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Geometric structures on 3–manifolds

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Handbook of Geometric Topology – Style files
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In 1977, W.P. Thurston stunned the world of low-dimensional topology by showing that 'many' (in a precise sense) compact 3–dimensional manifolds admitted a unique hyperbolic structure. Of course, hyperbolic 3–manifolds had been around since the days of Poincaré, as a subfield of complex analysis. Work of Andreev [6, 7] in the mid-sixties, Riley [111] and Jørgensen [64] in the early seventies, had provided hyperbolic 3–manifolds of increasing complex topology. In a different line of inquiry, W. Jaco and P. Shalen had also observed in the early seventies that the fundamental groups of atoroidal 3–manifolds shared many algebraic properties with those of hyperbolic manifolds. However, the fact that hyperbolic metrics on 3–manifolds were so common was totally unexpected, and their uniqueness had far reaching topological consequences.

At about the same time, the consideration of the deformations and degenerations of hyperbolic structures on non-compact 3–manifolds led Thurston to consider other types of geometric structures. Building on the existing topological technology of characteristic splittings of 3–manifold, he made the bold move of proposing his Geometrization Conjecture which, if we state it in loose terms, says that a 3–manifold can be uniquely decomposed into pieces which each admit a geometric structure. For the so-called Haken 3–manifolds, which at the time were essentially the only 3–manifolds which the topologists were able to handle, this Geometrization Conjecture was a consequence of Thurston's original Hyperbolization Theorem. But for non-Haken 3–manifolds, the conjecture was clearly more ambitious, for instance because it included the Poincaré Conjecture on homotopy 3–spheres as a corollary. Nevertheless, the Hyperbolic Dehn Surgery Theorem and, later, the Orbifold Geometrization Theorem provided a proof of many more cases of the Geometrization Conjecture.

This influx of new ideas completely revolutionized the field of 3–dimensional topology. In addition to the classical arguments of combinatorial topology, many proofs in low–dimensional topology now involve techniques borrowed from differential geometry, complex analysis or dynamical systems. This interaction between topology and hyperbolic geometry has also proved beneficial to the analysis of hyperbolic manifolds and Kleinian groups, where topological insights have contributed to much progress.

Yet, twenty years later, it is still difficult for the non-expert to find a way through the existing and non-existing literature on this topic. For instance, complete expositions of Thurston's Hyperbolization Theorem and of his Orbifold Geometrization Theorem are only beginning to become available. The problem is somewhat different with the topological theory of the characteristic splittings of 3–manifolds. Several complete expositions of the corresponding results have been around for many years, but they are not very accessible because the mathematics involved are indeed difficult and technical.

We have tried to write a reading guide to the field of geometric structures on 3–manifolds. Our approach is to introduce the reader to the main definitions and concepts, to state the principal theorems and discuss their importance and interconnections, and to refer the reader to the existing literature for proofs and details. In particular, there are very few proofs (or even sketches of proof) in this chapter.

In a field where unpublished prepublications have historically been very common and important, we tried to only quote references which are widely available, but it was of course difficult to omit such an influential publication as Thurston's original lecture notes [138]. The selection of topics clearly follows the biases of the author, but we also made the deliberate choice of privileging those aspects of geometric structures which have applications to geometric topology. In particular, we eliminated from our discussion the analysis of the geometric properties of infinite volume hyperbolic 3–manifolds, and its relation to complex analysis and complex dynamical systems; we can refer the reader to [17, 23, 77, 78, 82, 85, 138] for some details on this very active domain of research.

1. Geometric structures

1.1. The case of surfaces

As an introduction to geometric structures, we first consider a classical property of *surfaces*, namely (differentiable) manifolds of dimension 2. Before going any further, we should mention that we will use the usual implicit convention that a manifold is connected unless specified otherwise; however, submanifolds will be allowed to be disconnected. Also, a manifold will be without boundary, unless it is explicitly identified as a manifold with boundary (or perhaps we should say manifold-with-boundary). Manifolds with boundary will not occur until Section 2.5.

Any (connected) surface S admits a complete Riemannian metric which is locally isometric to the euclidean plane $\mathbb{E}^2$, the unit sphere $\mathbb{S}^2$ in euclidean 3–space $\mathbb{E}^3$, or the hyperbolic plane $\mathbb{H}^2$. There are two classical methods to see this: one based on complex analysis, and another one based on the topological classification of surfaces of finite type. We now sketch both, since they each are of independent interest.

Any orientable surface S admits a *complex structure* (or a *Riemann surface structure*), namely an atlas which locally models the surface over open subsets of $\mathbb{C}$, where all changes of charts are holomorphic, and which is maximal for these two properties; see for instance Ahlfors-Sario [2, Chap. III] or Reyssat [109]. The key idea is that, in dimension 2, any Riemannian metric is conformally flat. In other words, if we endow S with an arbitrary Riemannian metric, any point admits a neighborhood which is diffeomorphic to an open subset of $\mathbb{C}$ by an angle preserving diffeomorphism; since the changes of charts respect angles, the Cauchy-Riemann equation then implies that they are holomorphic. Similarly, a possibly non-orientable surface S admits a *twisted complex structure*, defined by a maximal atlas locally modeling S over open subsets of $\mathbb{C}$ and such that all changes of charts are holomorphic or antiholomorphic. This structure lifts to a twisted complex structure on the universal covering $\widetilde{S}$ of S. Since $\widetilde{S}$ is simply connected, we can choose an orientation for it. Then, composing orientation-reversing charts with the complex conjugation $z \mapsto \bar{z}$, we can arrange that all charts are orientation-preserving, so that all changes of charts are holomorphic. We now have a complex structure on

$\widetilde{S}$.

The construction of a twisted complex structure on S was only local. The Uniformization Theorem (see [2] or [109] for instance), a global property, asserts that every simply connected complex surface is biholomorphically equivalent to one of the following three surfaces: the complex plane $\mathbb{C}$, the half-plane $\mathbb{H}^2 = \{z \in \mathbb{C}; \operatorname{Im} z > 0\}$, and the complex projective line $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$. Therefore, $\widetilde{S}$ is biholomorphically equivalent to one of these three surfaces.

The surface S is the quotient of its universal covering $\widetilde{S}$ under the natural action of the fundamental group $\pi_1(S)$. Since the complex structure of $\widetilde{S}$ comes from a twisted complex structure on S, the covering automorphism defined by every element of $\pi_1(S)$ is holomorphic or anti-holomorphic with respect to this complex structure. Also, note that every element of $\pi_1(S)$ acts on $\widetilde{S}$ without fixed points since it is a covering automorphism. We now distinguish cases, according to whether $\widetilde{S}$ is biholomorphically equivalent to $\mathbb{C}$, $\mathbb{H}^2$ or $\mathbb{CP}^1$.

First consider the case where $\widetilde{S}$ is biholomorphically equivalent to $\mathbb{C}$. Every holomorphic or antiholomorphic automorphism of $\mathbb{C}$ is of the form $z \mapsto az + b$ or $z \mapsto c\bar{z} + d$, with a, b, c, $d \in \mathbb{C}$. For a fixed point free automorphism, we must have $a = 1$, or $|c| = 1$ and $c^{\frac{1}{2}}d \in \mathbb{R}$. In particular, every element of $\pi_1(S)$ respects the euclidean metric of $\widetilde{S}$ coming from the identifications $\widetilde{S} \cong \mathbb{C} \cong \mathbb{E}^2$. This induces on S a metric which, because the metric of $\widetilde{S} \cong \mathbb{E}^2$ is complete, is also complete. Note that this metric on S is *euclidean*, in the sense that every point of S has a neighborhood which is isometric to an open subset of the euclidean plane $\mathbb{E}^2$.

Every holomorphic or antiholomorphic automorphism of $\mathbb{H}^2$ is of the form $z \mapsto (az + b)/(cz + d)$ or $z \mapsto (a\bar{z} - b)/(c\bar{z} - d)$, with a, b, c, $d \in \mathbb{R}$ and $ad - bc = 1$. Poincaré observed that such an automorphism preserves the *hyperbolic metric* of $\mathbb{H}^2$, defined as the Riemannian metric which at $z \in \mathbb{H}^2$ is $1/\operatorname{Im} z$ times the euclidean metric of $\mathbb{H}^2 \subset \mathbb{C} \cong \mathbb{E}^2$. (We are here using the topologist's convention for the rescaling of metrics: When we multiply a metric by $\lambda > 0$, we mean that the distances are locally multiplied by λ; in the same situation, a differential geometer would say that the Riemannian metric is multiplied by λ^2.) The metric of $\mathbb{H}^2$ is easily seen to be complete. Therefore, if $\widetilde{S}$ is biholomorphically equivalent to $\mathbb{H}^2$, the hyperbolic metric of $\mathbb{H}^2$ induces a complete metric on $S = \widetilde{S}/\pi_1(S)$. By construction, this metric on S is *hyperbolic*, namely locally isometric to $\mathbb{H}^2$ at each point of S.

