Mathematics 8803

Characteristic Classes
Homework 1
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1. Show that a product of spheres has trivial tangent bundle if at least one of the spheres has odd dimension.

2. Find a nonvanishing vector field on S^n when n is odd.

3. Find explicit Poincaré duals for the Stiefel–Whitney classes of the closed nonorientable surfaces.

4. Prove that the number of triple points of an immersion of a surface M into $\mathbb{R}^3$ is congruent modulo 2 to $w_1^2(M) \in H^2(M; \mathbb{Z}_2) \cong \mathbb{Z}_2$.

5. Use the splitting principle to show that the first Stiefel–Whitney class of a bundle is trivial if and only if the bundle is orientable.

6. Show that the second Stiefel–Whitney class of an orientable surface is trivial. What about the nonorientable surfaces?

7. Show that, in the case of a complex vector bundle, w_{2i} is the image of c_i under the coefficient homomoprhism $H^{2i}(B; \mathbb{Z}) \to H^{2i}(B; \mathbb{Z}_2)$. What does c_1 measure?

8. Show that the projection $V_n(\mathbb{R}^k) \to G_n(\mathbb{R}^k)$ is a fiber bundle with fiber $O(n)$ by showing that it is the orthonormal n-frame bundle associated to the vector bundle $E_n(\mathbb{R}^k) \to G_n(\mathbb{R}^k)$. Here $V_n(\mathbb{R}^k)$ is the Stiefel manifold of orthonormal n-frames on $\mathbb{R}^k$ and $E_n(\mathbb{R}^k)$ is the canonical bundle.

9. Determine the Stiefel–Whitney classes of a product of two bundles. Do the same for the tensor product of two bundles. Show that the tensor product of two even-dimensional vector bundles is orientable.

10. If an n-manifold M can be immersed in $\mathbb{R}^{n+1}$ show that $w_i(M) = w_1(M)^i$ for all i. Use this to show that if $\mathbb{R}\mathrm{P}^n$ can be immersed in $\mathbb{R}^{n+1}$ then n must be of the form $2^r - 1$ or $2^r - 2$.

11. Show that if the tangent bundle to a complex manifold M of odd dimension is isomorphic to its conjugate bundle, then $\chi(M) = 0$.

12. Express the mod 2 euler class as a polynomial in terms of the Steifel–Whitney classes.