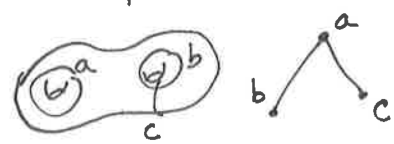
MATH 8803

LOW-DIMENSIONAL TOPOLOGY AND HYPERBOLIC GEOMETRY

Dan Margalit
Fall 2014
Georgia Tech

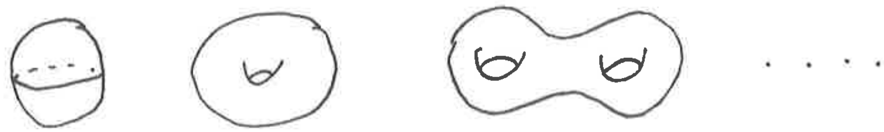
This course has two parts: I. 3-manifolds
II. Complex of curves

Topological objects studied via geometry.

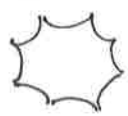


3-MANIFOLDS, OVERVIEW

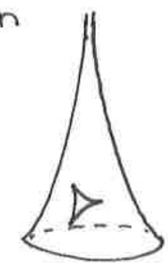
Classification of 2-manifolds mid 19^{th} cent. (closed, orient.)



χ	2	0	-2	
geometry	spherical	Euclidean	hyperbolic.	



regular octagon in $\mathbb{H}^2$



Gauss-Bonnet: $2\pi\chi = \int K$.

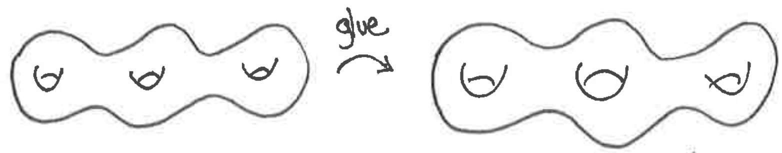
Examples of 3-manifolds

1. S^3

2. $S \times S^1$ e.g. T^3

3. $S^3 \setminus K$

4. Heegaard decompositions



all 3-mans arise this way!

5. Dehn surgery

Cut out solid torus, glue back in.

Lickorish-Wallace: all 3-mans arise from Dehn surgery on S^3.

6. Branched covers

$S^3 \setminus K \longrightarrow$ cov. space $\longrightarrow$ glue $D^2 \times S^1$ back.

Montesinos-Hilden: every 3-man is a 3-fold cover over S^3.

7. Gluing polyhedra

glue faces in pairs, delete vertices if nec.

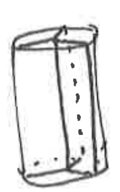
$\frac{8!}{2^4 4!} 3^4 = 8{,}505$ ways to glue faces of octahedron

surface case: $(2n)!/2^n n!$ ways to glue $2n$-gon, most are same!

Later: $S^3 \setminus \text{fig} 8 = 2$ tetrahedra

8. Seifert manifolds

Start with $S \times S^1$, twist by rat'l amount around some fibers



Classification of 3-manifolds – geometrization.

M

$\downarrow$ cut along spheres, glue balls in — Kneser 1930's

prime pieces — cut along tori — Jaco-Shalen, Johannson 1970s

Seifert $\swarrow$ $\searrow$ atoroidal

Seifert $\downarrow$ complete list 1930s

atoroidal $\downarrow$ hyperbolic — Thurston's geometrization conj. $\longrightarrow$ proved for Haken 80's; proved by Perelman in general '03

hyperbolic $\longrightarrow$ Mostow rigidity (60s): hyp struct. is !

Consequences:

① Poincaré conjecture: only simply conn (closed, or.) M is S^3.

Because: no counterexamples among Seifert manifolds (we have a list) or hyperbolic manifolds (π_1 infinite).

② Knot complements are Seifert, toroidal, hyperbolic according to whether the knot is torus, satellite, other.

③ Borel conjecture ($n=3$): homotopy equiv $\Rightarrow$ homeomorphic.

PRIME DECOMPOSITION FOR 3-MANIFOLDS

Connect sum

M_1, M_2 closed, conn, oriented n-mans

$M_i' = M_i \setminus B^n$

$$M_1 \# M_2 = M_1' \coprod_{B^3} M_2' \qquad \text{"connect sum"}$$

Properties: commutative

associative

identity: S^n.

e.g. 

Primes

M is <u>prime</u> if it cannot be written as a nontrivial connect sum ($M \# S^n$ is trivial)

eg. 

Thm (Kneser 1930's) M = closed, conn, or 3-man

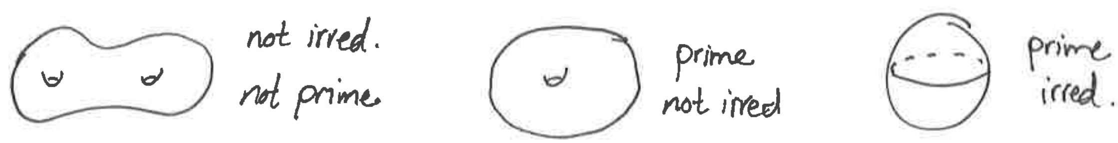
M has a unique prime decomposition.

Preliminaries

Alexander's Thm. Every smoothly embedded S^2 in $\mathbb{R}^3$ bounds a ball.

beware: horned sphere (youtube)
(there are no horned circles: Schönflies thm).

Irreducibles. M is *irreducible* if every S^{n-1} bounds a B^n.



Prop. The only orientable prime, reducible 3-man is $S^2 \times S^1$.

Pf. M prime, reducible
$\longrightarrow$ M has nonseparating sphere S.
Let α = arc in M connecting two sides of S.
$\rightsquigarrow N(S \cup \alpha) \cong (S^2 \times S^1) \setminus B^3$
M prime $\Longrightarrow M = S^2 \times S^1$.

Still need: $S^2 \times S^1$ is prime. Any separating sphere S lifts to $\widetilde{S^2 \times S^1} \cong \mathbb{R}^3 \setminus \{0\}$. By Alexander, the lift bounds a ball. One side of S, is simply conn (since $\pi_1(S^2 \times S^1) = \mathbb{Z}$) so it lifts to $\widetilde{S^2 \times S^1}$. This lift is the ball we found. So one side of S is a ball.

EXISTENCE OF PRIME DECOMP.

Step 1. Eliminate $S^2 \times S^1$ summands

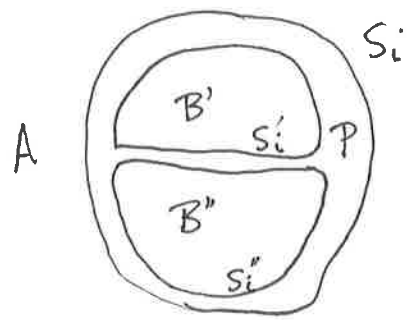
- If M has any nonsep. S^2 then as above there is an $S^2 \times S^1$ summand.
- At most finitely many for homological reasons:

$$H_1(\# M_i) = \oplus H_1(M_i)$$
$$\& \quad H_1(S^2 \times S^1) = \mathbb{Z}.$$

Step 2. $\{S_i\}$ = collection of disjoint spheres with no punctured sphere complementary regions.

D = disk, $D \cap \{S_i\} = \partial D \subseteq S_i$.

S_i', S_i'' obtained from S_i by surgery along D:



Can replace S_i with S_i' or S_i'' to get collection of disjoint spheres with no punc. sphere regions.

Indeed: · If B', B'' both punc. spheres then S_i bounds a punc. sphere. Say B' not a punc. sphere.

· Then $A \cup B'' \cup P$ also not a punc. sphere. Because $B'' \cup P$ is one. so this means A was a punc. sphere.

Step 3. There is a bound on the # of S_i so $\{S_i\}$ is a collection of disjoint spheres with no punc. sphere regions.

- $\mathcal{T}$ = smooth triangulation of M, say, N simplices.
- Make the S_i transverse to every simplex (induct on skeleta).

Eliminate:

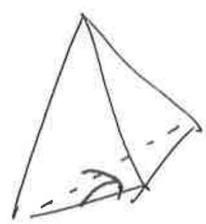
(i) spheres entirely in 3-cell

Alexander thm.

(ii) circles in 2-cell not bounding disk in 3-cell

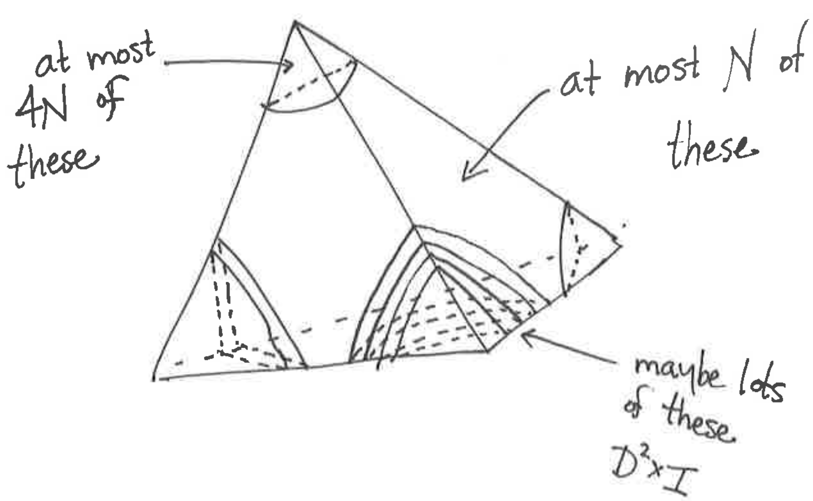
Step 2.

(iii) arcs in 2-cell connecting edge to self



Isotopy.

Now intersections look like:



We'll show the complementary regions containing these $D^2 \times I$ each contribute a $\mathbb{Z}_2$ to $H_1(M)$, so there are finitely many.

Each such region is an I-bundle over a surface with boundary a union of at most 2 spheres.

Two possibilities: ① $S^2 \times I$ = punc. sphere ruled out!

② Mapping cylinder of $S^2 \longrightarrow \mathbb{RP}^2$ (collapsing I to $\{0\}$ is covering map) $= \mathbb{RP}^3 \setminus B^3$

Since $H_1(\mathbb{RP}^3) = \mathbb{Z}_2$ we are done.

Uniqueness of Prime Decomp.

Idea. Given two sphere systems giving two decomps, use surgery a la Step 2 to make them disjoint. At this point the sphere systems must be parallel.

Torus Decompositions

Last time: cut M along spheres $\rightsquigarrow$ prime pieces

This time: cut irred M along tori $\rightsquigarrow$ atoroidal pieces

Next time: uniqueness

Incompressible surfaces

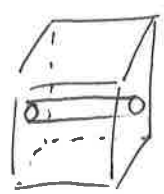
M = closed, conn, or 3-man

$S \subseteq M$ closed, conn, or surface. $S \neq S^2$.

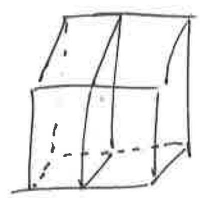
S is <u>incompressible</u> if $\forall\ D \subseteq M$ with $D \cap S = \partial D$

$\exists\ D' \subset S$ with $\partial D' = \partial D$.

e.g. $T^2 \subseteq T^3$:



compressible



incompressible

Some facts:

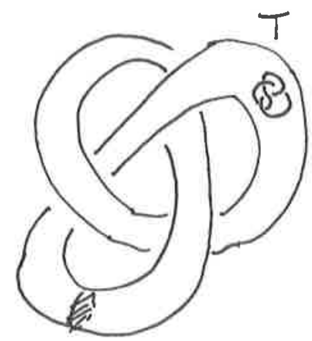
① $\pi_1(S) \hookrightarrow \pi_1(M) \Rightarrow S$ incompressible

(converse also true but harder).

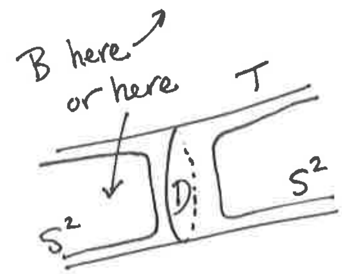
② No incompressible surfaces in S^3.

③ $T \subseteq M$ irred, or.
T compressible $\iff$ T bounds a solid torus
or lies in a ball.

example of 2^{nd} type :



Pf. T compressible along D
$\rightsquigarrow$ surger T along D to produce S^2
$\rightsquigarrow$ ball B bounded by S^2 (irreducibility)



Case 1. $B \cap D = \emptyset$
$\rightsquigarrow$ reverse surgery to get solid torus.

Case 2. $D \subseteq B$
$\rightsquigarrow$ $T \subseteq B$.

④ $T \subseteq S^3$ bounds a solid torus on one side or other.
Use ②+③. In Proof of ③ have a ball on both sides
by Alexander, so suffices to consider Case 1.

Exercise. $S^3 \setminus K$ toroidal $\Rightarrow$ K satellite.

⑤ $S \subseteq M$ incompressible. M irred $\iff$ $M|S$ irred (M cut along S.)

⑥ $S \subseteq M$ incomp or S^2. $T \subseteq M$ incompressible $\iff$ $T \subseteq M|S$ incompressible.
$T \cap S = \emptyset$.

Existence of Torus Decomps

Irreducible M is atoroidal if every incompressible torus is ∂-parallel.

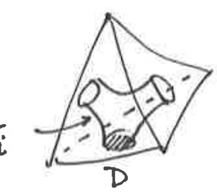
Thm. M = closed, conn, or, irred 3-man
There is a finite collection T of disjoint incompressible tori s.t. $M \setminus T$ is atoroidal.

Pf. Want a bound on # components in a system $T = T_1 \cup \cdots \cup T_n$ of disjoint, non-parallel incomp. tori in M (similar to prime decomp).