Every holomorphic or antiholomorphic automorphism of $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ is of the form $z \mapsto (az + b)/(cz + d)$ or $z \mapsto (a\bar{z} + b)/(c\bar{z} + d)$, with a, b, c, $d \in \mathbb{C}$ and $ad - bc = 1$. In particular, every holomorphic automorphism of $\mathbb{C} \cup \{\infty\}$ has fixed points. It follows that, either the fundamental group $\pi_1(S)$ is trivial, or it is isomorphic to the cyclic group $\mathbb{Z}_2$ and its generator acts antiholomorphically on $\widetilde{S} \cong \mathbb{C} \cup \{\infty\}$. If $z \mapsto (a\bar{z} + b)/(c\bar{z} + d)$ is a fixed point free involution, it is conjugated by a biholomorphic automorphism to the map $z \mapsto -1/\bar{z}$. (Hint: First conjugate it so that it exchanges 0 and ∞). Therefore, we can choose the holomorphic identification $\widetilde{S} \cong \mathbb{C} \cup \{\infty\}$ so that, either $\pi_1(S)$ is trivial, or $\pi_1(S) \cong \mathbb{Z}_2$ is generated by $z \mapsto -1/\bar{z}$. Identify $\mathbb{C} \cup \{\infty\} = \mathbb{R}^2 \times \{0\} \cup \{\infty\} \subset \mathbb{R}^3 \cup \{\infty\}$

to the unit sphere $\mathbb{S}^2$ of $\mathbb{R}^3$ by stereographic projection. For this identification, the antiholomorphic involution $z \mapsto -1/\bar{z}$ of $\mathbb{C} \cup \{\infty\}$ corresponds to the isometry $x \mapsto -x$ of $\mathbb{S}^2$. Then, the metric of $\widetilde{S} \cong \mathbb{S}^2$ induces a metric on $S = \widetilde{S}/\pi_1(S)$. Note that this metric on S is *spherical*, namely locally isometric to $\mathbb{S}^2$ everywhere, and is necessarily complete by compactness of $\mathbb{S}^2$.

The model spaces $\mathbb{E}^2$, $\mathbb{H}^2$ and $\mathbb{S}^2$ have a property in common: They are all *homogeneous* in the sense that, for any two points in such a space, there is an isometry sending one point to the other. As a consequence, the metrics we constructed on S are *locally homogeneous*: For any two x, $y \in S$, there is a *local isometry* sending x to y, namely an isometry between a neighborhood U of x and a neighborhood V of y which sends x to y. In other words, such a metric locally looks the same everywhere.

We should note that the topology of S is very restricted when $\widetilde{S}$ is biholomorphically isomorphic to $\mathbb{C}$. Indeed, the easy classification of free isometric actions on $\mathbb{E}^2$ shows that S must be a plane, a torus, an open annulus, a Klein bottle, or an open Möbius strip. The topology of S is even more restricted when $\widetilde{S}$ is isomorphic to $\mathbb{CP}^1$: We saw that in this case S must be homeomorphic to the sphere $\mathbb{S}^2$ or to the real projective plane $\mathbb{RP}^2 = \mathbb{S}^2/\{\pm\mathrm{Id}\}$. On the other hand, the case where $\widetilde{S}$ is biholomorphically isomorphic to $\mathbb{H}^2$ covers all the other surfaces. In this 'generic' case, we saw that S admits a complete hyperbolic metric, namely a metric which is locally isometric to the hyperbolic metric of $\mathbb{H}^2$.

Once we know what to look for, there is a more explicit construction of geometric structures, which is based on the topological classification of surfaces of finite type. Recall that a surface has *finite type* if it is diffeomorphic to the interior of a compact surface with (possibly empty) boundary. If S is a surface of finite type, it has a well-defined finite Euler characteristic $\chi(S) \in \mathbb{Z}$.

If $\chi(S) > 0$, the topological classification of surfaces (see for instance Seifert-Threlfall [128, Kap. 6] or Massey [83, Chap. II]) says that S is a plane, a 2–sphere or a projective plane. In the first case, S is diffeomorphic to the euclidean plane $\mathbb{E}^2$ and to the hyperbolic plane $\mathbb{H}^2$, and therefore admits a complete euclidean metric as well as a complete hyperbolic metric. In the remaining two cases, S is diffeomorphic to $\mathbb{S}^2$ or $\mathbb{RP}^2 = \mathbb{S}^2/\{\pm\mathrm{Id}\}$, and therefore admits a (complete) spherical metric.

When $\chi(S) = 0$, S is diffeomorphic to the open annulus, the open Möbius strip, the 2–torus or the Klein bottle. For the classical description of these surfaces as quotients of $\mathbb{E}^2$, we conclude that they all admit a complete euclidean metric. Considering the quotient of $\mathbb{H}^2 = \{z \in \mathbb{C}; \mathrm{Im}\, z > 0\}$ by a cyclic group of isometries generated by $z \mapsto \lambda z$ with $\lambda > 1$, or by another cyclic group of isometries generated by $z \mapsto -\lambda\bar{z}$ with again $\lambda > 1$, we can see that the open annulus and the open Möbius strip also admit complete hyperbolic metrics.

Finally, we can consider the case where $\chi(S) < 0$. Then, the classification of surfaces shows that we can find a compact 1–dimensional submanifold γ of S such that each component of $S - \gamma$ is, either a 'pair of pants' (namely an open annulus minus a closed disk) or a 'pair of Möbius pants' (namely an open Möbius strip minus a closed disk).

Consider a closed pair of pants P, namely the complement of three disjoint open disks in the 2–sphere. By an explicit construction involving right angled hexagons in $\mathbb{H}^2$, one can endow P with a hyperbolic metric for which the boundary ∂P is geodesic. In addition, this hyperbolic metric can be constructed so that the length of each boundary component of P can be an arbitrarily chosen positive number (up to isotopy, the hyperbolic metric is actually uniquely determined by the lengths of the boundary components). See [10, Sect. B.4] for details. In addition, there is a limiting case as we let the length of some boundary components tend to 0, which gives a complete hyperbolic metric on P minus 1, 2 or 3 boundary components, and where the remaining boundary components are still geodesic and of arbitrary lengths; in addition, such a metric has finite area.

There is a similar construction for the closed pair of 'Möbius pants', namely the complement of two disjoint open disks in the projective plane. Such a pair of Möbius pants P can be endowed with a hyperbolic metric for which the boundary is geodesic; in addition the length of each boundary component can be arbitrarily chosen (and there actually is an additional degree of freedom). Again, letting one or two of these lengths tend to 0, one obtains a finite area complete hyperbolic metric on P minus one of two boundary components.

Now, consider an arbitrary surface S of finite type, without boundary and with $\chi(S) < 0$. Using the classification of surfaces, one easily finds a compact 2–sided 1–submanifold C of S such that each component of $S - C$ is, either a pair of pants, or a pair of Möbius pants. For each component S_i of $S - C$, let $\widehat{S}_i$ be the surface with boundary formally obtained by adding to each end of S_i the component of C that is adjacent to it, with the obvious topology. We saw that we can endow each $\widehat{S}_i$ with a complete hyperbolic metric with geodesic boundary. Now, the surface S is obtained from the disjoint union of the $\widehat{S}_i$ by gluing back together the boundary components which correspond to the same component of C. We can choose the hyperbolic metric on the $\widehat{S}_i$ so that, when two boundary components are to be glued back together, they have the same length and the gluing map is an isometry. Then, one easily checks that the resulting metric on S is hyperbolic, even along C.

In this way, we can explicitly endow any surface S of finite type such that $\chi(S) < 0$ with a complete hyperbolic metric with finite area. Note that the isotopy class of this metric is in general far from being unique. Indeed, we were able to freely choose the length of the components of the 1–submanifold C. In a metric of negative curvature, every homotopy class of simple closed curves contains at most one closed geodesic. It follows that if, in the construction, we start from two hyperbolic metrics on the $\widehat{S}_i$ which give different lengths to some boundary components, the resulting two hyperbolic metrics on S cannot be isotopic. There is an additional degree of freedom associated to each component of C: when we glue back together the corresponding boundary components of the $\widehat{S}_i$, we can vary the gluing map by pre-composing it with an orientation-preserving isometry of one of these boundary components. If we add to this the degree of freedom hidden in the Möbius pant components of $S - C$ which we mentioned earlier, this clearly indicates that the hyperbolic metric of S is far from being unique.

However, the *Teichmüller space* $\mathcal{T}(S)$ of S, defined as the space of isotopy classes of all finite area complete hyperbolic metric on S, can be completely analyzed along these lines. In particular, it is homeomorphic to a Euclidean space of dimension $3|\chi(S)|-e$, where e is the number of ends of S. Good references include Benedetti-Petronio [10, Sect. B.4] or Fathi-Laudenbach-Poenaru [36, Exp. 7].

1.2. General definitions

The above analysis of surfaces suggests the following definition. A *geometric structure* on a connected manifold M without boundary is a locally homogeneous Riemannian metric m on M. As usual, the Riemannian metric turns M into a metric space, where the distance from x to y is defined as the infimum of the lengths of all differentiable arcs going from x to y. A geometric structure is *complete* when the corresponding metric space is complete. We just saw that every connected surface admits such a complete geometric structure.

Given a geometric structure m on M, we can always rescale the metric by a constant to obtain a new geometric structure. More generally, we can change m in the following way. For $x \in M$, consider all local isometries φ sending x to itself; the corresponding differentials $T_x\varphi : T_xM \to T_xM$ form a group G_x of linear automorphisms of the tangent space T_xM, called the *isotropy group* of the geometric structure at $x \in M$. Note that the isotropy group respects the positive definite quadratic form m_x defined by m on T_xM, and is therefore compact. Also, if φ is a local isometry sending x to y, the differential of φ sends the isotropy group of x to the isotropy group of y. The isotropy group of x is therefore independent of x up to isomorphism. If we fix a point $x_0 \in M$, let m'_{x_0} be another positive definite quadratic form on $T_{x_0}M$ which is respected by the isotropy group G_{x_0}. We can then transport m'_{x_0} to any other tangent space T_xM by using the differential $T_x\varphi : T_{x_0}M \to T_xM$ of any local m–isometry φ sending x_0 to x; the fact that G_{x_0} preserves m'_{x_0} guarantees that this does not depend on the choice of φ. We define in this way a new Riemannian metric on M, which is locally homogeneous by construction.

