & H_{n-i}(N) \arrow[l]
\end{tikzcd}$$

where all cohomology is with integer coefficients, and the upper right arrow is the tensor product with an inverse excision isomorphism and an excision isomorphism. The diagram commutes by the naturality of the cap product. Now start with $[X] \otimes u_A^X$ in the lower left. By definition, this lifts to $[X] \otimes u_A^{X,X\setminus A}$ in the upper left. The top row maps this to $[N] \otimes u_A$ in the upper right (because it follows from the defining property of the fundamental class that the composition

$$H_n(X) \longrightarrow H_n(X, X \setminus A) \xleftarrow{\simeq} H_n(N, N \setminus A)$$

maps $[X]$ to $[N]$). By Lemma 4.1, we know that $[N] \otimes u_A$ in the upper right maps down to $\left(\imath_A^N\right)_* [A] \in H_{n-i}(N)$, and the lower right arrow then sends this to $\left(\imath_A^X\right)_* [A] \in H_{n-i}(X)$. By commutativity of the diagram, this equals the image of $[X] \otimes u_A^X$ under the lower left arrow, namely $[X] \frown u_A^X$. $\square$

Proof of Theorem 1.1. To start, recall that we are given orientations of X, A, and B, and then $A \cap B$ is oriented according to the convention in Theorem 1.1. Using these orientations, orient the normal bundles N_A^X, N_B^X, $N_{A\cap B}^X$, and $N_{A\cap B}^A$ according to the "fiber first" convention in Lemma 4.1, so that Lemma 4.2 holds for their Thom classes without any sign.

Observe that there is a canonical isomorphism of vector bundles

$$N_{A\cap B}^A = \left(N_B^X\right)|_{A\cap B}.$$

Moreover, with the above orientation conventions, this is an isomorphism of oriented vector bundles. It then follows from the characterizing property of the Thom class that

$$u_{A\cap B}^A = \left(\imath_A^X\right)^* u_B^X. \tag{3}$$

Now to prove Theorem 1.1, by Lemma 4.2 it is enough to show that $u_{A\cap B}^X = u_A^X \smile u_B^X$. Equivalently, since Poincaré duality is an isomorphism, it is enough to show that

$$[X] \frown u_{A\cap B}^X = [X] \frown (u_A^X \smile u_B^X).$$

Using equations (2) and (3), we compute

$$\begin{aligned}
[X] \frown u_{A\cap B}^X &= \left(\imath_{A\cap B}^X\right)_* [A\cap B] \\
&= \left(\imath_A^X\right)_* \left(\imath_{A\cap B}^A\right)_* [A\cap B] \\
&= \left(\imath_A^X\right)_* \left([A] \frown u_{A\cap B}^A\right) \\
&= \left(\imath_A^X\right)_* \left([A] \frown \left(\imath_A^X\right)^* u_B^X\right) \\
&= \left(\left(\imath_A^X\right)_* [A]\right) \frown u_B^X \\
&= \left([X] \frown u_A^X\right) \frown u_B^X \\
&= [X] \frown \left(u_A^X \smile u_B^X\right).
\end{aligned}$$

□

5 The Euler class and the zero section

Definition 5.1. If $E \to B$ is an oriented rank n real vector bundle, define the *Euler class*

$$e(E) \in H^n(B;\mathbb{Z})$$

to be the image of the Thom class u under the composition

$$H^n(E, E\setminus B;\mathbb{Z}) \longrightarrow H^n(E;\mathbb{Z}) \xleftarrow{\pi^*} H^n(B;\mathbb{Z}).$$

It follows directly from the definition that e is natural, and that $e(E)=0$ if E has a nonvanishing section. If B is a CW complex, then it can be shown using naturality that $e(E)$ is the primary obstruction to the existence of a nonvanishing section.

A related fact is that the Euler class of a smooth vector bundle over a closed oriented manifold is Poincaré dual to the zero set of a generic section. We can now prove this using the machinery introduced above. Let ψ be a section of E, let $\Gamma=\{\psi(x)\mid x\in B\}\subset E$ denote the image of ψ (which we will call the "graph"), and let $Z=\psi^{-1}(0)=\Gamma\cap B$ denote the zero set of ψ. Standard transversality arguments show that for a generic section ψ, the graph Γ is transverse to the zero section $B\subset E$. It then follows that the zero set Z is a submanifold of B, and the derivative of ψ along the zero section defines an isomorphism of vector bundles

$$N_Z^B\simeq E|_Z. \tag{4}$$

We use this isomorphism to orient $N_Z B$, and then by the convention in Lemma 4.1 this orients Z.

Theorem 5.2. *Let $E\to B$ be a smooth oriented rank n real vector bundle over a closed oriented manifold B. Let ψ be a section whose graph is transverse to the zero section and let $Z=\psi^{-1}(0)$, oriented as above. Then*

$$e(E)=[Z]^*\in H^n(B;\mathbb{Z}).$$

Proof. Let $u\in H^n(E,E\setminus B;\mathbb{Z})$ denote the Thom class of E, and let $u|_E$ denote its image under the map $H^n(E,E\setminus B;\mathbb{Z})\to H^n(E;\mathbb{Z})$. Identify the normal bundle N_Z^B with an open tubular neighborhood N of Z in B, such that the zero section is identified with Z and the derivative of the identification along the zero section is the identity. Let $u_Z\in H^n(N,N\setminus Z;\mathbb{Z})$ denote the Thom class of N_Z^B.

Observe that the map of pairs $\psi|_N:(N,N\setminus Z)\to(E,E\setminus B)$ is homotopic through maps of pairs to the map $(N,N\setminus Z)\to(E|_Z,E|_Z\setminus Z)$ induced by the oriented bundle isomorphism (4). It then follows from the characterizing property of the Thom class that

$$\psi|_N^*u=u_Z\in H^n(N,N\setminus Z;\mathbb{Z}).$$

Applying the excision isomorphism to $H^n(B,B\setminus Z;\mathbb{Z})$ to this equation and then applying the natural map to $H^n(B;\mathbb{Z})$, we obtain, in the notation of Lemma 4.2, the identity

$$\psi^*(u|_E)=u_Z^B\in H^n(B;\mathbb{Z}).$$

In this equation, the left hand side is $e(E)$ by definition, while the right hand side is $[Z]^*$ by Lemma 4.2. $\square$

Example 5.3. If M is a closed oriented connected smooth manifold, then it follows from Theorem 5.2 and Example 2.2 that

$$e(TM) = \chi(M)[pt]^* \in H^n(M;\mathbb{Z}).$$

References

[1] Bott and Tu, *Differential forms in algebraic topology.*

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