Make T transverse to triangulation. Two simplifications

① Make each intersection of T with 3-cell union of disks.

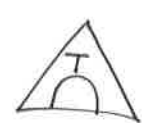
If see



incompressibility $\Rightarrow$ disk $D' \subseteq T_i$
irreducibility $\longrightarrow$ ball with $\partial = D \cup D'$
$\rightsquigarrow$ can push this intersection away (no surgery needed!).

Note: ① $\Rightarrow$ no intersection of T with 2-cell is circle (would get disk on both sides, hence sphere)

② Eliminate intersections of T with 2-cells like this:



again, by pushing off.

On each 2-cell, have:

Regions of $M \setminus T$ that only intersect 2-cells in strips are I-bundles.

Trivial bundles $\leftrightarrow$ parallel tori ruled out

For nontrivial bundle bounded by T_i, let $T_i' =$ 0-section (Klein bottle)

$T' = T$ with T_i replaced by T_i'.

$M' = M \setminus Nbd(T')$

$\quad = M$ with nontrivial I-bundles deleted.

components of M' $\leq 4(\#\,2\text{-cells}) = N$

Have:

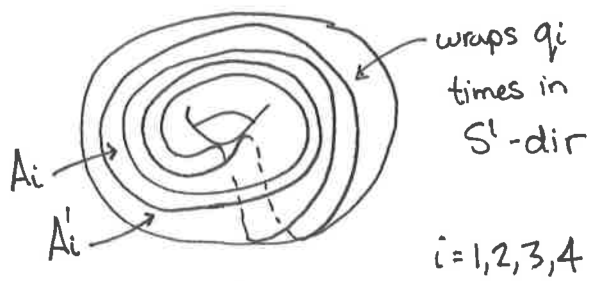
$$\begin{tikzcd}
H_3(M, T'; \mathbb{Z}/2) \arrow[r] \arrow[d, phantom, "\cong \text{ excision}"] & H_2(T'; \mathbb{Z}/2) \arrow[r] \arrow[d, phantom, "\cong"] & H_2(M; \mathbb{Z}/2) \\
H_3(M', \partial M'; \mathbb{Z}/2) & H_2(T; \mathbb{Z}/2) \arrow[d, phantom, "\parallel"] & \text{only depends on } M \\
\text{bounded by } N \text{ i.e. only depends on } M & {|T|} &
\end{tikzcd}$$

Thus $|T|$ is bounded by a # only depending on M. ▨

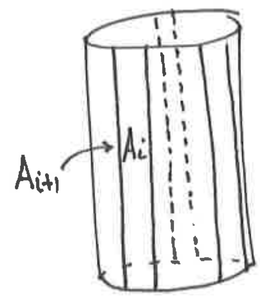
Non-uniqueness of torus decomps.

Will construct M with two very different torus decomps.

M_i =

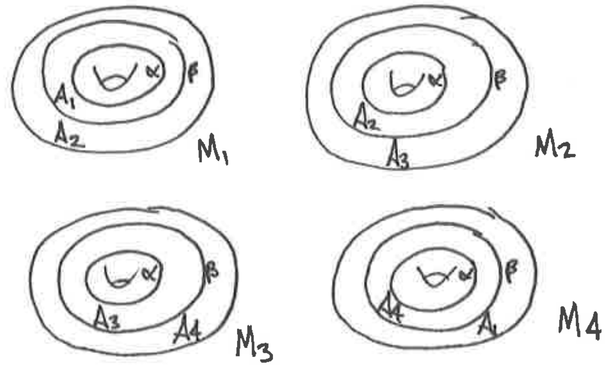


$i = 1,2,3,4.$



Glue A_i' to A_{i+1} mod 4.

Simplified picture:



$T_1 = A_1 \cup A_3 \qquad M \mid T_1$ is $M_1 \cup M_2 \amalg M_3 \cup M_4$

$T_2 = A_2 \cup A_4 \qquad M \mid T_2$ is $M_2 \cup M_3 \amalg M_4 \cup M_1$

Can show: M irred

T_i incompressible

$M \mid T_i$ atoroidal.

But: the two decompositions are very different.

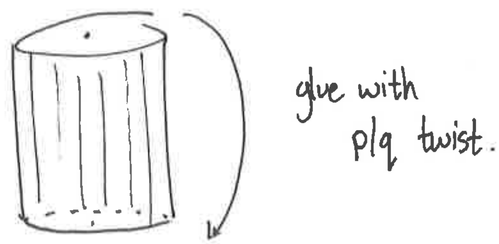
Van Kampen $\Rightarrow \pi_1(M_i \cup M_{i+1}) = \langle x_i, x_{i+1} \mid x_i^{q_i} = x_{i+1}^{q_{i+1}} \rangle$

These groups all different. The center is $\langle x_i^{q_i} \rangle$ and if we mod out we get $\mathbb{Z}/q_i * \mathbb{Z}/q_{i+1}$

Turns out: these are the only types of counterexamples!

Seifert Manifolds

A model Seifert fibering of $S^1 \times D^2$ is the decomp. into circles given by:



A Seifert fibering of a 3-man is a decomp. into disjoint circles so each circle has a nbd that is a model Seifert fibering.

A Seifert manifold is one with a Seifert fibering

↳ multiplicity of a fiber is q.

Collapsing each circle to a pt, get a map $M \to S$ = surface.

Thm. M = closed, or, irred 3-man.

$\exists$ collection T of disjoint incomp. tori s.t. each component of $M \setminus T$ is either ① atoroidal, or ② Seifert

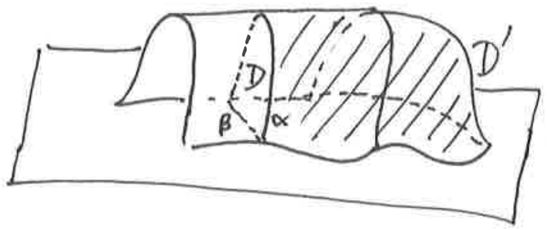
A minimal such collection is unique up to isotopy.

UNIQUENESS OF TORUS DECOMPS

∂- incompressible surfaces

$S \subset M$ is ∂-incomp. if $\forall\ D \subseteq M$ s.t. $\partial D = \alpha \cup \beta$, $D \cap S = \alpha$, $D \cap \partial M = \beta$

$\exists\ D' \subset S$ with $\alpha \subseteq \partial D'$, $\partial D' - \alpha \subset \partial S$.



Warmup. The only ∂-incomp, incomp surfaces in $S^1 \times D^2$ are disks isotopic to meridional disks.

Pf. Let S = connected, incomp, ∂-incomp.

Modify S so ∂S either meridians or transverse to meridians

Make S transverse to D_0 = fixed merid. disk.

Eliminate circles of $S \cap D_0$ using incomp & irreducibility.

Eliminate/rule out

$\Rightarrow S \cap D_0 = \emptyset$.

$\Rightarrow \partial S$ = union of meridian circles.

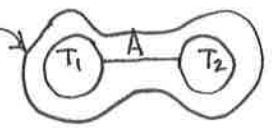
S incomp. in $M | D_0 = B^3 \Rightarrow S$ = union of disks

By Alexander's thm, a disk with meridional ∂ is isotopic to merid. disk with same ∂. ▨

Key Lemma. M = compact, conn, or., irred, atoroidal, torus boundary
If M contains an incomp., ∂-incomp annulus A
then M is Seifert.

Pf. Assume ∂A in two different tori (other case similar), say T_1 & T_2
Let $N = \mathrm{Nbd}(A \cup T_1 \cup T_2)$:

$T = \partial N - \partial M$

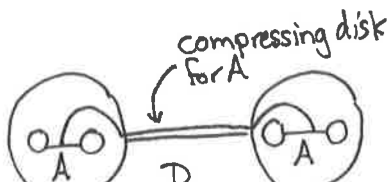


$\times S^1$

Seifert fibered!

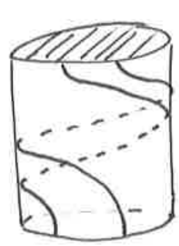
M atoroidal $\Rightarrow$ T either ① ∂ parallel, or
② compressible

In case ① $M \cong T$, so M is Seifert.
Now case ②. Let D = compressing disk
$\rightsquigarrow$ ∂D = nontrivial loop in T
Clearly $D \not\subseteq N$ (look at picture, or use π_1,
or Prop 1.13(a) in AH).
$\Rightarrow D \cap N = \partial D$.
Surgering T along D $\rightsquigarrow$ sphere
$\rightsquigarrow$ ball B (irreducibility)
B outside N since $N \neq$ solid torus.
$\Rightarrow$ $M - N$ = solid torus



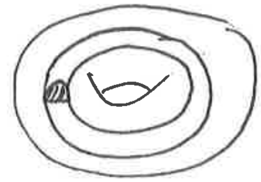
Claim: ∂D not ~~meridional~~ fibers in $T \subseteq N$
Pf. If it were, would give compressing disk for A.
Thus, S^1-fibers of N wrap at least once around
S^1-dir of $M - N = D^2 \times S^1$
$\rightsquigarrow$ can extend Seifert fibering from N to $M-N$.
$\Rightarrow$ M Seifert fibered. ▨

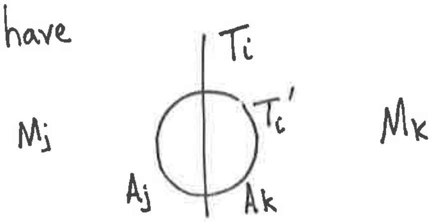
M−N



Thm (Uniqueness of Torus decomp) M = closed, or., irred. 3-man.
$\exists$ collection T of disjoint incomp tori s.t.
each component of $M \setminus T$ is either ① atoroidal or
② Seifert
A minimal such collection is unique up to isotopy.

Pf of uniqueness.
Say $T = T_1 \cup \cdots \cup T_m \longrightarrow$ split into M_j $\qquad m, n \neq 0$.
$T' = T_1' \cup \cdots \cup T_n' \longrightarrow$ split into M_j'
Make transverse
Eliminate:  push off.

So components of $T_i' \cap M_j$ are tori, annuli.
Annuli. Annulus components are incomp since the T_i' are.
If have ∂-incomp annulus:
the annulus is ∂-parallel
$\rightsquigarrow$ push off. (AH Lemma 1.10)

Now have



where A_j, A_k incomp.
∂-incomp.

(assume $M_j \neq M_k$ for simplicity).
Key Lemma $\Rightarrow M_j, M_k$ Seifert.
To show: can make the Seifert fiberings agree along T_i
$\rightsquigarrow T_i$ can be removed.

So $T \cap T' = \emptyset$.

Now assume $T \cap T' = \emptyset$.

If any T_i lies in M_j' then M_j' toroidal, hence Seifert fibered.

<u>Fact</u>. A surface in a Seifert man. is either isotopic to a horizontal one or a vertical one.

$\partial M_j' \neq \emptyset \Rightarrow T_i$ <u>vertical</u>.

Suppose $T_1' \subseteq M_j$. Want to argue the two sides of T_1' have compatible fiberings, so T_1' can be deleted.

Call the two sides M_k', M_ℓ'.

- If $\exists\, T_i \subseteq M_k'$ then $M_k' =$ Seifert as above $\Rightarrow M_j \cap M_k'$ has two Seifert fiberings, from M_j & M_k'. Seifert fiberings are (almost always) unique, so fibering of M_k' compatible with M_j.

- If no $T_i \subseteq M_k'$ then $M_k' \subseteq M_j$ and so M_k' again has fibering from M_j.

Same for M_ℓ'. So $M_k' \cup M_\ell'$ has fibering from M_j $\rightsquigarrow$ T_1' can be deleted. ∎

SEIFERT MANIFOLDS

S^1 - bundles

A manifold M is an S^1-bundle over a manifold B if there is $p: M \to B$ and B covered by U with $p^{-1}(U) \cong U \times S^1$.

e.g. T^2, Klein bottle

Prop. B = closed orientable surface

$\forall k \in \mathbb{Z}$ $\exists!$ S^1-bundle $M_k \to B$

s.t. $k = i(B,B)$ in M_k.

(so $k = 0 \iff M_k$ has section)

Construction of M_k. Let $B^0 = B \setminus$ open disk

$M_k^0 = B^0 \times S^1$

$s: B^0 \to M_k^0$ any section.

Glue $D^2 \times S^1$ so $s(\partial B^0)$ wraps k times around S^1-dir.

e.g. $B = S^2$, $k = \pm 1 \rightsquigarrow$ Hopf fibration of S^3.

Model Seifert manifolds

B = compact surface, maybe not orient.

$B^0 = B \setminus$ several open disks

M^0 = orientable S^1-bundle over B^0 (twisted over 1-sided loops).

s = section (regard M^0 as two orientable I-bundles glued on ∂I by id).

On each T^2 boundary, $s(\partial B^0)$ = 0-curve fiber = ∞-curve

Glue $S^1 \times D^2$ to i^{th} T^2 sending meridian to s_i-curve.

The S^1-fibering extends to Seifert fibering

[Note: $s_i \in \mathbb{Z}$ means the meridian hits $s(\partial B_0)$ s_i times
fiber 1 time.

as in construction of M_K.

so $s_i \in \mathbb{Z} \iff$ locally have S^1-bundle (as opposed to Seifert).]

$\rightsquigarrow$ model $M(\pm g, b; s_1, \dots, s_k)$

- $s_1, \dots, s_k$: gluing slopes
- b: # boundary
- g: genus
- $\pm$: orientable or not

Prop. Every orientable Seifert manifold is $\cong$ to one of the models.
Further $M(\pm g, b; s_1, \dots, s_k) \overset{o.p.}{\cong} M(\pm g, b; s_1', \dots, s_k')$
iff the following hold ① $s_i \equiv s_i' \mod 1 \quad \forall i$
② $b > 0$ or $\sum s_i = \sum s_i'$ (euler number).