If the isotropy group G_x acts transitively on T_xM, the above construction simply yields a rescaling of the metric. For geometric structures with non-transitive isotropy groups, the modifications of the geometric structures can be a little more complex. However, they still do not differ substantially from the original geometric structure. This leads us to consider a weaker form of geometric structures, in order to neutralize these trivial deformations.

A complete geometric structure on M lifts to a complete geometric structure on the universal covering $\widetilde{M}$ of M. A result of Singer [130] asserts that a complete locally homogeneous Riemannian metric on a simply connected manifold is actually homogeneous. In particular, the isometry group of $\widetilde{M}$ acts transitively in the sense that, for every $\widetilde{x}, \widetilde{y} \in \widetilde{M}$, there exists an isometry g of $\widetilde{M}$ such that $g(\widetilde{x}) = \widetilde{y}$. We consequently have a Riemannian manifold $X = \widetilde{M}$ and a group G of isometries of X acting transitively on X. In addition, M admits an atlas $\{\varphi_i : U_i \to V_i\}_{i\in I}$ which locally models M over X and where all changes of charts are restrictions of elements of G. Namely, each φ_i is a diffeomorphism between an open subset U_i of M and an open subset V_i of X, the union of the U_i is equal to M, and each change of charts $\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ is the restriction of an element of G. Finally, note that an isometry g of X is completely determined by the image gx of a point x and by the differential $T_xg : T_xX \to T_{gx}X$ (Hint: Follow the geodesics). If we endow G with the compact open topology, it follows that for every $x \in X$ the stabilizer $G_x = \{g \in G; gx = x\}$ is compact, since it is homeomorphic to its image in the orthogonal group of isometries of the tangent space T_xM.

More generally, consider a group G acting effectively[1] and transitively on a connected manifold X, in such a way that the stabilizer G_x of each point $x \in X$ is compact for the compact open topology. An (X,G)–*structure* on a manifold M is defined by an atlas $\{\varphi_i : U_i \to V_i\}_{i\in I}$ which locally models M over X and where all changes of charts are restrictions of elements of G, as defined above. More precisely, such an (X,G)–*atlas* is contained in a unique maximal (X,G)–atlas, and an (X,G)–structure on M is defined as a maximal (X,G)–atlas.

In this situation, the hypothesis that the stabilizers G_x are compact guarantees the existence of a Riemannian metric on X which is invariant under the action of G. Indeed, if we fix a base point $x_0 \in X$, we can average an arbitrary positive definite quadratic form on $T_{x_0}X$ with respect to the Haar measure of G_{x_0} to obtain a G_{x_0}–invariant positive definite quadratic form. If we use $g \in G$ to transport this quadratic form to $T_{gx_0}X$, we now have a well defined Riemannian metric on X which is invariant under the action of G. This metric is homogeneous by construction, and therefore complete. Also, note that this construction establishes a one-to-one correspondence between G–invariant Riemannian metrics on X and G_{x_0}–invariant positive definite quadratic forms on the tangent space $T_{x_0}X$.

If M is endowed with an (X,G)–structure and if we choose a G–invariant Riemannian metric on X, we can pull back the metric of X by the charts of the (X,G)–atlas. This gives a locally homogeneous metric on M, namely a geometric structure on M.

An (X,G)–structure is *complete* if, for an arbitrary choice of a G–invariant metric on X, the associated geometric structure on M is complete. Note that different choices of a G–invariant metric on X give geometric structures on M which are Lipschitz equivalent, so that this notion of completeness is independent of the choice of the G–invariant metric on X.

As a summary, a complete geometric structure on M defines a complete (X,G)–structure on M, where X is the universal covering of M and where G is the isometry group of X. Conversely, a complete (X,G)–structure on M defines a complete geometric structure on M, modulo the choice of a G_{x_0}–invariant positive definite quadratic form on the tangent space on $T_{x_0}X$. So, intuitively, a complete (X,G)–structure corresponds to a metric independent version of a complete geometric

[1] Recall that a group G acts *effectively* on a set X if no non-trivial element of G acts by the identity on X.

structure. The reader should however beware of a few phenomena such as the fact that, if we start from a (X, G)–structure and a G–invariant metric on M, associate to them a complete geometric structure, and then consider the corresponding (X', G')–structure, the final geometric model may be much more symmetric than the original one in the sense that the stabilizers $G'_{x'}$ may be larger than the stabilizers G_x.

A *geometry* consists of a pair (X, G) as above, namely where X is a connected manifold, where the group G acts effectively and transitively on X, and where all stabilizers G_x are compact. This is also equivalent to the data of a connected Lie group G and of a compact Lie subgroup H of G, if we associate to this data the homogeneous space $X = G/H$ endowed with the natural left action of G.

We identify two geometries (X, G) and (X', G') if there is a diffeomorphism from X to X' which sends the action of G to the action of G'. An (X, G)–structure on M naturally lifts to an $(\widetilde{X}, \widetilde{G})$–structure where $\widetilde{G}$ consists of all lifts of elements of G to the universal covering $\widetilde{X}$ of X. Therefore, we can restrict attention to geometries (X, G) where X is simply connected. Also, if the geometry (X, G) can be enlarged to a more symmetric geometry (X, G') with $G \subset G'$, every (X, G)–structure naturally defines an (X, G')–structure. Consequently, if we want to classify all possible geometries in a given dimension, it makes sense to restrict attention to geometries (X, G) which are *maximal*, namely where X is simply connected and where there is no larger geometry (X', G') with $G \subset G'$ and $G \neq G'$.

2. The eight 3–dimensional geometries

We now focus on the dimension 3, and want to list all maximal geometries (X, G) where X is 3–dimensional. As indicated above, this amounts to listing all pairs (G, H) where G is a Lie group, H is a compact Lie subgroup of G, and the quotient G/H has dimension 3 and is simply connected (we let the reader translate the maximality condition into this context). Note that H must be isomorphic to a closed subgroup of $O(3)$. Listing all such geometries now becomes a relatively easy exercise using the Lie group machinery.

However, it is convenient to decrease the list even further. We will see that complete geometric structures of finite volume tend to have better uniqueness properties. Therefore, it makes sense to restrict attention to geometries for which there is at least one manifold admitting a complete (X, G)–structure of finite volume; note that this finite volume property does not depend on the choice of a G–invariant metric on X.

In this context, Thurston observed that there are exactly 8 maximal geometries (X, G) for which there is at least one finite volume complete (X, G)–structure. This section is devoted to a description of these eight geometries and of their first properties. The article by Scott [125] constitutes a very complete reference for this material.

2.1. The three isotropic geometries

The three 2–dimensional geometries (X, G) which we encountered are *isotropic* in the sense that, for any two points x, $x' \in X$ and any half-lines $\mathbb{R}^+ v \subset T_x X$ and $\mathbb{R}^+ v' \subset T_{x'} X$ in the tangent spaces of X at x and x', there is an element of G sending x to x' and v to v'. This is equivalent to the property that the stabilizer G_x acts transitively on the set of half-lines in the tangent space $T_x X$. In other words the geometry (X, G) is isotropic if, not only does X look the same at every point, but it also looks the same in every direction.

In dimension 3 (and actually in any dimension), there similarly are three isotropic maximal geometries. If, for an isotropic geometry (X, G), we endow X with a G–invariant Riemannian metric, passing to the orthogonal shows that we can send any plane tangent to X at $x \in X$ to any other plane tangent to X at $x' \in X$ by an element of G. As a consequence, any G–invariant metric on X must have constant sectional curvature. A classical result in differential geometry says that, for every $K \in \mathbb{R}$ and every dimension n, there is only one simply connected complete Riemannian manifold of dimension n and of constant sectional curvature K, up to isometry; see for instance Wolf [154]. Since rescaling the metric by $\lambda > 0$ multiplies the curvature by λ^{-2}, this leaves us with only 3 possible models for X, according to whether the curvature is positive, 0 or negative.

When the curvature is positive, we can rescale the metric so that the curvature is $+1$. Then, X is isometric to the unit sphere

$$\mathbb{S}^3 = \left\{ (x_0, x_1, x_2, x_3) \in \mathbb{R}^4; \sum_{i=0}^{3} x_i^2 = 1 \right\}$$

with the Riemannian metric induced by the euclidean metric of $\mathbb{R}^4 = \mathbb{E}^4$. By maximality, G is equal to the isometry group $\mathrm{Isom}\left(\mathbb{S}^3\right)$ of $\mathbb{S}^3$. This isometry group clearly contains the orthogonal group $\mathrm{O}(4)$. Since $\mathrm{O}(4)$ acts transitively on the space of orthonormal frames[2] of $\mathbb{S}^3$, this inclusion is actually an equality, namely $G = \mathrm{Isom}\left(\mathbb{H}^3\right) = \mathrm{O}(4)$.

When the curvature is 0, X is isometric to the euclidean space $\mathbb{E}^3$, with the usual euclidean metric. Again, G coincides with the isometry group $\mathrm{Isom}\left(\mathbb{E}^3\right)$, which is described by the exact sequence

$$0 \to \mathbb{R}^3 \to \mathrm{Isom}\left(\mathbb{E}^3\right) \to \mathrm{O}(3) \to 0$$

where the subgroup $\mathbb{R}^3$ consists of all translations, and where the map $\mathrm{Isom}\left(\mathbb{E}^3\right) \to \mathrm{O}(3)$ is defined by considering the tangent part of an isometry. Any choice of a base point $x_0 \in \mathbb{E}^3$ defines a splitting of this exact sequence, by sending $g \in \mathrm{O}(3)$ to the isometry of $\mathbb{E}^3$ that fixes x_0 and is tangent to g. In particular, this describes $\mathrm{Isom}\left(\mathbb{E}^3\right)$ as the semi-direct product of $\mathbb{R}^3$ and of $\mathrm{O}(3)$, twisted by the usual action

[2] Recall that an orthogonal frame is an orthonormal basis in the tangent space $T_x\mathbb{S}^3$ of some $x \in \mathbb{S}^3$

of $\mathrm{O}(3)$ on $\mathbb{R}^3$.

When the curvature is negative, we can again rescale the metric so that the curvature is -1. Then, X is isometric to the hyperbolic 3–space

$$\mathbb{H}^3 = \left\{(u,v,w) \in \mathbb{R}^3; w > 0\right\}$$

endowed with the Riemannian metric which, at (u,v,w), is $1/w$ times the euclidean metric. Among the three isotropic geometries, the geometry of $\mathbb{H}^3$ is probably the least familiar, but it is also the richest. For instance we will see that, as in the case of surfaces, there are many more 3–manifolds which admit a geometry modelled over $\mathbb{H}^3$ than over $\mathbb{E}^3$ or $\mathbb{S}^3$. An isometry of $\mathbb{H}^3$ continuously extends to its closure in $\mathbb{R}^3 \cup \{\infty\}$. The boundary of $\mathbb{H}^3$ in $\mathbb{R}^3 \cup \{\infty\}$ is $\mathbb{R}^2 \times \{0\} \cup \{\infty\}$, which the standard isomorphism $\mathbb{R}^2 \cong \mathbb{C}$ identifies to the complex projective line $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$. It can be shown that any homeomorphism of $\mathbb{C} \cup \{\infty\}$ that is induced by an isometry of $\mathbb{H}^3$ is holomorphic or antiholomorphic, and therefore is of the form $z \mapsto (az+b)/(cz+d)$ or $z \mapsto (a\bar{z}+b)/(c\bar{z}+d)$ with a, b, c, $d \in \mathbb{C}$ with $ad-bc=1$. Conversely, every holomorphic or antiholomorphic homeomorphism φ of $\mathbb{C} \cup \{\infty\}$ extends to an isometry of $\mathbb{H}^3$. The easier way to see this is probably to remember that such a φ can be written as a product of inversions across circles, to extend an inversion of $\mathbb{C} \cup \{\infty\}$ across the circle C to the inversion of $\mathbb{R}^3 \cup \{\infty\}$ across the sphere that has the same center and the same radius as C, and to check that the inversion across such a sphere respects $\mathbb{H}^3$ and the metric of $\mathbb{H}^3$.

2.2. The four Seifert type geometries

In contrast to the dimension 2, there is enough room in dimension 3 to allow maximal geometries (X,G) which are not isotropic. Namely, for such a geometry, there is at each point x a preferred line L_x in the tangent space T_xX such that, for each $g \in G$ and each $x \in X$, the differential $T_xg : T_xX \to T_xX$ sends the line L_x to L_{gx}.

The first two such geometries are provided by the Riemannian manifolds $\mathbb{S}^2 \times \mathbb{E}^1$ and $\mathbb{H}^2 \times \mathbb{E}^1$, endowed with the product metric.

For $X = \mathbb{H}^2 \times \mathbb{E}^1$, say, consider the natural action of the group $G = \mathrm{Isom}\left(\mathbb{H}^2\right) \times \mathrm{Isom}\left(\mathbb{E}^1\right)$, where $\mathrm{Isom}(Y)$ denotes the isometry group of Y. This action respects the metric of X, and is clearly transitive. Note that, for every $(x,y) \in \mathbb{H}^2 \times \mathbb{E}^1$, the differential of any element of the stabilizer $G_{(x,y)}$ respects the line $L_{(x,y)} = 0 \times T_y\mathbb{E}^1 \subset T_{(x,y)}\mathbb{H}^2 \times \mathbb{E}^1$. Therefore, the geometry (X,G) is non-isotropic, and of the type mentioned above.

It remains to check that this geometry (X,G) is maximal. This is clearly equivalent to the property that, for every G–invariant metric m on X, the isometry group of m cannot be larger than G. At each point $(x,y) \in X$, the metric m must be invariant under the action of the stabilizer $G_{(x,y)}$. In particular, since $G_{(x,y)}$ contains maps which rotate X around $\{x\} \times \mathbb{E}^1$, the bilinear form induced by m on $T_{(x,y)}X$ must be invariant under rotation around $L_{(x,y)} = 0 \times T_y\mathbb{E}^1$. It follows that m must be obtained from the product metric m_0 by rescaling it by a factor of $\lambda_1 > 0$ in the direction of the line $L_{(x,y)} = 0 \times T_y\mathbb{E}^1$ and by a factor of $\lambda_2 > 0$ in the direction of the orthogonal plane $L^\perp_{(x,y)} = T_x\mathbb{H}^2 \times 0$ (keeping these two subspaces orthogonal). To show that G is the whole isometry group of such a metric m, note that the sectional curvature of m along a plane $P \subset T_{(x,y)}X$ is 0 if P contains the line $L_{(x,y)}$, is $-\lambda_2^{-2}$ if P is equal to the orthogonal $L^\perp_{(x,y)}$, and is strictly between 0 and $-\lambda_2^{-2}$ otherwise. It follows that the differential of every m–isometry φ must send $L_{(x,y)}$ to $L_{\varphi(x,y)}$ and $L^\perp_{(x,y)}$ to $L^\perp_{\varphi(x,y)}$. In particular, at an arbitrary point $(x_0, y_0) \in X$, there is an isometry $\varphi' \in G$ such that $\varphi'(x_0, y_0) = \varphi(x_0, y_0)$ and $T_{(x_0,y_0)}\varphi' = T_{(x_0,y_0)}\varphi$, which implies that $\varphi = \varphi' \in G$. Therefore, every m–isometry φ is an element of G.

Replacing $\mathbb{H}^2$ by $\mathbb{S}^2$, we similarly prove that the manifold $X = \mathbb{S}^2 \times \mathbb{E}^1$ endowed with the natural action of $G = \mathrm{Isom}\left(\mathbb{S}^2\right) \times \mathrm{Isom}\left(\mathbb{E}^1\right)$ defines a maximal geometry (X, G) (the only difference being that the sectional curvature along a plane is now between 0 and $+\lambda_2^{-2}$).

Note that the geometry where $X = \mathbb{E}^2 \times \mathbb{E}^1$ and $G = \mathrm{Isom}\left(\mathbb{E}^2\right) \times \mathrm{Isom}\left(\mathbb{E}^1\right)$ is conspicuously absent. This is because $\mathbb{E}^2 \times \mathbb{E}^1$ is identical to the euclidean 3–space $\mathbb{E}^3$, and G can therefore be extended to the larger group $\mathrm{Isom}\left(\mathbb{E}^3\right)$. Therefore, this geometry is not maximal.

There are also twisted versions of these product geometries. We first describe an explicit model for the twisted product $\mathbb{H}^2\widetilde{\times}\mathbb{E}^1$. Let $T^1\mathbb{H}^2$ be the unit tangent bundle of $\mathbb{H}^2$, consisting of all tangent vectors of length 1 of $\mathbb{H}^2$. Consider the natural projection $p : T^1\mathbb{H}^2 \to \mathbb{H}^2$, associating its base point to each $v \in T^1\mathbb{H}^2$.

The metric of $\mathbb{H}^2$ determines a metric on $T^1\mathbb{H}^2$ as follows: The tangent space of $T^1\mathbb{H}^2$ at $v \in T^1\mathbb{H}^2$ naturally splits as the direct sum of a line L_v and of a plane P_v, where L_v is the tangent line to the fiber $p^{-1}(p(v))$, and where P_v consists of all infinitesimal parallel translations of v along geodesics passing through the point $p(v) \in \mathbb{H}^2$. The norm defined by the metric of $\mathbb{H}^2$ on $T_{p(v)}\mathbb{H}^2$ induces a metric on the fiber $p^{-1}(p(v)) \subset T_{p(v)}\mathbb{H}^2$, making it isometric to the unit circle $\mathbb{S}^1$, and this metric induces a norm on the line L_v tangent to $p^{-1}(p(v))$. Also, the restriction of the differential dp_v identifies the plane P_v to the tangent space $T_{p(v)}\mathbb{H}^2$, and the metric of $\mathbb{H}^2$ then defines a norm on P_v. The Riemannian metric of $T^1\mathbb{H}^2$ is defined by the property that, at each $v \in T^1\mathbb{H}^2$, it restricts to the above norms on L_v and P_v and it makes these two spaces orthogonal.

The construction of this metric is intrinsic enough that it is respected by the natural lift $v \mapsto T_{p(v)}\varphi(v)$ of each isometry $\varphi : \mathbb{H}^2 \to \mathbb{H}^2$. It is also respected by the other natural transformations of $T^1\mathbb{H}^2$ that rotate each vector v by a fixed angle θ, for every θ. In particular, this metric makes $T^1\mathbb{H}^2$ a homogeneous Riemannian manifold.

The space $T^1\mathbb{H}^2$ has the homotopy type of a circle. The model for $\mathbb{H}^2\widetilde{\times}\mathbb{E}^1$ is its universal covering $\widetilde{T}^1\mathbb{H}^2$.