Prop. $M(\pm g, b; s_i)$ has a section iff $b > 0$ or $\sum s_i = 0$.

Examples: Lens Spaces

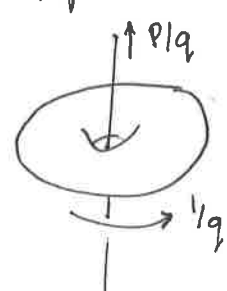
T, T' solid tori

meridian of T = ∞-curve, longitude = 0-curve.

glue meridian of T' to p/q curve in T

$\rightsquigarrow$ Lens space $L_{p/q}$

As quotient of S^3:



slope p curves invariant
$\rightsquigarrow$ longitudes on quotient.

Proof of classification of Seifert man's in terms of models

M = Seifert

$M^\circ = M \setminus$ nbds of special fibers

$\rightsquigarrow S^1 \to M^\circ \to B^\circ$

Let $s : B^\circ \to M^\circ$ section.

$\rightsquigarrow s(\partial B^\circ)$ = circles of slope 0 in $\partial M^\circ = \coprod T^2$

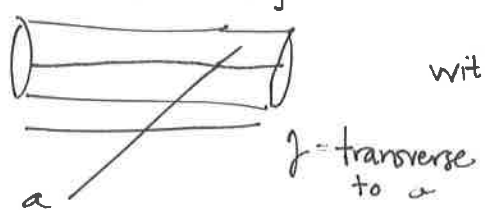
fibers = circles of slope ∞.

$\rightsquigarrow$ slopes s_i for gluing the Seifert fibered pieces back.

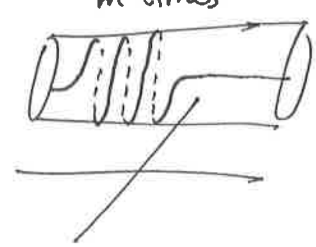
Changing the s_i by twisting:

a = arc connecting ∂B°

replace



with



Changes $s_i \to s_i + m$ at one end (the basis $(1,0),(0,1)$ gets replaced with $(1,m),(0,1)$)

$s_j \to s_j - m$ at other.

So if $b \neq 0$ can connect one end of a to ∂M, modifying one s_i by m.

Remains to check: any two sections differ by these twist moves. Indeed, cut ∂B° along arcs to get a disk. Away from arcs, one choice of section. Near arcs, only have twisting. $\blacksquare$

CLASSIFICATION OF SEIFERT FIBERINGS

<u>Thm</u>. Seifert fiberings of orientable Seifert man's are unique up to isomorphism, except:

(a) $M(0,1;\alpha/\beta)$ the fiberings of $S^1\times D^2$

(b) $M(0,1;1/2,1/2) = M(-1,1;)$ fiberings of $S^1 \tilde{\times} S^1 \tilde{\times} I$

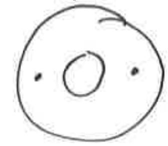
(c) $M(0,0;s_1,s_2)$ various fiberings of S^3, $S^1\times S^2$, lens sp

(d) $M(0,0;1/2,-1/2,\alpha/\beta) = M(-1,0;\beta/\alpha)$ $\alpha,\beta\neq 0$.

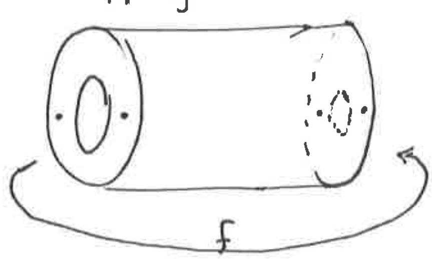
(e) $M(0,0;1/2,1/2,-1/2,-1/2) = M(-2,0)$ fiberings of $S^1\tilde{\times}S^1\tilde{\times}S^1$

The two fiberings of $S^1\tilde{\times}S^1\tilde{\times}I$.

Let $f: S^1\times I \to S^1\times I$ reflection in both factors.
f has 2 fixed pts



$S^1\tilde{\times}S^1\tilde{\times}I$ is mapping torus:



fibering by horizontals has two special fibers.
fibering by verticals has no special fibers.

Note c,d,e come from a,b: specifically the fiberings in c come from different fiberings in a, d comes from gluing a model solid torus to b and e is the double of b.

HYPERBOLIC SPACE

Disk model B^n = open unit ball in $\mathbb{R}^n$, dx^2 = Euclidean metric

$$ds^2 = dx^2 \left(\frac{2}{1-r^2}\right)^2 \rightsquigarrow \mathbb{H}^n$$

Note: ① Since ds^2 is dx^2 scaled, hyp. angles = Euc. angles

② Distances large as $r \to 1$

③ Inclusions $D^1 \subset D^2 \subset \cdots$ induce isometries $\mathbb{H}^1 \subset \mathbb{H}^2 \subset \cdots$

∂B^n is sphere at infinity, denoted $\partial\mathbb{H}^n$.

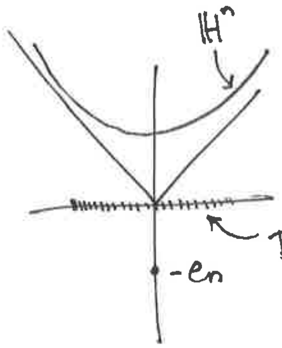
Upper half-space model $U^n = \{(x_1,\dots,x_n) \in \mathbb{R}^n : x_n > 0\}$

$$ds^2 = \frac{1}{x_n^2}\, dx^2$$

Check: Inversion in sphere of rad $\sqrt{2}$ centered at $-e_n$ is an isometry $B^n \to U^n$.

Here, $\partial\mathbb{H}^n$ is $x_n = 0$ plane plus pt at ∞.

Hyperboloid model



$\mathbb{R}^{n,1}$, Lorentz metric $x_1^2 + \cdots + x_n^2 - x_{n+1}^2$

Sphere of radius $\sqrt{-1}$ is hyperboloid

Upper sheet with induced metric is $\mathbb{H}^n$.

By defn, $\mathrm{Isom}^+ \mathbb{H}^n = SO(n,1)$

Isometry with B^n via stereographic proj from $-e_n$

Isometries of $\mathbb{H}^n$

Examples ① Orthogonal maps of $\mathbb{R}^n$ restricted to B^n
$\rightsquigarrow$ all possible rotations about e_n in U^n.
② Translation of U^n by $v = (v_1, \ldots, v_{n-1}, 0)$
③ Dilation of U^n about 0.
③' Rotation about e_n axis.

Easy from defn of ds^2 that these are isometries.

Thm. The above isometries generate $\text{Isom}(\mathbb{H}^n)$

Pf. Use: if two isometries of a Riem. manifold agree at a point, they are equal.

Consequences: Any isometry of $\mathbb{H}^n$
① extends continuously to $\partial\mathbb{H}^n$
② preserves $\{\text{spheres}\} \cup \{\text{planes}\}$
③ preserves angles between arcs in $\mathbb{H}^n$ and $\partial\mathbb{H}^n$.
consequence of pf of Thm. → ④ In U^n model, each isometry of form $\lambda A x + b$ $\lambda > 0$, A orthogonal & fixes e_n
$b = (b_1, \ldots, b_{n-1}, 0)$

Geodesics

Prop. In U^n $\exists!$ geodesic from e_n to λe_n.
Pf. Given any path, its projection to e_n-axis is shorter.
Geodesics in $\mathbb{R}$ are unique.

Length is $\int_1^\lambda \frac{1}{y}\, dy = \ln \lambda$.

Consequences: ① $\mathbb{H}^n$ is a unique geodesic space (use change of coords + Prop)

② The geodesics in $\mathbb{H}^n$ are exactly the straight lines and circles $\perp$ to $\partial\mathbb{H}^n$.

③ Given a geodesic L and $x \notin L$ $\exists$ infinitely many L' with $x \in L'$, $L \cap L' = \emptyset$.

④ Between any pts of $\partial\mathbb{H}^n$ $\exists!$ geodesic

(geodesic rays asymp $\Leftrightarrow$ endpts same)

⑤ Geodesics are infinitely long in both directions.

exercise: space of geodesics in $\mathbb{H}^2$ is homeo to Möbius strip.

CLASSIFICATION OF ISOMETRIES

Via fixed pts: ① elliptic - fixes pt of $\mathbb{H}^n$

② parabolic - fixes 1 pt of $\partial\mathbb{H}^n$, no pt of $\mathbb{H}^n$

③ hyperbolic - fixes 2 pts of $\partial\mathbb{H}^n$, no pt of $\mathbb{H}^n$

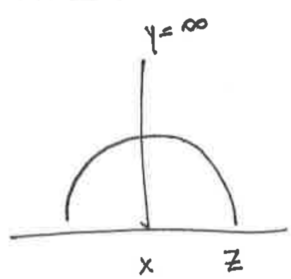
<u>Thm</u>. Each elt of $\mathrm{Isom}(\mathbb{H}^n)$ is one of these.

<u>Pf</u>. Brouwer $\Rightarrow$ at least one fixed pt.

Suppose f fixes $x, y, z \in \partial\mathbb{H}^n$

$\Rightarrow f$ ~~fixes~~ preserves $\overline{xy}$ and since $f(z) = z$, f fixes $\overline{xy}$ ptwise

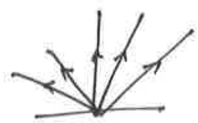
~~and does not rotate about $\overline{xy}$~~. $\Rightarrow f$ elliptic.



Can give explicit descriptions of 3 types. Using change of coords, can assume a fixed pt in $\mathbb{H}^n$ is e_n and a fixed pt in $\partial\mathbb{H}^n = \infty$ in U_n model.

elliptic: rotation 

parabolic: $Ax+b$  A = orthogonal, preserves e_n
$b = (b_1, \dots, b_{n-1}, 0)$

hyperbolic: λAx  A as above
$\lambda \in \mathbb{R}_{>0}$

Via translation length $\quad \tau(f) = \inf\{d(x, f(x)) : x \in \mathbb{H}^n\}$

Prop. Let $f \in \operatorname{Isom}(\mathbb{H}^n)$

① f elliptic $\Leftrightarrow \tau(f) = 0$, realized

② f parabolic $\Leftrightarrow \tau(f)$ not realized

③ f hyperbolic $\Leftrightarrow \tau(f) > 0$, realized.

Pf. All $\Rightarrow$ follow from above descriptions.

First $\Leftarrow$ by defn

Second $\Leftarrow$ find x_n s.t. $d(x_n, f(x_n)) \longrightarrow \tau(f)$

note x_n leave every compact set

$\rightsquigarrow$ convergent seq $\rightsquigarrow$ limit $x \in \partial\mathbb{H}^n$.

Third $\Leftarrow$ If $d(x, f(x)) = \tau(f)$ then f preserves geodesic through $x, f(x), f^2(x), \dots$

$\rightsquigarrow$ 2 fixed pts in $\partial\mathbb{H}^n$.

Dimensions 2 & 3

Thm. $\mathrm{Isom}^+(\mathbb{H}^2) \cong PSL_2\mathbb{R}$
$\mathrm{Isom}^+(\mathbb{H}^3) \cong PSL_2\mathbb{C}$

Pf. $\mathbb{H}^3$ case first.
By above, there is:
$$\mathrm{Isom}^+(\mathbb{H}^3) \longrightarrow \mathrm{Homeo}(\partial\mathbb{H}^3) \cong \mathrm{Homeo}(\hat{\mathbb{C}})$$ and this is injective.
By Möbius transformations
$$PSL_2\mathbb{C} \longrightarrow \mathrm{Homeo}(\hat{\mathbb{C}})$$ injective.
Suffices to show images are same.
First, $PSL_2\mathbb{C}$ gen. by $\begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}$
$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$
$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

exercise: realize each by $\mathrm{Isom}^+(\mathbb{H}^3)$.
For other dir, show each elt of $\mathrm{Isom}^+(\mathbb{H}^3)$ fixes a pt in $\partial\mathbb{H}^3$. Change of coords: this pt is ∞.
By above, an isometry fixing ∞ is of form $z \mapsto \lambda A z + b$,
or $z \longmapsto wz + b$, $w, b \in \mathbb{C}$

but this is Möbius.

$\mathbb{H}^2$ case. $PSL_2\mathbb{R}$ = subgp of $PSL_2\mathbb{C}$ preserving $\mathbb{R}$ with orientation.
$\Longrightarrow \mathrm{Isom}^+(\mathbb{H}^2) \subseteq PSL_2\mathbb{R}$
For other inclusion, show every isometry of $\mathbb{H}^2$ extends to $\mathbb{H}^3$. (check on generators). ∎

Loose Ends

Intrinsic defn of $\partial \mathbb{H}^n$

$$\partial \mathbb{H}^n = \{\text{based geodesic rays in } \mathbb{H}^n\}/\sim$$

$$\gamma \sim \gamma' \quad \text{if} \quad \lim d_{\mathbb{H}^n}(\gamma(t), \gamma'(t)) = 0.$$

topology: for open half-space $S \subseteq \mathbb{H}^n$
$V_S = \{ [\gamma] : \gamma \text{ positively asymptotic into } S \}$
$\rightsquigarrow$ basis

(check this is same topology as before!)

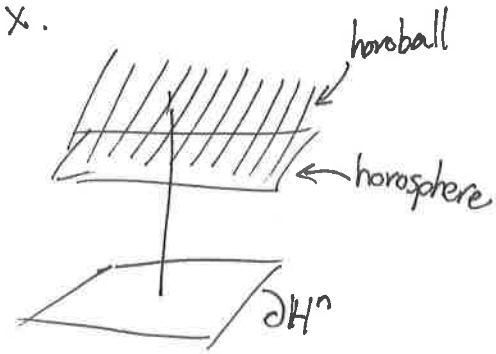
This also gives topology on $\mathbb{H}^n \cup \partial \mathbb{H}^n$
By defn, $\mathrm{Isom}(\mathbb{H}^n)$ acts continuously on the union.