Topologically, $\mathbb{H}^2\widetilde{\times}\mathbb{E}^1$ is homeomorphic to $\mathbb{H}^2 \times \mathbb{E}^1$, by a homeomorphism which conjugates the submersion $\widetilde{p} : \mathbb{H}^2\widetilde{\times}\mathbb{E}^1 \to \mathbb{H}^2$ lifting p to the projection $\mathbb{H}^2 \times \mathbb{E}^1 \to \mathbb{H}^2$. However, the situation is metrically very different. Indeed, if α is an oriented differentiable curve going from x to itself in $\mathbb{H}^2$ and if v is in the fiber $\widetilde{p}^{-1}(x)$, there is a unique way of lifting α to a curve $\widetilde{\alpha}$ in $\mathbb{H}^2 \widetilde{\times} \mathbb{E}^1$ that begins at v and is everywhere orthogonal to the fibers $p^{-1}(\alpha(t))$. It immediately follows from the Gauss Bonnet Formula that, in contrast to what happens in the case of $\mathbb{H}^2 \times \mathbb{E}^1 \to \mathbb{H}^2$ (where $\widetilde{\alpha}$ returns to its starting point v), the end point of $\widetilde{\alpha}$ sits at a signed distance of $-A$ from v in the fiber $\widetilde{p}^{-1}(x)$, where A is the signed area enclosed by α in $\mathbb{H}^2$ and where $\widetilde{p}^{-1}(x)$ is oriented by the orientation of $\mathbb{H}^2$.

Since the Riemannian manifold $T^1\mathbb{H}^2$ is homogeneous, so is its universal covering $\widetilde{T}^1\mathbb{H}^2 = \mathbb{H}^2 \widetilde{\times} \mathbb{E}^1$. In particular, the isometry group of $X = \mathbb{H}^2 \widetilde{\times} \mathbb{E}^1$ contains the group G generated by the vertical translations along the fibers and by the lifts of the isometries of $T^1\mathbb{H}^2$ associated to the isometries of $\mathbb{H}^2$.

It remains to see that the geometry (X, G) so defined is maximal. As usual, it suffices to prove that the isometry group of any G–invariant metric is equal to G. The action of the stabilizer G_v on the tangent space T_vX contains all rotations around the line L_v tangent to the fiber $p^{-1}(p(v))$. Therefore, any G–invariant metric m on X must be obtained by rescaling the original metric by a uniform factor along L_v and by another uniform factor along the plane P_v orthogonal to L_v. A straightforward computation shows that the sectional curvature of such a metric m along a plane $P \subset T_vX$ is maximal when P contains L_v, and is minimal when P is orthogonal to L_v. As a consequence, the differential of every isometry φ of m must send L_v to $L_{\varphi(v)}$, and therefore commutes with the projection $\widetilde{p} : \mathbb{H}^2 \widetilde{\times} \mathbb{E}^1 \to \mathbb{H}^2$. Also, considering the lift of a closed curve α enclosing a non-zero area in $\mathbb{H}^2 \widetilde{\times} \mathbb{E}^1$, we see that an m–isometry φ respects the orientation of the fibers of $\widetilde{p}$ if and only if the induced map $\mathbb{H}^2 \to \mathbb{H}^2$ is orientation-preserving. At this point, for every m–isometry φ and for an arbitrary $v \in X$, one easily finds an element $\psi \in G$ with $\varphi(v) = \psi(v)$ and $T_v\varphi = T_v\psi$, from which we conclude that $\varphi = \psi$. Therefore, the geometry of $\mathbb{H}^2 \widetilde{\times} \mathbb{E}^1$ with the transformation group G is maximal.

To conclude this discussion of $\mathbb{H}^2 \widetilde{\times} \mathbb{E}^1$, we should note that the action of the orientation-preserving isometry group of $\mathbb{H}^2$ on $T^1\mathbb{H}^2$ is transitive and free, so that the choice of a base point identifies this group to $T^1\mathbb{H}^2$. We saw in Section 1.1 that every orientation-preserving isometry of $\mathbb{H}^2$ is a linear fractional map of the form $z \mapsto (az + b) / (cz + d)$ with a, b, c, $d \in \mathbb{R}$, which we can normalize so that $ad - bc = 1$. Associating to such a linear fractional the matrix with coefficients a, b, c, d defines a group isomorphism between the orientation-preserving isometry group of $\mathbb{H}^2$ and the matrix group $\mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R}) / \{\pm \mathrm{Id}\}$. For this reason, $\mathbb{H}^2 \widetilde{\times} \mathbb{E}^1 = \widetilde{T}^1\mathbb{H}^2$ is often denoted by $\widetilde{\mathrm{PSL}_2}(\mathbb{R})$.

There is a similar twisted product $\mathbb{E}^2 \widetilde{\times} \mathbb{E}^1$. An explicit model for $\mathbb{E}^2 \widetilde{\times} \mathbb{E}^1$ can be constructed as follows. Let $\mathbb{E}^2 \widetilde{\times} \mathbb{E}^1$ be $\mathbb{R}^3$ with the Riemannian metric

$$ds^2 = dx^2 + dy^2 + \left(dz - \tfrac{1}{2} y\, dx + \tfrac{1}{2} x\, dy\right)^2$$

There is a Riemannian submersion $p : \mathbb{E}^2 \widetilde{\times} \mathbb{E}^1 \to \mathbb{E}^2$ defined by $p(x, y, z) = (x, y)$. Any isometry $\varphi : \mathbb{E}^2 \to \mathbb{E}^2$ lifts to an isometry Φ of $\mathbb{E}^2 \widetilde{\times} \mathbb{E}^1$ defined by the formula

$$\Phi(x, y, z) = \left(\varphi(x, y), \varepsilon z + \tfrac{1}{2} bx - \tfrac{1}{2} ay\right)$$

where $\varepsilon = +1$ or -1 according to whether φ preserves or reverses the orientation of $\mathbb{E}^2$ and where $(a, b) = \varphi(0, 0)$. Also, every vertical translation of $\mathbb{E}^2 \widetilde{\times} \mathbb{E}^1 = \mathbb{R}^3$ is an isometry. It follows that the Riemannian manifold $\mathbb{E}^2 \widetilde{\times} \mathbb{E}^1$ is homogeneous.

The Riemannian submersion $p : \mathbb{E}^2 \widetilde{\times} \mathbb{E}^1 \to \mathbb{E}^2$ is 'twisted' in a way which is very similar to the one we observed for $\mathbb{H}^2 \widetilde{\times} \mathbb{E}^1$: If α is a curve going from x to x in $\mathbb{E}^2$ and if we lift α to a curve $\widetilde{\alpha}$ in $\mathbb{E}^2 \widetilde{\times} \mathbb{E}^1$ that is everywhere orthogonal to the fibers, the end point of $\widetilde{\alpha}$ is at oriented distance $-A$ from its starting point in the fiber $p^{-1}(x)$, where A is the signed area enclosed by α in $\mathbb{E}^2$; this immediately follows from the expression of the area A as the line integral of $\frac{1}{2} x\,dy - \frac{1}{2} y\,dx$ along α.

The manifold $X = \mathbb{E}^2 \widetilde{\times} \mathbb{E}^1$, endowed with the action of the group G generated by all vertical translations and by the lifts of isometries of $\mathbb{E}^2$ described above, defines a geometry (X, G). The fact that this geometry is maximal is proved by the same methods as for $\mathbb{H}^2 \widetilde{\times} \mathbb{E}^1$.

The isometry group G of $X = \mathbb{E}^2 \widetilde{\times} \mathbb{E}^1$ is easily seen to be nilpotent. It is the only nilpotent group among the groups associated to maximal 3–dimensional geometries. For this reason, the geometry of $X = \mathbb{E}^2 \widetilde{\times} \mathbb{E}^1$ is often called the Nil geometry.

We could also expect a similarly twisted geometry $\mathbb{S}^2 \widetilde{\times} \mathbb{E}^1$. However, a computation shows that the sectional curvature of such a homogeneous manifold would have to be everywhere positive, and the model space would consequently have to be compact. There is a twisted product $\mathbb{S}^2 \widetilde{\times} \mathbb{S}^1$ corresponding to the universal cover $\widetilde{T}^1 \mathbb{S}^2$ of the unit tangent bundle of $\mathbb{S}^2$, as in the case of $\widetilde{T}^1 \mathbb{H}^2 = \mathbb{H}^2 \widetilde{\times} \mathbb{E}^1$. Note that each fiber of $\widetilde{T}^1 \mathbb{S}^2 = \mathbb{S}^2 \widetilde{\times} \mathbb{S}^1$ double covers a fiber of $T^1 \mathbb{S}^2$. In this geometry, if we lift a closed curve α in $\mathbb{S}^2$ to a curve $\widetilde{\alpha}$ in $\mathbb{S}^2 \widetilde{\times} \mathbb{S}^1$ which is everywhere orthogonal to the $\mathbb{S}^1$ factor, the end points of $\widetilde{\alpha}$ are $+A$ apart in the fiber above the starting point of α, where A is the area enclosed by α in $\mathbb{S}^2$; note that A is defined modulo the area of $\mathbb{S}^2$, namely 4π, which is exactly the length of the fiber above the initial point of α. However, this geometry is not maximal. Indeed, it is well known that there is a diffeomorphism between $\widetilde{T}^1 \mathbb{S}^2$ and $\mathbb{S}^3$ which sends the projection $\widetilde{T}^1 \mathbb{S}^2 \to \mathbb{S}^2$ to the Hopf fibration. In addition, an immediate computation shows that the standard identification $\mathbb{S}^3 \cong \widetilde{T}^1 \mathbb{S}^2$ sends the metric of $\mathbb{S}^3$ obtained by rescaling the metric of $\widetilde{T}^1 \mathbb{S}^2$ by a factor of $\frac{1}{2}$ in the direction of the fibers (so that the fibers now have length 2π). As a consequence, the geometry of $\widetilde{T}^1 \mathbb{S}^2 = \mathbb{S}^2 \widetilde{\times} \mathbb{S}^1$ is actually contained in the geometry of $\mathbb{S}^3$, and is not maximal.

We will later see that the four geometries considered in this section mostly occur for manifolds which admit fibrations of a certain type, called Seifert fibrations. For this reason, these are often called Seifert-type geometries.

2.3. The Sol *geometry*

Finally, there is a maximal geometry (X, G) where all stabilizers G_x are finite.

Topologically, X is just $\mathbb{R}^3$, but is endowed with the Riemannian metric m_0 which at (x, y, z) is $ds^2 = \mathrm{e}^{2z} dx^2 + \mathrm{e}^{-2z} dy^2 + dz^2$. (There is nothing special about the number e; it can easily be replaced by any number greater than 1 by rescaling in the z–direction). This metric is respected by the group G of transformations of X

of the form

$$(x, y, z) \mapsto \big(\varepsilon \mathrm{e}^{-c} x + a, \varepsilon' \mathrm{e}^{c} y + b, z + c\big) \text{ or } \big(\varepsilon \mathrm{e}^{-c} y + a, \varepsilon' \mathrm{e}^{c} x + b, -z + c\big)$$

where a, b, $c \in \mathbb{R}$ and ε, $\varepsilon' = \pm 1$. Namely, G is generated by all horizontal translations, the reflections across the xz– and yz–coordinate planes, the vertical shifts $(x, y, z) \mapsto (e^{-c}x, e^{c}y, z + c)$, $c \in \mathbb{R}$, and the flip $(x, y, z) \mapsto (y, x, -z)$. Note that the stabilizer of the origin consists of the eight maps $(x, y, z) \mapsto (\pm x, \pm y, z)$ and $(x, y, z) \mapsto (\pm y, \pm x, -z)$.

Let us show that the geometry (X, G) is maximal. As usual, it suffices to show that, for any G–invariant metric m, the isometry group of m is not larger than G. Looking at stabilizers, we immediately see that such a G–invariant metric m must be obtained by rescaling the original metric m_0 in the horizontal and vertical directions, namely that there exist constants λ, $\mu > 0$ such that the metric m corresponds to $ds^2 = \lambda \mathrm{e}^{2z} dx^2 + \lambda \mathrm{e}^{-2z} dy^2 + \mu dz^2$. Then, the sectional curvature of m is equal to $+1/\mu$ along any horizontal (for the identification $X = \mathbb{R}^3$) tangent plane, is equal to $-1/\mu$ along any vertical tangent plane, and is strictly between these two values along a plane which is neither horizontal nor vertical. It follows that any m–isometry φ must respect up to sign the vector field U which is vertical pointing upwards in $X = \mathbb{R}^3$ and has norm 1 for m. If we consider the covariant differentiation $v \mapsto \boldsymbol{\nabla}_v U$ as an automorphism of each tangent space $T_{(x,y,z)}X = \mathbb{R}^3$, a straightforward computation shows that the direction of the x–axis is the eigenspace of a positive eigenvalue, and that the direction of the y–axis is the eigenspace of a negative eigenvalue. Therefore, the differential of an m–isometry φ must respect the three coordinate axes if φ sends U to U, and exchange the x– and y–axes if φ sends U to $-U$. It easily follows that there is an element φ' of G which has the same value and the same differential as φ at an arbitrary point $(x_0, y_0, x_0) \in X$, from which we conclude that $\varphi = \varphi' \in G$. This concludes the proof that the geometry (X, G) is maximal.

The group G is easily seen to be solvable, and is the only one with this property among the groups corresponding to the geometries we have encountered so far. For this reason, the geometry (X, G) is often called the Sol geometry.

At this point, we have described eight maximal 3–dimensional geometries. We will later see that, for each of these geometries, there is at least one manifold which admits a finite volume complete geometric structure corresponding to this geometry. Thurston showed that the list is complete, and that there is no other maximal 3–dimensional geometry with this property. We cannot give the details of the proof of this fact here, and refer to Scott [125, §5] for a discussion of this proof. However, it is probably worth mentioning that certain maximal geometries, such as the geometry of $X = \mathbb{R}^3$ endowed with the isometry group G of the metric $ds^2 = \mathrm{e}^{2\lambda z} dx^2 + \mathrm{e}^{-2\mu z} dy^2 + dz^2$ with $\lambda \neq \mu$ are excluded because no manifold admits a *finite volume* (G, X)–structure of this type.

2.4. Topological obstructions to the existence of complete geometric structures

We saw that every surface admits a complete geometric structure. In dimension 3, there are topological obstructions to the existence of a complete geometric structure on a given 3–manifold.

A simple observation restricts the geometric structures with which a non-orientable manifold M can be endowed. We saw that the geometries of $\mathbb{E}^2 \widetilde{\times} \mathbb{E}^1$ and $\mathbb{H}^2 \widetilde{\times} \mathbb{E}^1$ are *chiral*, in the sense that they admit no orientation-reversing isometries. In particular, in the atlas defining on M a geometric structure modelled over one of these two spaces, the changes of charts are orientation-preserving. It follows that M is orientable.

The same holds for a *complete* geometric structure modelled over $\mathbb{S}^3$. Indeed, if M is endowed with a complete geometric structure modelled over $\mathbb{S}^3$, any isometry between an open subset of the universal covering $\widetilde{M}$ of M and an open subset of $\mathbb{S}^3$ uniquely extends to an isometry between $\widetilde{M}$ and $\mathbb{S}^3$, by the result of Singer [130] which we already mentioned. As a consequence, M is isometric to a quotient $\mathbb{S}^3/\Gamma$, where $\Gamma \cong \pi_1(M)$ is a finite group acting freely and isometrically on $\mathbb{S}^3$. The Lefschetz Fixed Point Theorem, or inspection, shows that every orientation-reversing isometry of $\mathbb{S}^3$ must have fixed points. Therefore, Γ must respect the orientation of $\mathbb{S}^3$. This proves:

Proposition 2.1. *If the* 3*–manifold* M *admits a geometric structure modelled over* $\mathbb{E}^2 \widetilde{\times} \mathbb{E}^1$ *or* $\mathbb{H}^2 \widetilde{\times} \mathbb{E}^1$*, or a complete geometric structure modelled over* $\mathbb{S}^3$*, then* M *is orientable.*

The second restriction has to do with the fact that, for seven out of the eight 3–dimensional geometries, the model space X is homeomorphic to $\mathbb{R}^3$ or $\mathbb{S}^3$ and, as such, contain no essential 2–sphere. A 2–sphere S embedded in the 3–manifold M is *essential* if the closure of no component of $M - S$ is diffeomorphic to the 3–ball. By a theorem of Alexander [4], $\mathbb{R}^3$ and $\mathbb{S}^3$ contain no essential 2–sphere. Among the eight 3–dimensional geometries, the only model space that contain essential 2–spheres is therefore $\mathbb{S}^2 \times \mathbb{E}^1$.

An *essential projective plane* in the 3–manifold M is a surface P embedded in M which is diffeomorphic to the real projective plane $\mathbb{RP}^2$ and which is 2–sided, namely such that the normal bundle of P in M is trivial. Note that, if M contains a 1–sided projective plane, either M contains an essential 2–sphere, namely the boundary of a tubular neighborhood of P, or else M is diffeomorphic to the real projective 3–space $\mathbb{RP}^3$.

An easy covering theory argument shows that, if we have a covering $\widetilde{M} \to M$ of 3–manifolds and if $\widetilde{M}$ contains no essential 2–sphere or projective plane, then M also contains no essential 2–sphere or projective plane. The converse is actually true by a deeper result, the Equivariant Sphere Theorem of W. Meeks, L. Simon and S.T. Yau [88, 89, 87]. But we only need the result in the forward direction, which immediately shows:

Theorem 2.2. *If the* 3*–manifold* M *admits a complete geometric structure modelled over* $\mathbb{E}^3$, $\mathbb{S}^3$, $\mathbb{H}^3$, $\mathbb{H}^2 \times \mathbb{E}^1$, $\mathbb{E}^2 \widetilde{\times} \mathbb{E}^1$, $\mathbb{H}^2 \widetilde{\times} \mathbb{E}^1$ *or* Sol *(namely one of the eight* 3*–dimensional geometries except* $\mathbb{S}^2 \times \mathbb{E}^1$*), then* M *contains no essential* 2*–sphere or projective plane.*

More stringent restrictions occur for the Seifert-type geometries. Namely, the existence of such a geometry on a compact 3–manifold usually leads to a Seifert fibration on this manifold. But we first need to define Seifert fibrations. Seifert fibrations were introduced by H. Seifert[3] [127]; in addition to [127], useful references on Seifert fibrations include the books by Orlik [103] and Montesinos [96], as well as Scott's survey [125].

A *Seifert fibration* of the 3–manifold M is a decomposition of M into disjoint simple closed curves, called the *fibers* of the fibration, such that the following property holds: Every fiber has a neighborhood U which is diffeomorphic to the quotient of a solid torus $\mathbb{S}^1 \times \mathbb{B}^2$ by the free action of a finite group action respecting the product structure, in such a way that the fibers of the fibration correspond to the images of the circles $\{x\} \times \mathbb{B}^2$. Here, $\mathbb{B}^n$ denotes the closed unit ball in $\mathbb{R}^n$.