Horospheres

B = Euclidean ball in ball model of $\mathbb{H}^n$
tangent to boundary sphere at x.

$\partial B \setminus x$ = horosphere

$\mathrm{int}\, B$ = horoball.



note: horosphere has Euclidean metric

Areas in $\mathbb{H}^2$

Circles. $f(t) = re^{it}$ circle in disk model, hyp. radius $s = \ln\left(\frac{1+r}{1-r}\right)$

$$C = \int_0^{2\pi} \frac{2}{1-r^2}\, r\, dt = \frac{4\pi r}{1-r^2} = \frac{4\pi \tanh s/2}{1-(\tanh s/2)^2} = \frac{4\pi \tanh s/2}{(\operatorname{sech} s/2)^2} = 2\pi \sinh s$$

$$\sim e^s$$

$$A = \int_0^s 2\pi \sinh t\, dt = 2\pi(\cosh s - 1) = 2\pi(2\sinh^2 s/2) = 4\pi \sinh s/2$$

Ideal triangles. All are isometric to:

$$A = \int_{-1}^{1} \int_{\sqrt{1-x^2}}^{\infty} \frac{1}{y^2}\, dy\, dx$$

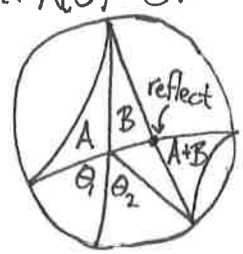
$$= \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\, dx = \pi$$

Polygons. Thm. $A(P) = (n-2)\pi$ - sum of int. angles

Step 1. 2/3 ideal Δ. $A(\theta)$ = area of Δ with angles $0, 0, \pi - \theta$.

Claim: $A(\theta) = \theta$.

Pf:



A continuous

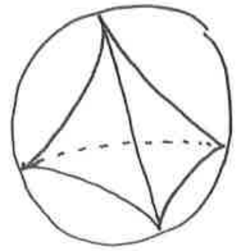
picture $\Rightarrow$ A linear

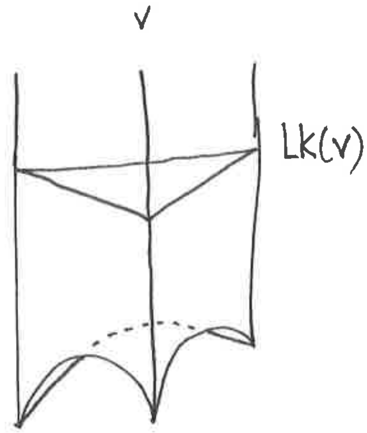
above $\Rightarrow A(\pi) = \pi$.

Step 2. Arbitrary Δ Hint:

Step 3. Cut P into Δs.

Ideal Tetrahedra





T = ideal tetrahedron in $\mathbb{H}^3$

S = horosphere based at ideal vertex v, disjoint from opp side

$Lk(v) = S \cap T$ = link of v in T

= Euclidean Δ, angles are dihedral angles of T, o.p. similarity class indep. of S.

Facts ① o.p. congruence class of (T,v) determined by $Lk(v)$

pf: similarities of $\mathbb{C}$ extend to isometries of $\mathbb{H}^3$

② If the dihedral angles corresp. to v are α, β, γ then $\alpha+\beta+\gamma=\pi$

pf: Euclid

③ The dihedral angles of opp. edges are equal

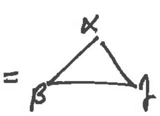
pf: 6 vars, 4 eqns

④ $Lk(v)$ same for all vertices of T

pf: ③

⑤ The o.p. ~~similarity~~ congruence class of T determ. by $Lk(v)$

pf: ① + ④

⑥ $\forall\ \alpha, \beta, \gamma$ s.t. $\alpha+\beta+\gamma=\pi$ $\exists\ T$ with $Lk(v)$ = 

pf: construct it. Notation $T_{\alpha,\beta,\gamma}$

⑦ Congruence class of T determ. by cross ratio of vertices.

pf: up to isometry, 3 vertices are $0, 1, \infty$.

Thm. $\text{Vol}(T_{\alpha,\beta,\gamma}) = Л(\alpha) + Л(\beta) + Л(\gamma)$ $\quad Л(\theta) = -\int_0^\theta \log|2\sin t|\,dt$

see Ratcliffe Thm 10.4.10 "Lobachevsky fn"

Consequences ① $\text{Vol}(T_{\pi/3,\pi/3,\pi/3})$ maximal (easy calculus)

② it equals $3Л(\pi/3) \approx \cancel{2.0298832}\ldots\ 1.01\ldots.$

Hyperbolic Manifolds

Goal: S_g has a hyp. structure $g \geq 2$
$S^3 \setminus \text{Fig8}$ has hyp. structure

A hyperbolic manifold is a topological manifold with a cover by open sets U_i and open maps $\varphi_i : U_i \to \mathbb{H}^n$ that are homeos onto their image and so for each component X of $U_i \cap U_j$,

$$\varphi_j \circ \varphi_i^{-1} : \varphi_i(X) \longrightarrow \varphi_j(X)$$

is the restriction of an elt of $\text{Isom}(\mathbb{H}^n)$.

Note: A hyp. man inherits a Riem. metric.

<u>Prop.</u> A Riem. manifold is a hyperbolic n-manifold iff each point has a nbd isometric to an open subset of $\mathbb{H}^n$.

<u>Pf.</u> $\Rightarrow$ by defn of inherited metric.

$\Leftarrow$ Take the local isometries as the charts $\varphi_i : U_i \to \mathbb{H}^n$
Let X = component of $U_i \cap U_j$
Then $\varphi_i \circ \varphi_j^{-1}|_{\varphi_j(X)}$ is an isometry $\varphi_j(X) \longrightarrow \varphi_i(X)$.
Want an elt of $\text{Isom}(\mathbb{H}^n)$ restricting to this.
But we can find an elt of $\text{Isom}(\mathbb{H}^n)$ that agrees with $\varphi_i \circ \varphi_j^{-1}$ at any $x \in \varphi_j(X)$.
This isometry then agrees on all of $\varphi_j(X)$. ▨

POLYHEDRA

Polyhedron: compact subset of $\mathbb{H}^n$, intersection of finitely many half-spaces.
Ideal polyhedron: intersection of finitely many half-spaces in $\mathbb{H}^n$, no vertices in $\mathbb{H}^n$, closure in $\mathbb{H}^n \cup \partial\mathbb{H}^n$ is a finite set of pts.

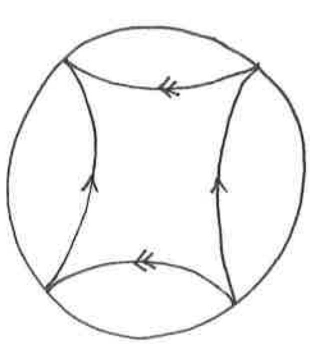
M = space obtained from a collection of (possibly ideal) hyp. polyhedra P_i by gluing codim 1 faces by isometries.
M° = image of $\bigcup \operatorname{int} P_i$.

<u>Thm</u>. M as above. Say each $x \in M$ has a nbd U_x and an open mapping $\varphi_x : U_x \longrightarrow B_{\varepsilon(x)}(0) \subseteq B^n$ (ball model) that is (1) a homeo onto its image (2) sends x to 0 and (3) restricts to isometry on each component of $U_x \cap M^\circ$. Then M is a hyperbolic manifold.

<u>Pf</u>. Need to check condition on overlaps.
This works because gluing maps are isometries (see Lackenby) ▨

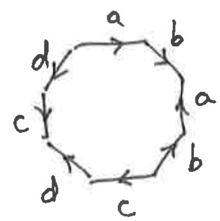
↖ or use the Prop.

A First example:



SURFACES

Will show S_g has hyp. structure $g \geq 2$.

Fact 1. S_2 given by  and similar for $g > 2$.

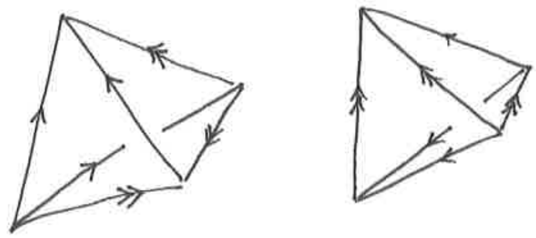
Fact 2. $\exists$ regular 4g-gon in $\mathbb{H}^2$ with angles $2\pi/4g$

Pf: IVT. Small 4g-gons are near Euclidean, angles $> 2\pi/4g$
Large 4g-gons are ideal, angle 0.

Apply the theorem. When we glue, nothing to check on interiors of 1- and 2-cells. At 0-cells, angle condition is exactly what is needed.

Figure-Eight Knot Complement

$K =$ 

Consider 

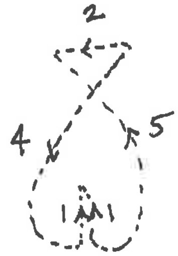
$\exists!$ way to glue faces so edges match up

$\rightsquigarrow$ cell complex M. with one vertex v.

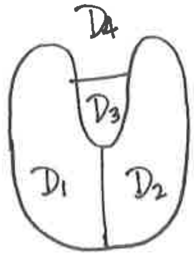
Will show: $M - v \cong S^3 \setminus K$

First note M is not a manifold. In fact, a neighborhood of v is a cone on T^2. To see this, the boundary of a nbd of v is a union of 8 triangles. Label the 24 edges, glue in pairs, result is T^2. (tedious but easy).

Γ = 2-complex in S^3 obtained by attaching 4 2-cells to

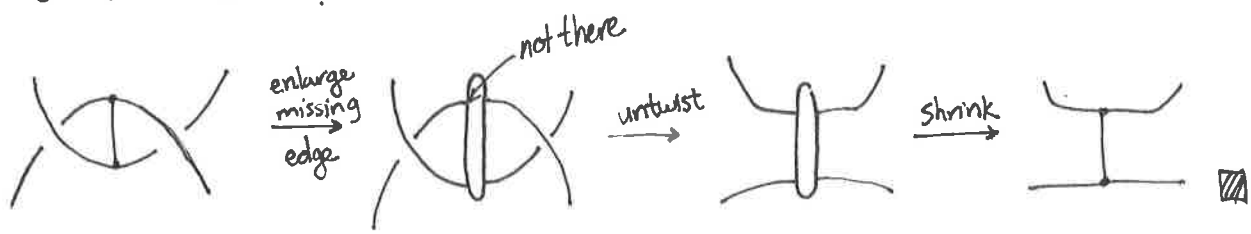
Sample 2-cell: 

(find the other three!)

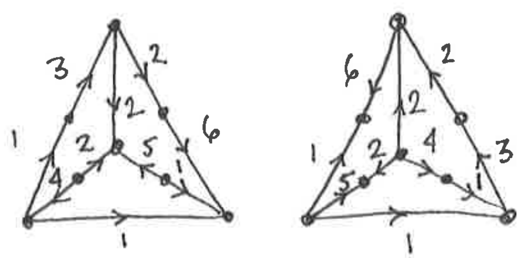
Let $\Gamma' =$ 

$\cong S^2$ Note. $S^3 - \Gamma' \cong \mathrm{int}(B^3) \amalg \mathrm{int}(B^3)$

Claim. $S^3 - \Gamma \cong S^3 - \Gamma'$.

Pf. 

Now go back to Γ picture. The claim tells us the 4 disks of Γ cover S^2. We can read off the gluing:



hard to see!

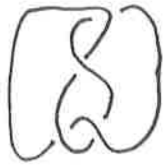
Note K is the union of the edges 3, 4, 5, 6.
So to remove K, can collapse these edges, then delete.
But this is $M \setminus v$!

THE HYPERBOLIC STRUCTURE

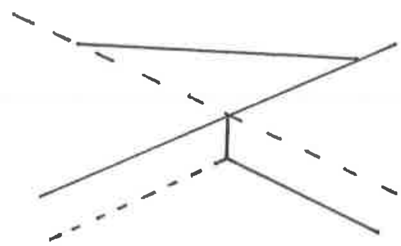
$M \setminus v$ has 2 edges, each with 6 dihedral angles around.
So if we glue two regular ideal tetrahedra, get angle 2π around each edge. Thm $\Rightarrow$ result is hyperbolic.

Hyperbolic volume ≈ 2.0298832 smallest among knot complements

Figure Eight Knot Complement – Reboot

K = 

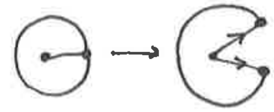
Idea: Simultaneously inflate balloons above and below. (3-cells). These press against each other in each planar region (2-cells). At crossings, the balloons compete:



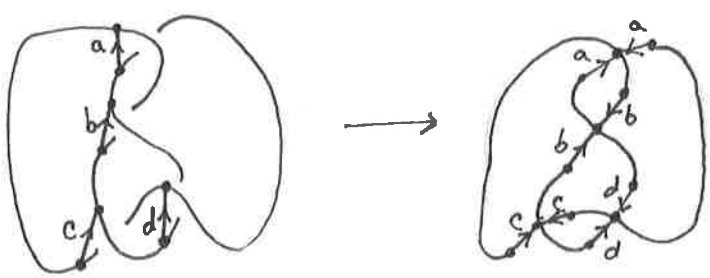
$\rightsquigarrow$ 1-cells

see paper model on Purcell p. 11

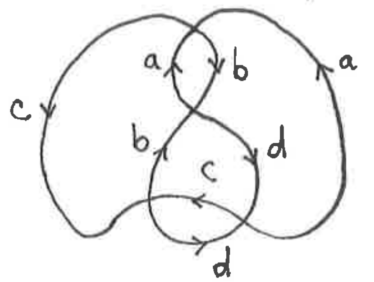
$\rightsquigarrow$ S^3 (with K) as a 3-complex. The 2-skeleton is a 2-sphere pinched near the crossings. To understand the attaching map we unpinch.