By inspection, the above fibered neighborhood U has to be of one of the following two types:

If U is orientable, then there is a diffeomorphism between U and $\mathbb{S}^1 \times \mathbb{B}^2$ for which, if we identify $\mathbb{S}^1$ and $\mathbb{B}^2$ to the unit circle and unit disk in $\mathbb{C}$, the fibers of U all have a parametrization of the form $z \mapsto (z^p, z_0 z^q)$, where z ranges over $\mathbb{S}^1$, $z_0 \in \mathbb{B}^2$ depends on the fiber, and the coprime integers $p > 0$ and q depend only on U and on its parametrization. Namely, the fibers wrap p times in the $\mathbb{S}^1$–direction and q times in the $\mathbb{B}^2$–direction, except the central fiber corresponding to $z_0 = 0$, which wraps only once in the $\mathbb{S}^1$–direction.

If U is non-orientable, it admits a diffeomorphism with $[0,1] \times \mathbb{B}^2/\sim$, where $\sim$ identifies $\{1\} \times \mathbb{B}^2$ to $\{0\} \times \mathbb{B}^2$ by complex conjugation, and where the fibers correspond to the sets $[0,1] \times \{z_0, \bar{z}_0\}$. Note that most fibers are orientation-preserving, with the exception of those corresponding to $z_0 \in \mathbb{R}$.

A fiber is *generic* if it admits an orientable fibered neighborhood U as above with $p = 1$ and $q = 0$, namely if the fibration is a locally trivial bundle near this fiber; otherwise, the fiber is *exceptional*. Orientation-preserving exceptional fibers are isolated. Orientation-reversing exceptional fibers form a 2–dimensional submanifold of M whose components are open annuli, tori and Klein bottles (since they are locally trivial circle bundles).

Now, consider the space Σ of fibers of a Seifert fibration of the 3–manifold M. Near a fiber f which is orientation-preserving in M, a fibered solid torus neighborhood U of f in M provides a neighborhood of the point $f \in \Sigma$ in Σ which is homeomorphic to a quotient $\mathbb{B}^2/\mathbb{Z}_p$, where the cyclic group $\mathbb{Z}_p$ (possibly with $p = 1$) acts on $\mathbb{B}^2$ by rotation; note that this quotient $\mathbb{B}^2/\mathbb{Z}_p$ is homeomorphic to a disk centered at f. Similarly, the point of Σ corresponding to a fiber f which is orientation-reversing in M has a neighborhood homeomorphic to $\mathbb{B}^2/\mathbb{Z}_2$, where $\mathbb{Z}_2$

[3] The fibrations considered by Seifert had only orientation-preserving fibers, but the generalization given below is not intrinsically different.

acts on $\mathbb{B}^2$ by complex conjugation; note that $\mathbb{B}^2/\mathbb{Z}_2$ is in this case homeomorphic to a half-disk with f on its boundary. As a consequence, Σ is a topological surface with boundary, where the boundary points correspond to orientation-reversing fibers.

When we consider Σ only as a topological surface, we unfortunately loose a lot of information regarding the differentiable structure of M. For instance, we can endow the surface Σ with a differentiable structure for which the natural projection $p : M \to \Sigma$ is differentiable but, if the fibration has at least one exceptional fiber, there is no differentiable structure on Σ for which $M \to \Sigma$ is a submersion, in contrast to what is usually expected of a fibration. For this reason, it is much better to consider the natural orbifold structure of Σ. This leads us to digress into a brief discussion of orbifolds.

Orbifolds were first introduced in the 1950s by I. Satake [120, 121] under the name of V–manifolds, and later rediscovered and popularized by Thurston [138] under the name of orbifolds. In addition to these references, some basic facts about orbifolds can also be found in Montesinos [96] or Bonahon-Siebenmann [19].

Roughly speaking, an orbifold is a topological space endowed with an atlas which locally models it over quotients of manifolds by properly discontinuous group actions. More precisely, let a (differentiable) *folding map* be a continuous map $f : \widetilde{U} \to U$ from a differentiable manifold $\widetilde{U}$ to a topological space U such that the *folding group* G_f, defined as the group of diffeomorphisms g of $\widetilde{U}$ such that $f \circ g = f$, acts properly discontinuously[4] on $\widetilde{U}$ and such that the induced map $\widetilde{U}/G_f \to U$ is a homeomorphism.

A (differentiable) *orbifold* is defined as a metrizable topological space O endowed with an atlas of *folding charts* $f_i : \widetilde{U}_i \to U_i$, $i \in I$, where the U_i form an open covering of O and where the f_i are compatible in the following sense: For every $x_i \in \widetilde{U}_i$ and $x_j \in \widetilde{U}_j$ with $f_i(x_i) = f_j(x_j)$, there is a diffeomorphism ψ_{ij} from an open neighborhood $\widetilde{V}_i$ of x_i in $\widetilde{U}_i$ to an open neighborhood $\widetilde{V}_j$ of x_j in $\widetilde{U}_j$ such that $f_j \circ \psi_{ij} = f_i$ over $\widetilde{V}_i$. More formally, an orbifold is a topological space O with a maximal atlas of compatible folding charts as above. Note that it is always possible to restrict the folding charts so as to obtain an atlas where all folding charts have finite folding group. So, the definition of orbifolds would be unchanged if we restricted attention to folding charts with finite folding groups, which is what many authors do. To alleviate the notation, we will often use the same symbol to represent an orbifold and its underlying topological space; in theory, this is somewhat dangerous since the structure of an orbifold involves much more data than its underlying topological space, but we will try to make sure that the context clearly identifies the interpretation which has to be used. When we really need to emphasis the distinction, we will denote by $|O|$ the topological space underlying the orbifold O.

A typical example of orbifold is provided by the properly discontinuous action of a group Γ over a manifold M. Then, the quotient map $M \to M/\Gamma$ is a folding

[4] Recall that a group G acts *properly discontinuously* on a locally compact space X if every $x \in X$ admits a neighborhood V such that $\{g \in G; V \cap gV \neq \emptyset\}$ is finite.

chart, and defines an orbifold structure on M/Γ. An orbifold obtained in this way is said to be *uniformizable*. Although many (and perhaps most) important orbifolds are uniformizable, there exists orbifolds which are not; we will encounter some non-uniformizable 2–orbifolds in Proposition 2.6. In any case, it is always useful to keep the example of uniformizable orbifolds in mind, since any orbifold is *locally* of this type.

An *orbifold covering map* between two orbifolds is a continuous open map $\varphi : O \to O'$ between their underlying topological spaces such that, if $\{f_i : \widetilde{U}_i \to U_i; i \in I\}$ is the maximal atlas defining the first orbifold, $\{\varphi \circ f_i : \widetilde{U}_i \to \varphi(U_i); i \in I\}$ is an atlas defining the second orbifold. If, in addition, the map $\varphi : O \to O'$ is a homeomorphism, then φ^{-1} is also an orbifold covering map, and this defines an *isomorphism* between the two orbifolds.

If the group Γ acts properly discontinuously on M and if Γ' is a subgroup of Γ, the canonical map $M/\Gamma' \to M/\Gamma$ is an orbifold covering map. By definition, an orbifold covering map is always *locally* of this type.

For every point x of the topological space O underlying an orbifold, every folding chart $f_i : \widetilde{U}_i \to U_i$ of the orbifold atlas with $x \in U_i$ and every $\widetilde{x} \in f_i^{-1}(x)$, it immediately follows from definitions that the action on the tangent space $T_{\widetilde{x}}\widetilde{U}_i$ of the (finite) stabilizer of $\widetilde{x}$ in the folding group G_{f_i} depends only on x, up to conjugation. The corresponding finite linear group G_x, well defined up to linear conjugation, is the *isotropy group* of x. The point x is *regular* if the isotropy group G_x is trivial, and *singular* otherwise. For instance, for the uniformizable orbifold M/Γ arising from a properly discontinuous action of a group Γ on a manifold M, the set of singular points of M/Γ is exactly the image of the union of the fixed point sets of the non-trivial elements of Γ.

By straightforward generalization of the case of manifolds, we can define geometric structures on orbifolds. Namely, an orbifold admits an (X, G)–*structure* if its maximal orbifold atlas contains an atlas $\{f_i : \widetilde{U}_i \to U_i; i \in I\}$ where the $\widetilde{U}_i$ are open subsets of X, the folding groups G_{f_i} consist of restrictions to $\widetilde{U}_i$ of elements of G, and the change of charts ψ_{ij} are also restrictions of elements of G. If X is endowed with a G–invariant Riemannian metric, this metric induces a Riemannian metric on the set of regular points of the orbifold, and therefore a metric space structure on the topological space underlying the orbifold, by defining the distance from x to y as the infimum of the lengths of those paths which go from x to y and which consist only of regular points, with the possible exception of x and y. By definition, the corresponding geometric structure is *complete* if this underlying metric space is complete.