2D pic: 

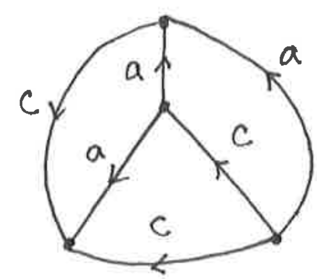
Unpinching from point of view of top ball:



Unlabeled edges make up K. To remove K, collapse each to a pt, think of as ideal vertices:

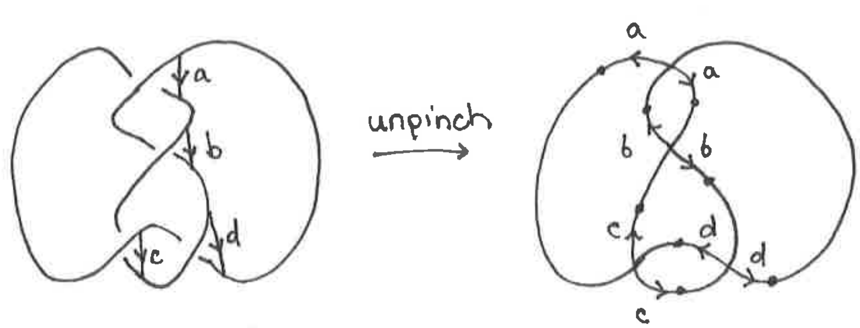


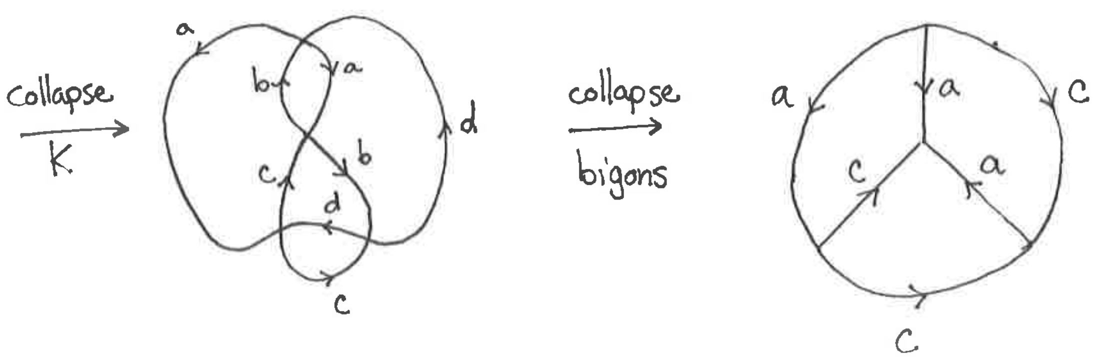
Next, gluing along a bigon is same as gluing along edge. Collapsing both bigons, we identify a with $\bar{b}$, c with $\bar{d}$ and get:



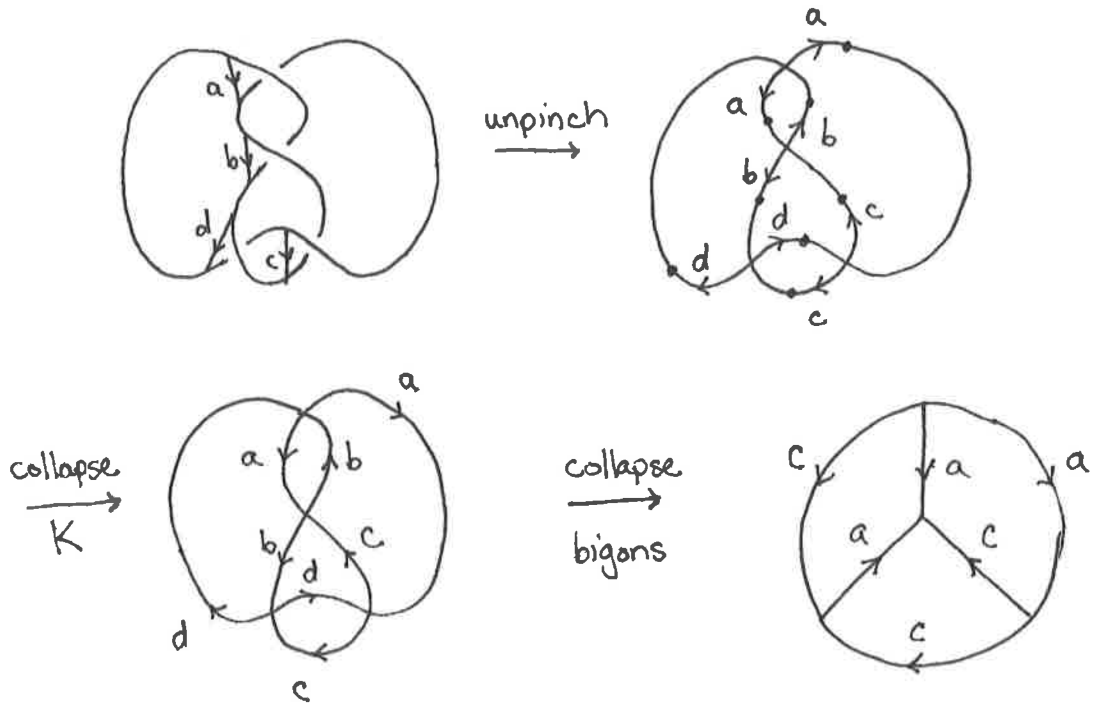
Doing same from the point of view of the bottom.

This is wrong! See next page.





Corrected bottom view:

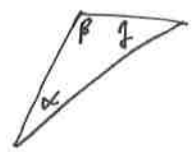


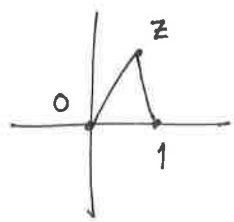
Hyperbolic Structures on Ideal Triangulations

Say M = top. manifold obtained by gluing ideal simplices, e.g. $S^3 \setminus K$.

Q1. Which shapes of tetrahedra give hyp. structures?
Q2. Which give <u>complete</u> hyp. structures? (Cauchys converge)

Again, by above thm, need angle 2π around each edge.
Recall: ideal  determined by its link



This is congruent to 

z = the complex parameter for the tetrahedron.

Note, z, $\frac{1}{1-z}$, $1-\frac{1}{z}$ all give congruent triangles.
But if we distinguish one vertex of the link (because it is on the edge we are focusing on) there is a unique complex param.

Let w_{ij} = complex param. for j^{th} tetrahedron around i^{th} edge.

<u>Thm.</u> M inherits a hyp. structure $\iff \prod_j w_{ij} = 1 \quad \forall i$.

~~Easier version: M inherits a hyp str~~ $\cancel{\iff \prod_j w_{ij} = 1}$ and $\sum_j \arg(w_{ij}) = 2\pi \ \forall i$.

"gluing equations"

Pf. Claim 1. M a man $\Leftrightarrow$ $|\prod_j w_{ij}| = 1 \ \forall i$.

Claim 2. M has angle 2π around i^{th} edge $\Leftrightarrow$ $\sum_j \arg(w_{ij}) = 2\pi$ and ~~Claim 3. $|\prod w_{ij}| = 1$ & $\sum \arg(w_{ij}) = 2\pi$ $\forall i$~~ $\prod_j w_{ij} = 1 \ \forall i$.

~~Note Claims 1,2 give easier version.~~

Pf of Claim 1. Let $e_1, \dots, e_k$ be the edges of ideal tets that get identified to i^{th} edge of M.

$\rightsquigarrow$ isometries $e_1 \to e_2 \to \cdots \to e_k \to e_1$ induced by face gluings.

$\rightsquigarrow$ $e_1 \to e_1$ isometry

Subclaim. $e_1 \to e_1$ is id $\Leftrightarrow$ M a man.

pf. If $e_1 \to e_1$ is translation then each pt of i^{th} edge has ∞ many preimages
$\Rightarrow$ M not locally compact.
If $e_1 \to e_1$ is reflection, $\exists$ fixed pt
$\rightsquigarrow$ pt in M with link $\cong$ cone on $\mathbb{RP}^2$

Subclaim. $e_1 \to e_1$ is id $\Leftrightarrow$ $|\prod_j w_{ij}| = 1$.

pf. place tetrahedra around i^{th} edge in U^3 around line from 0 to ∞.
and so first has vertices $0, \infty, 1, w_{i1}$
Then second has vertices $0, \infty, w_{i1}, w_{i1}w_{i2}$
Last face $0, \infty, \prod_j w_{ij}$ gets glued to first face $0, \infty, 1$ in a unique way by isometry.
The isometry fixes $0, \infty$ so it is dilation, which ~~So $e_1 \to e_1$ is id~~ is trivial iff $|\prod_j w_{ij}| = 1$.

$\blacksquare$

Claim 2 now evident.

Gluing Eqns for Fig 8

If the 3 complex parameters for the link of a tetrahedron in $S^3 \setminus K$ are $\quad z_1, \; z_2 = 1 - \frac{1}{z}, \; z_3 = \frac{1}{1-z}$ (first tet)

and $\quad w_1, \; w_2 = 1 - \frac{1}{w}, \; w_3 = \frac{1}{1-w}$ (second)

then the two sets of gluing eqns are:

$$z_1^2 z_2 w_1^2 w_2 = 1$$

$$z_3^2 z_2 w_3^2 w_2 = 1$$

Set $z_1 = z$, $w_1 = w$. First eqn gives:

$$z^2\left(1 - \tfrac{1}{z}\right) w^2 \left(1 - \tfrac{1}{w}\right) = 1$$

$$z(z-1)\, w(w-1) = 1$$

$$\rightsquigarrow \quad z = \frac{1 \pm \sqrt{1 + 4/(w(w-1))}}{2}$$

parameter space has one complex dim.

~~Need imag. parts of z, w [illegible]~~

Note $z = w = e^{i\pi/3}$ is a solution. But there are many others.

Will show this is the only solution giving a complete metric.

Completeness

Last time: family of hyp. structures on $S^3 \setminus K$
Q. Which are complete? Who cares?

<u>Complete hyperbolic manifolds</u>

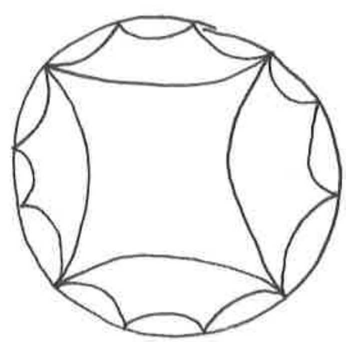
<u>Thm</u>. If M is a simply conn. complete hyp. n-man then M is isometric to $\mathbb{H}^n$.

<u>Cor</u>. The universal cover of a complete hyp. n-man is isometric to $\mathbb{H}^n$.

So we now have 3 ways to think about hyp mans:

1. topological charts with $\mathrm{Isom}(\mathbb{H}^n)$ transitions
2. locally isometric to $\mathbb{H}^n$
3. quotient of $\mathbb{H}^n$ by free, proper disc. action.

e.g.



Special case of Mostow Rigidity. If a hyp. n-man $(n \geq 3)$ has a hyp. metric that is complete and has finite volume, then the metric is unique.

Fig 8 Knot Complement as a complete manifold

Prop. M a metric space

S_t = family of compact subsets, $t \geq 0$ that cover M, and

$S_{t+a} \supseteq \mathrm{Nbd}(S_t, a)$

Then M is complete.

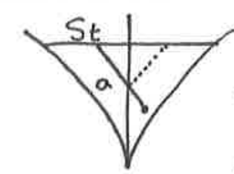
Pf. exercise.

Consider the hyp structure on $S^3 \setminus K$ given by two regular, ideal tetrahedra. Put ideal vertices of one tetrahedron T_i on vertices of regular (Euclidean) tetrahedron. (ball model).

Let $S_t^{(i)}$ = intersection of T_i with $B(0,t)$

$S_t = S_t^{(1)} \cup S_t^{(2)}$

exercise: these S_t satisfy the Prop (use the fact that both tetrahedra are regular & that the pic is symmetric! Hint: at each ideal vertex have reflection:



)

Cor. K = fig 8 knot.

The universal cover of $S^3 \setminus K$ with above metric is $\mathbb{H}^3$.

In particular, the univ cover of $S^3 \setminus K$ is homeo to $\mathbb{R}^3$.

Other Consequences

① A complete finite vol. hyp. man has infinite π_1. (must show $\mathrm{vol}(\mathbb{H}^n) = \infty$)

② S^n has no hyp. structure, $n > 1$.

③ A compact hyp. man has no $\mathbb{Z}^2 < \pi_1$.

so, e.g. T^n not hyperbolic

more generally a closed, hyp. 3-man is atoroidal.

④ A complete hyp. 3-man is irred.

Pf of ③.: Step 1. Universal cover is $\mathbb{H}^n$ (by completeness)

Step 2. Deck trans are hyperbolic

- ellipitics have fixed pts
- parabolics violate compactness (can find arbitrarily short loops)

Step 3. Commuting hyp. isometries have same axis

Step 4. Two translations of $\mathbb{R}$ either ① have a common power or ② have dense orbits.

Pf of ④. Let $S^2 \subseteq M$

Preimage in $\mathbb{H}^3$ is a collection of spheres. (using completeness here).

Alexander $\rightarrow$ each bounds a ball

Compactness $\Rightarrow \exists$ innermost lift of S^2, call it $\widetilde{S}^2$

$\rightsquigarrow$ ball in $\mathbb{H}^3$ with $\partial B = \widetilde{S}^2$

Translates of $\widetilde{S}^2$ all disjoint

$\Rightarrow B$ projects homeomorphically to closed ball $\overline{B}$ in M with $\partial \overline{B} = S^2$

Complete structures on surfaces

An example of an incomplete structure.

Let $B = \{(x,y) \in U^2 : 1 \le x \le 2\}$

Glue sides of B by $z \mapsto 2z$.

Result is incomplete: let $z_i = (1, 2^i) \sim (2, 2^{i+1})$

$$d(z_i, z_{i+1}) \le d_{\mathbb{H}^2}((2,2^{i+1}),(1,2^{i+1})) < 1/2^{i+1}$$

$\rightsquigarrow$ z_i Cauchy, does not converge since y-values $\longrightarrow \infty$.

More generally.

M = oriented hyp. surf. obtained by gluing ideal polygons

v = ideal vertex of M

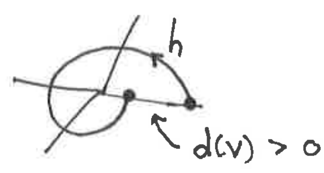
h = counterclockwise horocycle centered at v on one of the polygons P incident to v.

h meets ∂P in right angles

$\rightsquigarrow$ can continue h into next polygon.

$\rightsquigarrow$ eventually return to P.

$\rightsquigarrow$ $d(v)$ = resulting signed distance along ∂P (oriented to v).



exercise: $d(v)$ well defined.

Prop. M complete $\iff$ $d(v) = 0$ $\forall\, v$.

Pf. $d(v) \ne 0$ some v $\rightsquigarrow$ find nonconvergent Cauchy seq. as above.

$d(v) = 0$ $\forall\, v$ $\rightsquigarrow$ can make horocycles around each v.

S_t = subset of M obtained by deleting interior of horoballs bounded by horocycles distance t from originals.

Apply Prop. ∎

COMPLETE HYPERBOLIC 3-MANIFOLDS

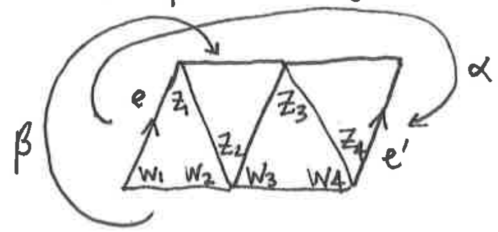
Overview

M = orientable hyp. 3-man obtained by gluing ideal tetrahedra
The link of any ideal vertex is a torus.
The intersection of any such torus with a tetrahedron is
a triangle (or more than one) cf. $S^3 \setminus K$ example.
$\rightsquigarrow$ triangulation of the torus into Euclidean triangles.
Will show: M complete $\Leftrightarrow$ each such torus is Euclidean
(angle 2π around each vertex).

The two sides are related by the developing map.

Completeness Equations

M as above. Say the triangulation of some torus link is



Choose two gluing maps α, β so the surface obtained by doing both gluings is a torus (possibly with holes).

Consider α. Say it glues e to e'.
Choose a path from e to e' in 1-skeleton.

$\rightsquigarrow$ sequence of edges $e = e_0, \dots, e_k = e'$

$\rightsquigarrow$ sequence of edge invariants $z_1, \dots, z_k$. (vertices of the Δs are edges in M)

Raise z_i to $+1$ power if $e_{i-1} \longrightarrow e_i$ is counterclockwise

-1 otherwise

$\rightsquigarrow$ product of $z_i^{\pm 1}$, call it H.

forgot: multiply by -1 if the seq of edge swings takes e to reverse of e'.

In above example: $H(\alpha) = z_1 z_2^{-1} z_3 z_4^{-1}$

or $H(\alpha) = w_1^{-1} w_2^{-1} z_2^{-1} w_3^{-1} w_4^{-1} z_4^{-1}$

exercise: $H(\alpha)$ is well defined.

Completeness Equations

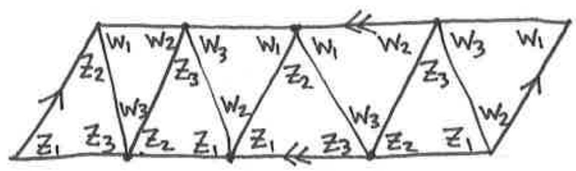
Proposition. The torus is Euclidean iff $H(\alpha) = H(\beta) = 1$.

Pf idea. $H(\alpha) = 1 \iff$ edges e, e' being glued are $\parallel$ and same length.

So $H(\alpha) = H(\beta) = 1 \iff$ corresponding deck trans are Euc. isometries. ▨

Figure 8 Example

Triangulation:



$\rightsquigarrow$ completeness eqns: $z_1^2 (w_2 w_3)^2 = (z/w)^2 = 1$

$w_1 / z_3 = w(1-z) = 1$.

first eqn $\rightsquigarrow$ $z = w$ (recall edge invariants have $\mathrm{Im} > 0$)

plugging into gluing eqn $\rightsquigarrow$ $(z(z-1))^2 = 1$

into second completeness eqn $\rightsquigarrow$ $z(z-1) = -1$

$\Rightarrow z = w = e^{i\pi/3}$ unique!

Developing Maps (Combinatorial Version)

M = hyperbolic (or Euclidean) manifold obtained by gluing (possibly ideal) polyhedra.

Will define $D: \tilde{M} \longrightarrow \mathbb{H}^n$ (or $\mathbb{E}^n$).

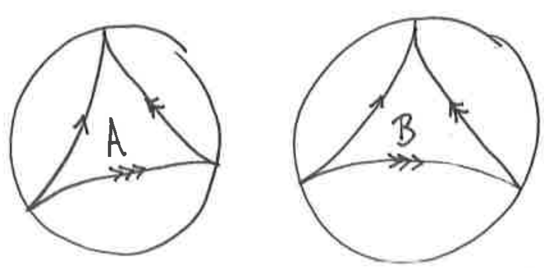
First, a description of $\tilde{M}$: glue polyhedra using same instructions as for M <u>except</u> each time we do a new gluing we take a new copy of the polyhedron

exercise: make sense of this and show the result is indeed $\tilde{M}$ (think of torus example).

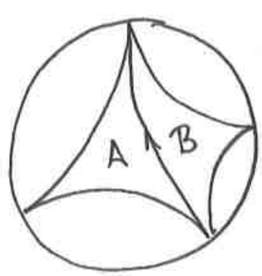
The map D is now evident: put the first polyhedron anywhere. Then glue in the rest of $\tilde{M}$ inductively.

The resulting map $\pi_1(M) \longrightarrow \mathrm{Isom}(\mathbb{H}^n)$ is called the <u>holonomy</u>.

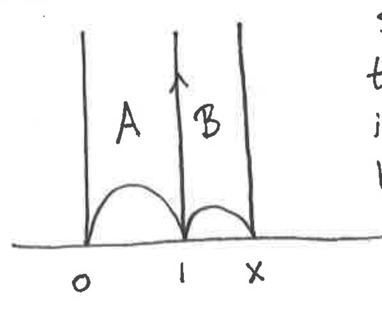
Example: sphere with punctures.



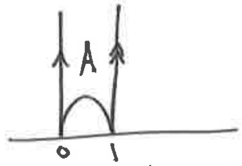
a gluing is prescribed by a picture like:



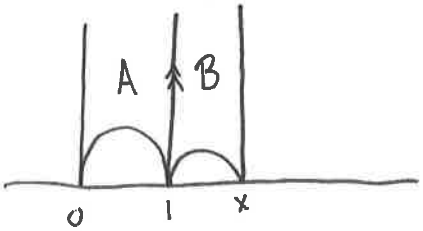
or

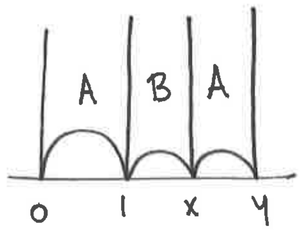


so a gluing of two ideal Δs is determined by $x > 1$.

We first put A in $\mathbb{H}^2$: 

Then put B in according to the prescribed gluing:



Then glue in A... 

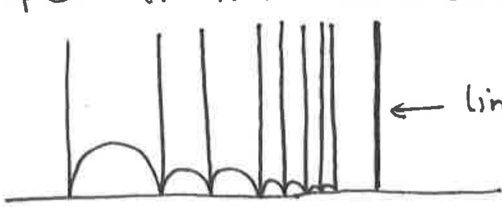
etc.

Recall our condition for completeness: horocycles obtained by extending the horocycle from one triangle should close up.

exercise: in our example this works iff $x=2, y=3$.*

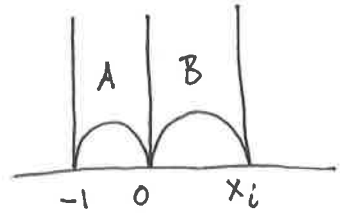
$\Rightarrow$ exactly one complete structure on 

An incomplete example. If in the above construction we take $x=3/2, y=2$ we get

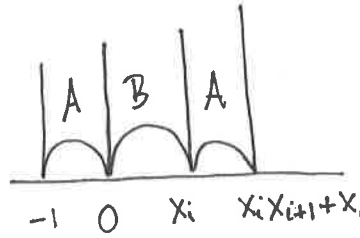


$\leftarrow$ limiting lines.

---

* More carefully: Say the 3 gluings are given by: 

The condition for completeness at a single cusp is $x_i x_{i+1} = 1$ (indices mod 3). Indeed, this is equivalent to the two copies of A differing by horizontal translation. The three eqns together imply $x_1 = x_2 = x_3 = 1$.



Developing Maps and Completeness

Theorem. M = hyp. n-man.
M is complete iff $D : \tilde{M} \to \mathbb{H}^n$ is a covering map
(iff D is a homeo)

This works more generally for (G,X)-structures on manifolds.

Pf. $\boxed{\Rightarrow}$ Say M complete.

D is a local homeo, so suffices to show D has the path lifting property.
Let α_t = path in M
D a local homeo $\Rightarrow$ can lift α_t to path $\tilde{\alpha}_t$ in $\tilde{M}$ for $t \in [0, t_0)$ $t_0 > 0$.
$\tilde{M}$ complete $\Rightarrow$ $\tilde{\alpha}_t$ extends to $[0, t_0]$.
D local homeo $\Rightarrow$ $\tilde{\alpha}_t$ extends to $[0, t_0 + \varepsilon)$
So $\tilde{\alpha}_t$ extends to $[0,1]$.
Converse similar. ▨

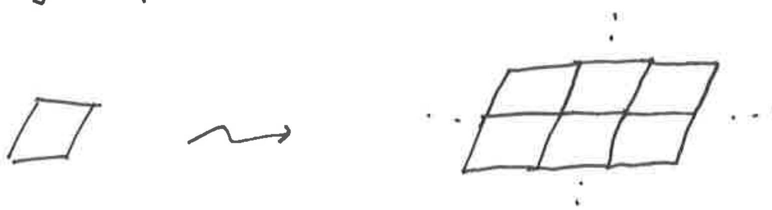
Compare with  example.

Prop. B = locally simply conn (any nbd ~~contai~~ of any pt contains a simply conn one)
$\tilde{B}$ = locally arcwise conn. (any nbd of any pt contains an arcwise conn. one)
$\pi : \tilde{B} \to B$ local homeo s.t. every arc in B lifts to $\tilde{B}$.
Then π is a covering map.

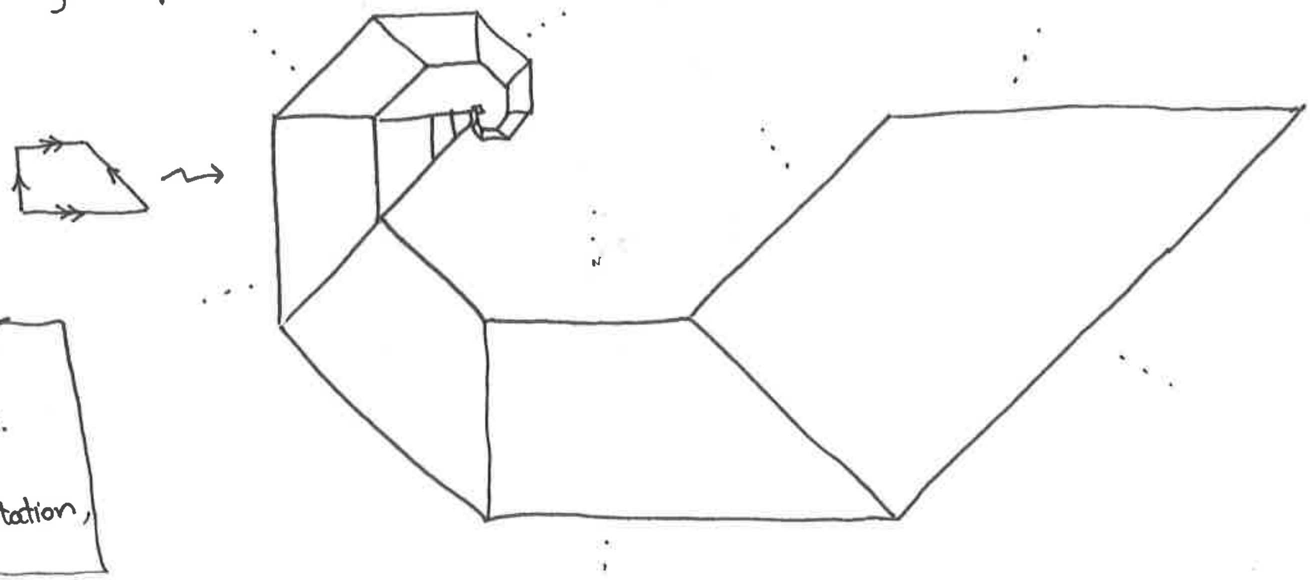
Pf. exercise (see baby do Carmo p. 383)

Affine Tori

Can do developing map with Euclidean tori:



Also makes sense with affine tori: arbitrary quadrilateral with gluing maps that are orient. pres. similarities of $\mathbb{E}^n$ instead of isometries.



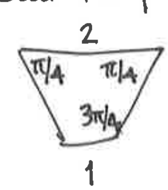
Classification of orientation pres. similarities: translation, rotation, spiral

If the quadrilateral is not a parallelogram, holonomy will have similarities that are not translations $\rightsquigarrow$ $\exists$ global fixed pt. (commuting similarities have same fixed pt).

~~spiral~~

~~To see that a similarity with nontrivial scaling has a fixed pt, assume the scaling is < 1 (up to taking inverses). Iterate on a disk. It converges to a point.~~ Summarizing:

Good example



<u>Prop.</u> $D: \tilde{T} \to \mathbb{E}^2$ is surjective iff T Euclidean.

Can show: if not surjective, D misses exactly one pt.

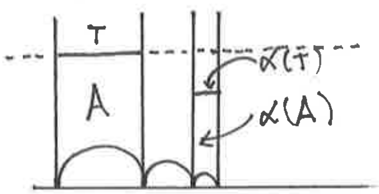
Complete Manifolds, Euclidean Cusps

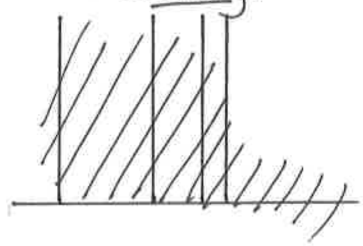
M = hyp 2- or 3-manifold obtained by gluing polyhedra.

v = ideal vertex

L = link of v (torus or circle)

L has a Euclidean similarity structure: under the developing map, simplices of L might change horocycles. To get any kind of Euclidean structure must project to a fixed horocycle. The cost of this is <u>scaling</u>.





<u>Thm</u>. M complete $\Leftrightarrow$ induced structure on each L is Euclidean.

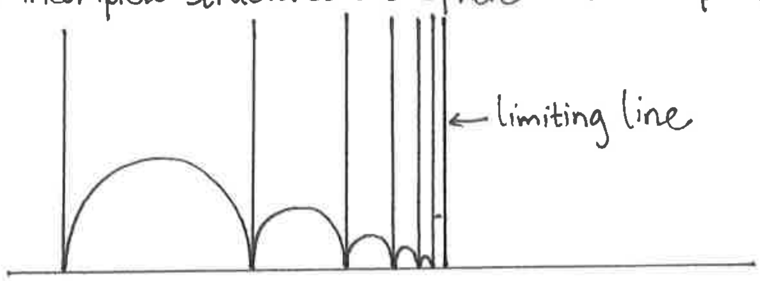
<u>Pf</u>. M complete $\Leftrightarrow$ developing map preserves horocycles

$\Leftrightarrow$ L Euclidean. $\boxtimes$

Completions

Surfaces

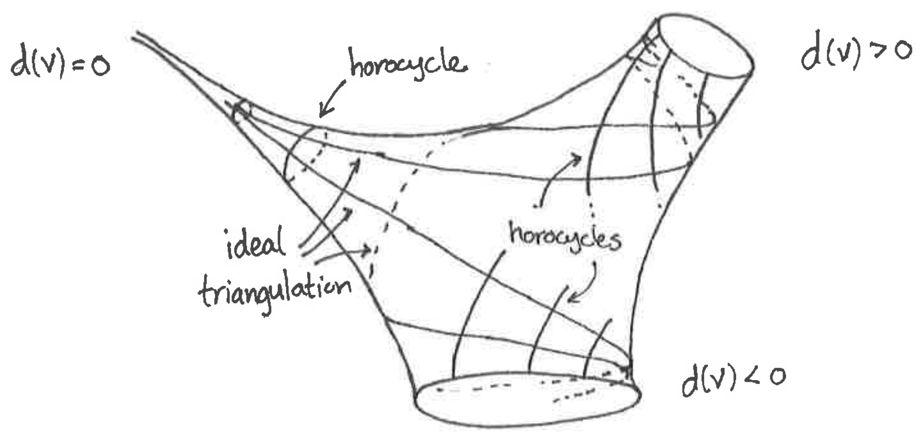
Recall incomplete structures on sphere with 3 punctures:



A horocycle ~~about~~ converging to the limiting line gives a nonconvergent Cauchy seq.

Horocycles at (oriented) distance $d(v)$ are *identified*

$\rightsquigarrow$ need to adjoin a segment of length $d(v)$.



COMPLETIONS: 3-MANIFOLDS.

M = hyp. 3-man obtained by gluing polyhedra.
G = holonomy gp corresponding to cusp torus T
about ideal vertex v

M incomplete $\Rightarrow G(\tilde{T}) = \mathbb{R}^2 \setminus pt$
$\Rightarrow G(\tilde{M})$ misses a line L

Case 1. G has dense orbits in L
$\rightsquigarrow$ completion is 1 pt compactification, not a mnfld.

Case 2. G has discrete orbits in L.

Pts in each orbit have distance $d(v)$ apart.
$\rightsquigarrow$ completion obtained by adding geodesic circles of length $d(v)$.

What does the completion look like?
Any elt of $\ker(G \to \text{Isom}(L))$ acts by rotation by θ.
$\Rightarrow$ cross sections of completion are 2D hyp. cones.
Completion is a cone manifold.

When $\theta = 2\pi$, completion is a hyp. manifold. If we remove a nbd of completion pts, we recover M.

We say the completion is obtained by Dehn filling on M.

Hyperbolic Dehn Surgery Space

Next big goal: Which Dehn fillings of $S^3 \setminus K$ are hyperbolic?

M = orientably hyp 3-man of ideal tetrahedra
v = ideal vertex (assume only 1 for simplicity).
$T = \text{Link}(v)$ torus
$\rightsquigarrow \pi_1(T) = \mathbb{Z}^2$

Dehn ~~Surgery~~ filling

Choose coords on $\pi_1(T^2)$. The (p,q) Dehn filling of M, written $M_{(p,q)}$ is the mnfld obtained by gluing solid torus s.t. ∂ of meridian disk attaches to (p,q)-curve in T.

clasps the knot

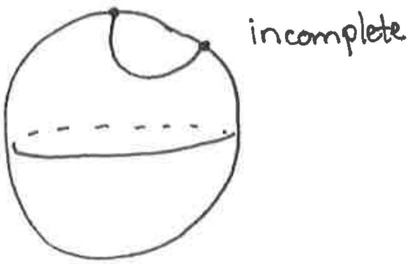
For $M = S^3 \setminus K$ there are canonical coords: meridian m is $1 \in H_1(M)$, longitude ℓ is 0.
$\searrow$ follows K.

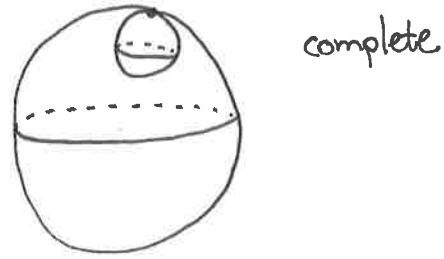
Holonomy

$\pi_1(T)$ abelian $\Rightarrow$ $\pi_1(T)$ fixes 1 or 2 pts of $\mathbb{H}^3$ (under holonomy)

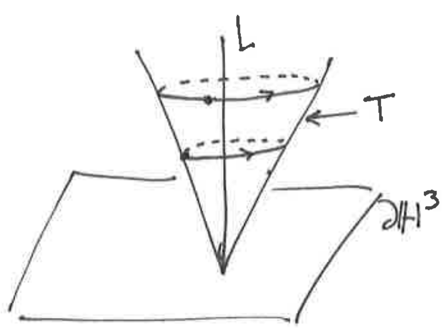
Fixes 1 pt $\Rightarrow$ image of $\pi_1(T)$ parabolic $\Rightarrow$ M complete.
Fixes 2 pt $\Rightarrow$ image of $\pi_1(T)$ consists of hyp. isometries. along single axis L. L is the pts missing from developing map of T in each horocycle. $\Rightarrow$ M incomplete.

Can see now why there is a 2D space of incomplete structures and one complete one:





Note: T is quotient of tube around L:



Complex Length

Any $\gamma \in \pi_1(T)$ translates L by d, rotates by $\theta \in \mathbb{R}$ — to get a real number, need to keep track of the number of times it goes around L.

$\ell(\gamma) = d + i\theta$ "complex length"

$\rightsquigarrow \ell : H_1(T;\mathbb{Z}) \longrightarrow \mathbb{C}$ linear

$\rightsquigarrow \ell : H_1(T;\mathbb{R}) \longrightarrow \mathbb{C}$ linear

~~We are more interested in $\tilde{\ell} : H_1(T) \longrightarrow \tilde{\mathbb{C}}^*$ where you keep track of the number of times a loop circles L, not just angle.~~

Note: If we want a discrete action, $\pi_1(T) \longrightarrow \mathrm{Isom}(L)$ has nontrivial kernel.

Dehn Surgery Coefficients

In general $\exists!\ c \in H_1(T;\mathbb{R})$ s.t. $\mathcal{L}(c) = 2\pi i$

This is the Dehn surgery coeff of T.

If $c = (p,q)$ & $\gcd(p,q) = 1$ then c is a curve in T that bounds a hyp disk and $\overline{M} = M_{(p,q)}$ is hyperbolic.

Thurston's Hyp. Dehn Surgery Thm

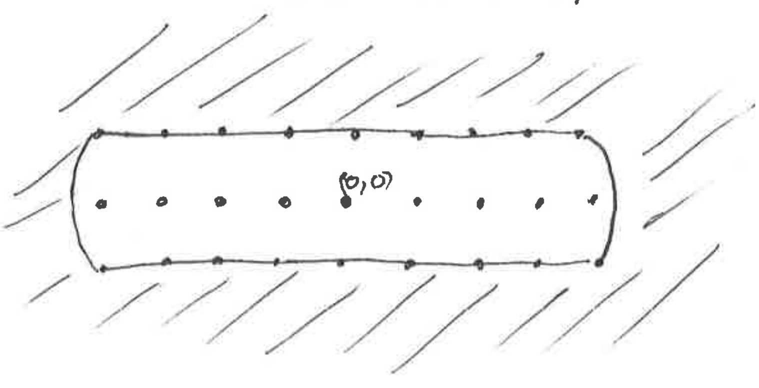
hyperbolic ~~topological~~

The hyp. Dehn surgery space for M is the set of all Dehn surgery coeffs, e.g. the Dehn fillings that give hyp. mans.

Thm (Thurston). The Dehn surgery space contains a nbd of ∞ in $\mathbb{C}$. Moreover $M_{(p_i,q_i)} \longrightarrow M_\infty$ as $(p_i,q_i) \longrightarrow \infty$.

(Analogous statement for multiple cusps: finitely ~~exce~~ many exceptional slopes on each ~~one~~ torus).

Example. $S^3 \setminus$ Fig 8 :



Idea: Explicitly analyze the map

$$\{\text{solutions to gluing eqns}\} \longrightarrow \{\text{Dehn surgery coeffs}\}$$

ie deform the triangles in T, then find the elements of $\pi_1(T)$ with complex length $2\pi i$.

Mostow Rigidity

<u>Thm.</u> M, N complete, finite vol, hyp n-mans $n > 2$
Any isomorphism $\pi_1 M \to \pi_1 N$ is induced by a unique isometry $M \to N$

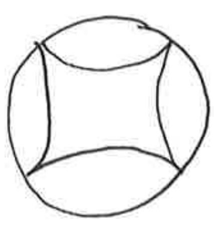
In particular: ① $\pi_1(M) \cong \pi_1(N) \Rightarrow M \equiv N$
② volume, diam, inj rad are invariants of M.

<u>Cor.</u> M closed, hyp n-man $n > 2$
$\mathrm{Isom}(M) \cong \mathrm{Out}(\pi_1 M)$
and these gps are finite.

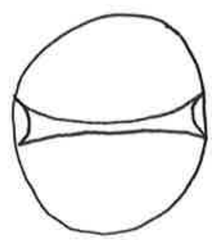
Pf idea. Mostow $\Rightarrow$ $\mathrm{Isom}(M) \to \mathrm{Out}(\pi_1 M)$ is surjective.

<u>Non-rigidity</u>

① Mostow not true for $n=2$:



$\not\equiv$



$\rightsquigarrow$ Teichmüller space

② Mostow not true for non-hyp mans

$\pi_1 L(7,1) \cong \pi_1 L(7,2) \cong \mathbb{Z}/7$
but $L(7,1) \not\equiv L(7,2)$ (Reidemeister)

OUTLINE OF PROOF

Assume M, N compact.

Start with $F: \pi_1 M \xrightarrow{\cong} \pi_1 N$

Want to promote F to an isometry $M \longrightarrow N$

Step 1. Homotopy equivalence

M, N are $K(G,1)$ spaces since $\tilde{M} \cong \tilde{N} \cong \mathbb{H}^n$

$\Rightarrow \exists \; f: M \longrightarrow N$

$g: N \longrightarrow M$

s.t. $g \circ f \simeq \mathrm{id}$.

Step 2. Lift

$\rightsquigarrow \tilde{f}: \mathbb{H}^n \longrightarrow \mathbb{H}^n$ (lifting criterion)

Step 3. Extend

$\rightsquigarrow \partial\tilde{f}: \partial\mathbb{H}^n \longrightarrow \partial\mathbb{H}^n$

Step 4. Show $\partial\tilde{f}$ is conformal.

Step 5. Extend

$\rightsquigarrow \varphi: \mathbb{H}^n \longrightarrow \mathbb{H}^n$ isometry

Step 6. φ descends to $\bar{\varphi}: M \longrightarrow N$.

Step 2. Properties of $\tilde{f}$

① $\tilde{f}$ is $\pi_1(M)$-equivariant:

$$\tilde{f}(g\cdot x) = f_*(g)\cdot \tilde{f}(x) \qquad \text{(exercise)}.$$

② $\tilde{f}$ is a quasi-isometry: $\exists\ K, C$ s.t.

$$\frac{1}{K} d(x,y) + C \leq d(\tilde{f}(x), \tilde{f}(y)) \leq K\, d(x,y) + C \qquad \left(\begin{array}{l}\text{and } \exists \\ \text{qi inverse}\end{array}\right)$$

pf of ②. Compactness + continuity $\Longrightarrow$ $\tilde{f}, \tilde{g}$ Lipschitz, i.e. $\exists\ K > 0$ s.t.

$$d(\tilde{f}(x), \tilde{f}(y)) \leq K\, d(x,y)$$

Other inequality. For $x, y \in \tilde{M}$ have

$$d(\tilde{g}\tilde{f}(x), \tilde{g}\tilde{f}(y)) \leq K\, d(\tilde{f}(x), \tilde{f}(y))$$

But $\tilde{g}\tilde{f}$ equiv. homotopic to id & M compact

$\rightsquigarrow$ $d(\tilde{g}\tilde{f}(z), z) \leq C$ for some C indep of z.

$$\Longrightarrow d(\tilde{f}(x), \tilde{f}(y)) \geq 1/K\ d(\tilde{g}\tilde{f}(x), \tilde{g}\tilde{f}(y))$$
$$\geq 1/K\ (d(x,y) - 2C). \qquad \blacksquare$$

Step 3. Quasigeodesics and the boundary map

Thm. Any quasi-isometry $h: \mathbb{H}^n \to \mathbb{H}^n$ extends to a homeo $\partial\mathbb{H}^n \to \partial\mathbb{H}^n$

Note: h need not be continuous!

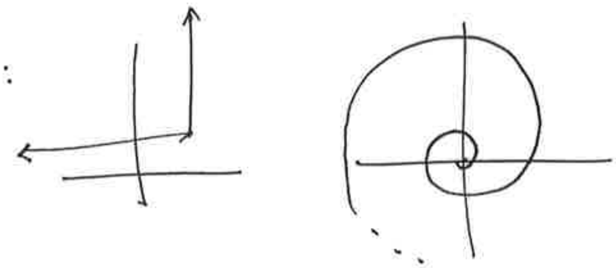
This works for $n = 2$.

Quasigeodesics

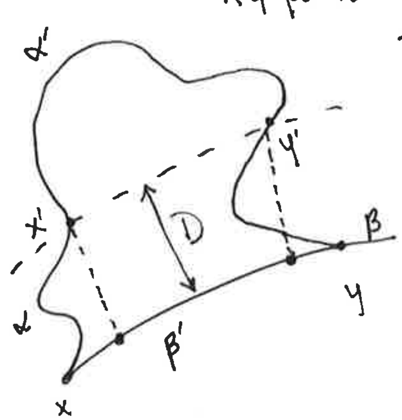
A geodesic in a metric space X is an isometric embedding $I \to X$.

A quasigeodesic is a quasi-isometric embedding $I \to X$.

examples in $\mathbb{R}^2$:



Morse-Mostow Stability Lemma. If $\alpha : \mathbb{R} \to \mathbb{H}^n$ is a quasigeod, $\exists !$ geod γ s.t. α lies in bdd nbd of γ.



Key point: Let $I = [a,b]$, $x = \alpha(a)$, $y = \alpha(b)$, β the geodesic from x to y.

Pick $D \gg K$ and suppose α does not stay within D of β.

Let x', y' be distinct pts of α at distance D from β.

Let β' be ~~projection~~ segment of β from projs of x' & y'.

Projections in $\mathbb{H}^n$ decrease length exponentially

$$\rightsquigarrow \ell(\alpha') \leq K^2(\ell(\beta') + 2D) + CK$$
$$\leq K^2(e^{-D} d(x', y') + 2D) + CK$$

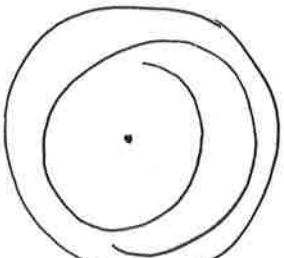
Now, $d(x', y') \leq \ell(\alpha')$ and $D \gg K$

$$\Rightarrow \ell(\alpha') \leq \frac{2DK^2 + CK}{1 - K^2 e^{-D}} \leq 4D^2$$

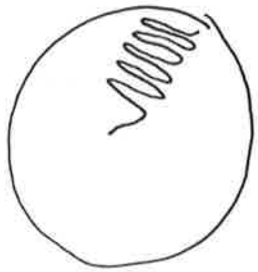
$\Rightarrow \alpha$ stays in $D + 4D^2$ nbd of β.

This only depends on K so works for any bdd interval $[a,b]$

Any quasigeodesic leaves every ball around O in $\mathbb{H}^n$, and this argument rules out spiralling:



or



The Extension

Recall $\partial\mathbb{H}^n = \{\text{geodesic rays}\}/\sim$

$\alpha \sim \beta$ if $d(\alpha(t), \beta(t))$ bounded

By the Lemma, h takes rays to rays (after straightening) and preserves $\sim$

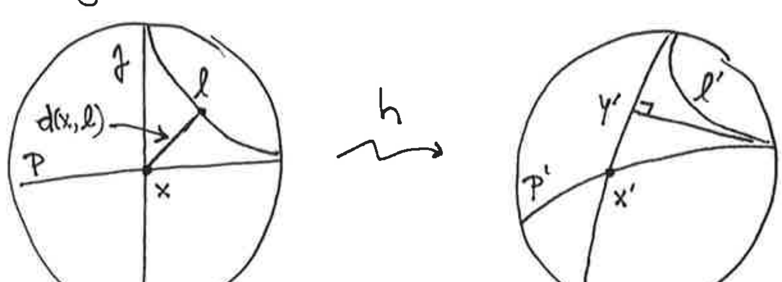
$\rightsquigarrow \partial h : \partial\mathbb{H}^n \longrightarrow \partial\mathbb{H}^n$

Check: ∂h is well def and 1-1.

Want to show ∂h is continuous.

"No tilting" **Lemma.** $\exists\, D = D(K)$ s.t. for any hyperplane $P \subseteq \mathbb{H}^n$ and any geod $\gamma \perp P$ we have $\operatorname{diam} \operatorname{Proj}_\gamma(h(P)) \leq D$.

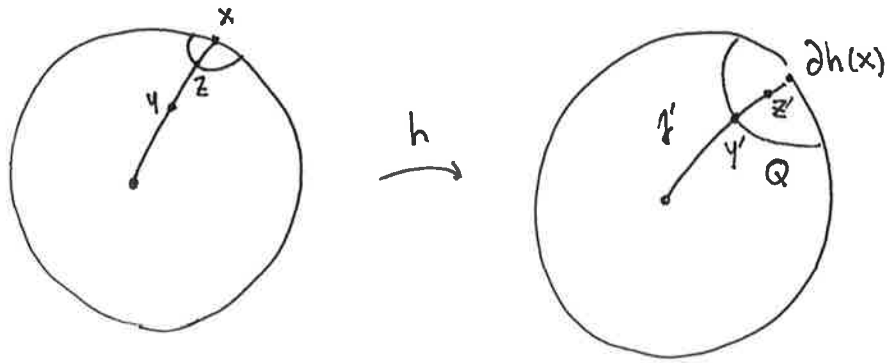
Pf.



prime' means: apply h then straighten.

$$d(x', y') \leq d(x', \ell') \leq K\, d(x, \ell) + C.$$

Proof that $\partial\tilde{f}$ is continuous:



Open half-spaces $\perp$ to γ' form a nbd basis around $\partial h(x)$.

Pick such a half space Q.

Choose z on γ s.t. $d(z', \partial Q) > 100D$ ← as in lemma

Then the half-space $\perp$ to γ through z maps into Q. ∎

Mostow Rigidity via Gromov Norm

Thm. M, N complete, finite vol, hyp mans $n>2$
Any isomorphism $\pi_1 M \to \pi_1 N$ is induced by
a unique isometry $M \to N$

Step 1. $\exists\ f: M \to N$ homotopy equiv. (uses completeness!)

Step 2. Lift to $\tilde{f}: \mathbb{H}^n \to \mathbb{H}^n$ quasi-isometry

Step 3. Extend to $\partial\tilde{f}: \partial\mathbb{H}^n \to \partial\mathbb{H}^n$ continuous

Gromov Norm

Norm on real singular n-chains: $\| \Sigma t_i \sigma_i \| = \Sigma |t_i|$

$\rightsquigarrow$ pseudo-norm on $H_n(X;\mathbb{R})$:

$$\|\alpha\| = \inf_{[\Sigma t_n \sigma_n] = \alpha} \| \Sigma t_i \sigma_i \|$$ "Gromov norm"

Lemma. $f: X \to Y$ cont, $\alpha \in H_n(X;\mathbb{R})$
then $\|f_*(\alpha)\| \leq \|\alpha\|$

Cor. f a homot. equiv $\Rightarrow \|f_*(\alpha)\| = \|\alpha\|$.

For M closed, orientable: $\|M\| = \|[M]\|$

Fact. If M admits deg >1 self-map then $\|M\| = 0$.

Step 4. Gromov norm vs. volume

Thm. M = closed, hyp n-man

$$\|M\| = \mathrm{Vol}(M)/v_n$$

v_n = max vol of a simplex

Cor. ① M has no self-maps of deg > 1

② volume is an invariant.

Step 5. $\partial\tilde{f}$ preserves regular ideal tetrahedra ($n=3$).

Step 6. $\partial\tilde{f}$ is conformal (hence agrees with some isometry).

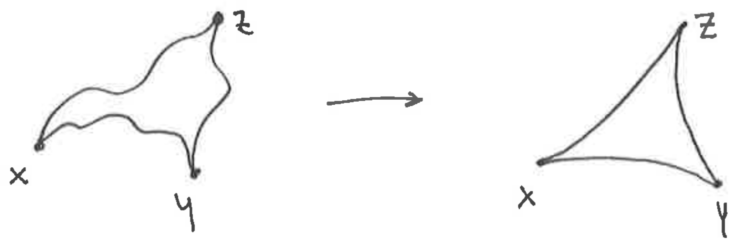
Fact. Let $n > 2$, σ reg. ideal tet, τ = face.
$\exists!$ reg ideal tet σ' s.t. $\sigma \cap \sigma' = \tau$.

Let σ = any reg ideal tetrahedron.
Step 5 $\Rightarrow$ $\partial\tilde{f}_*(\sigma)$ regular
$\Rightarrow$ Up to postcomposing with conformal map can assume $\partial\tilde{f}_*(\sigma) = \sigma$.
Fact $\Rightarrow$ $\partial\tilde{f}_*$ fixes every simplex obtained from σ via the grp gen by reflections in faces of σ
But the vertices of these tetrahedra are dense in $\partial\mathbb{H}^3$
$\Rightarrow$ $\partial\tilde{f}_* = \mathrm{id}$, as desired. ■

Gromov's Thm

Straightening simplices

In $\mathbb{H}^n$ an arbitrary singular simplex can be straightened:



This works for simplices in M (lift, straighten, project)

Note: ① Straightening takes cycles to cycles
② $\|\text{straight}(z)\| \leq \|z\|$ (some simplices might cancel/vanish).

Lower bound

Prop. $\|M\| \geq \mathrm{vol}(M)/v_n$

Pf. Let $z = \sum t_i \sigma_i$ straight cycle with $[z]=[M]$

$$\mathrm{vol}(M) = \int_M d\mathrm{Vol} = \sum t_i \int_{\Delta^n} \sigma_i^*(d\mathrm{Vol}) \leq \sum |t_i| v_n$$

$\Rightarrow \|z\| \geq \mathrm{vol}(M)/v_n$ take inf. ∎

Upper bound

Prop. $\|M\| \leq \mathrm{vol}(M)/V_n$

Need chains σ_L with $[\sigma_L] = [M]$
and $\|\sigma_L\| \longrightarrow \mathrm{vol}(M)/V_n$ as $L \longrightarrow \infty$.

Smearing.

D = fund. dom. for M
σ = simplex in M
$\rightsquigarrow \tilde{\sigma}$ = simplex in $\tilde{M} = \mathbb{H}^n$
t = signed measure of simplices in $\mathbb{H}^n$ with vertices in same copies of D as $\tilde{\sigma}$ (sign means mult by -1 if σ reverses or.)
$\rightsquigarrow$ Smear$(\sigma) = t\sigma$

Defining σ_L.

Consider all regular straight simplices σ with side length L, zeroth vertex in D. Choose $x \in D$.
Let σ' be the straight simplex with vertices at corresponding translates of x.
$\sigma_L = \sum_{\sigma}$ Smear(σ').

Check: ① volume of each such σ is $V_n - \varepsilon(L)$ $\qquad \lim_{L\to\infty} \varepsilon(L) = 0.$

② each such sum is finite, moreover

③ σ_L is a cycle

In particular, some multiple of $[\sigma_L]$ is $[M]$.

Say this multiple is $z = \sum t_i \sigma_i$

$$\rightsquigarrow \|M\| \leq \sum t_i = \mathrm{vol}(M) / V_n - \varepsilon(L)$$

$\blacksquare$

Step 5. Regular ideal tetrahedra go to same.

If not, a definite fraction of σ_L loses a definite amount of volume, violating Step 4.