If an orbifold O admits a complete (G, X)–structure, consider a folding chart $f : \widetilde{U} \to U$ of the atlas defining this structure. By definition, $\widetilde{U}$ is an open subset of X and the folding group G_f is a subgroup of G. Then, the argument of Singer [130] immediately extends to give a global folding chart $X \to O$, whose folding group Γ is contained in G. In other words the orbifold O is isomorphic to the orbifold X/Γ, quotient of X by a subgroup Γ of G acting properly discontinuously on X (See also Thurston [138, Chap. 3] or Benedetti-Petronio [10, Sect. B.1]). This proves:

Lemma 2.3. *If an orbifold admits a complete geometric structure modelled over (X, G), then it is isomorphic to the orbifold X/Γ quotient of X by a subgroup Γ of G acting properly discontinuously on X, and this by an isomorphism respecting geometric structures. In particular, the orbifold is uniformizable.*

In the case of the base space Σ of a Seifert fibration of the 3–manifold M, any small 2–dimensional submanifold $\widetilde{U}$ of M which is transverse to the leaves projects to an open subset U of Σ, and the local models for the Seifert fibration show that the restriction of the projection p to $\widetilde{U} \to U$ locally is a folding chart. It is immediate that these folding charts are compatible, and therefore define an orbifold structure on Σ. This 2–dimensional orbifold is the *base orbifold* of the Seifert fibration. Note that the singular points of this orbifold are exactly those corresponding to exceptional fibers of the Seifert fibration; the isotropy group of such a singular point is cyclic, acting by rotations on $\mathbb{R}^2$, when the singular point corresponds to an orientation-preserving exceptional fiber, and is $\mathbb{Z}_2$ acting by reflection when it corresponds to an orientation-reversing exceptional fiber.

Up to orbifold isomorphism, this base 2–orbifold is completely determined by: the topological type of the topological surface Σ with boundary underlying the orbifold; the discrete subset S of Σ consisting of those singular points where the isotropy group is a finite rotation group; the assignment of this isotropy group G_x to each $x \in S$. This easily follows from local considerations near the singular set.

Seifert fibrations were classified by H. Seifert [127] in the 1930s. Namely, given two 3–manifolds M and M' each endowed with a Seifert fibration, he introduced invariants which enabled him to decide whether there exists a diffeomorphism between M and M' sending fibration to fibration. As indicated earlier, Seifert was only considering fibrations where the fibers are orientation-preserving, but his work straightforwardly extends to the case where we allow orientation-reversing fibers. We now discuss Seifert's classification, using a reformulation proposed by Thurston. The details of this classification can be found in Seifert [127], Orlik [103], Scott [125], Montesinos [96], Bonahon-Siebenmann [19].

We first consider the classification of Seifert fibrations of oriented 3–manifolds M.

In this case, there are no orientation-reversing fibers, so that the topological space underlying the base 2–orbifold Σ is a surface without boundary. The first invariant of the Seifert fibration is the orbifold isomorphism type of Σ.

Then, there is an invariant $\beta/\alpha \in \mathbb{Q}/\mathbb{Z}$ associated to each exceptional fiber f as follows: Let $U \cong \mathbb{S}^1 \times \mathbb{B}^2$ be a fibered neighborhood of f where the fibers have a parametrization of the form $z \mapsto (z^p, z_0 z^q)$, $z \in \mathbb{S}^1$, where f is the central fiber corresponding to $z_0 = 0$, and where the identification $U \cong \mathbb{S}^1 \times \mathbb{B}^2$ is consistent with the orientation of M and with the standard orientation of $\mathbb{S}^1$ and $\mathbb{B}^2$. Then $\alpha = p$ and $\beta \in \mathbb{Z}$ is such that $\beta q \equiv 1 \bmod p$. (In particular, the data of $\beta/\alpha \in \mathbb{Q}/\mathbb{Z}$ is equivalent to that of $q/p \in \mathbb{Q}/\mathbb{Z}$, but it turns out to be slightly more convenient). Note that, if we consider f as a point of Σ, the isotropy group of f in the base orbifold is $\mathbb{Z}_\alpha$, acting by rotations.

Finally, when M is compact, there is a globally defined Euler number $e_0 \in \mathbb{Q}$.

This invariant has the property that its $\bmod \mathbb{Z}$ reduction is equal to the sum of the invariants $\beta/\alpha \in \mathbb{Q}/\mathbb{Z}$ associated to all the exceptional fibers of the fibration. When the Seifert fibration is a locally trivial bundle where the fibers can be coherently oriented, this bundle has an Euler class defined in the cohomology group $H^2(\Sigma;\mathbb{Z})$; the orientations of M and of the fibers determine an orientation of Σ, which itself determines an isomorphism $H^2(\Sigma;\mathbb{Z}) \cong \mathbb{Z}$; then, e_0 is the integer corresponding to the Euler class through this isomorphism; note that reversing the orientation of the fibers multiplies the Euler class and the isomorphism $H^2(\Sigma;\mathbb{Z}) \cong \mathbb{Z}$ by -1, so that e_0 is unchanged. Also, this Euler number is well behaved with respect to coverings: If we have a finite covering $M' \to M$ of oriented manifolds and if M is endowed with a Seifert fibration of Euler number e_0, this fibration pulls back to a Seifert fibration of M' whose Euler number is $e_0 n^2/p$, where p is the degree of the covering and where n is the number of components of the preimage in M' of a generic fiber of M. With the fact that $e_0 = 1$ for every Seifert fibration of $\mathbb{S}^3$, these properties can actually be used to explicitly define e_0: Indeed, an easy exercise, based on the choice of suitable orbifold coverings of Σ, shows that, for every Seifert fibration of M, there is a finite covering of M where this Seifert fibration pulls back to a locally trivial bundle or to a Seifert fibration of $\mathbb{S}^3$. A more explicit definition of e_0 can be found in Neumann-Raymond [100], Montesinos [96], or Bonahon-Siebenmann [19].

When M is not compact, e_0 is not defined.

Seifert's classification of Seifert fibrations of oriented 3–manifolds can be rephrased as follows.

Theorem 2.4 (Oriented classification of Seifert fibrations)**.** *Consider two oriented 3–manifolds M and M', each endowed with a Seifert fibration. Then, there is an orientation-preserving diffeomorphism $M \to M'$ sending fiber to fiber if and only if there is an isomorphism between their base orbifolds which sends each singular point to a singular point with the same invariant $\beta/\alpha \in \mathbb{Q}/\mathbb{Z}$ and if, when the manifolds are compact, the two Seifert fibrations have the same Euler number $e_0 \in \mathbb{Q}$. Conversely, if Σ is a 2–orbifold where all isotropy groups are cyclic acting by rotation, if we assign to each singular point of Σ with isotropy group $\mathbb{Z}_\alpha$ an element $\beta/\alpha \in \mathbb{Q}/\mathbb{Z}$ with β coprime to α, and if, when Σ is compact, we pick a rational number $e_0 \in \mathbb{Q}$ whose* $\bmod \mathbb{Z}$ *reduction is equal to the sum of the β/α, there is a Seifert fibration of an oriented 3–manifold M which realizes this data.*

Note that reversing the orientation of the manifold M reverses the sign of the invariants $\beta/\alpha \in \mathbb{Q}/\mathbb{Z}$ associated to their exceptional fibers and, if applicable, of the Euler number $e_0 \in \mathbb{Q}$. Therefore, Theorem 2.4 also provides the *unoriented* classification of Seifert fibrations of orientable 3–manifolds.

The classification of Seifert fibrations of non-orientable manifolds is somewhat harder to state, but it essentially follows the lines of the oriented classification. We can summarize it by saying that such a Seifert fibration is characterized by the following data: the base orbifold Σ; invariants $\beta/\alpha \in \mathbb{Q}/\mathbb{Z}$ associated to the orientation-preserving exceptional fibers; orientability data for the locally trivial bundle obtained by removing the exceptional fibers; when the manifold is compact, a global obstruction in $\mathbb{Z}_2$. However these data tend to be interdependent. We refer to Seifert [127], Scott [125], Montesinos [96], Bonahon-Siebenmann [19] for precise statements.

We can now state the restrictions which a Seifert-type geometry imposes on a 3–manifold. Complete proofs and details can be found in Scott [125].

Theorem 2.5. *If the manifold M admits a complete geometric structure modelled over $\mathbb{S}^2 \times \mathbb{E}^1$, $\mathbb{H}^2 \times \mathbb{E}^1$, $\mathbb{H}^2 \widetilde{\times} \mathbb{E}^1$ or $\mathbb{E}^2 \widetilde{\times} \mathbb{E}^1$, then one of the following occurs:*

(i) *The foliation of M by the $\mathbb{E}^1$ factors is a Seifert fibration. In this case, all generic fibers of the Seifert fibration have the same length l, and the metric of the $\mathbb{S}^2$, $\mathbb{H}^2$ or $\mathbb{E}^2$ factors projects to a complete spherical, hyperbolic or euclidean geometric structure on the base orbifold Σ of this fibration. If M is compact, orientable, and oriented so that the charts of its geometric structure are orientation-preserving, then the Euler number $e_0 \in \mathbb{Q}$ is equal to 0 for the product geometry of $\mathbb{S}^2 \times \mathbb{E}^1$ and $\mathbb{H}^2 \times \mathbb{E}^1$, and is negative for the twisted geometries of $\mathbb{H}^2 \widetilde{\times} \mathbb{E}^1$ and $\mathbb{E}^2 \widetilde{\times} \mathbb{E}^1$. In addition, when e_0 is defined and non-zero, the length l of the generic fibers is equal to $-e_0 \operatorname{area}(\Sigma)$, where the area is that of the geometric structure induced on the base 2–orbifold Σ.*

(ii) *The foliation of M by the $\mathbb{E}^1$ factors is a (locally trivial) $\mathbb{E}^1$–bundle over a surface Σ. In this case, the metric of the $\mathbb{S}^2$, $\mathbb{H}^2$ or $\mathbb{E}^2$ factors projects to a spherical, hyperbolic or euclidean complete geometric structure on Σ.*

(iii) *At most two leaves of the foliation by the $\mathbb{E}^1$ factors are closed subsets of M. In this case, M is diffeomorphic to one of eleven manifolds: the two $\mathbb{S}^2$–bundles over $\mathbb{S}^1$, the connected sum $\mathbb{RP}^3 \# \mathbb{RP}^3$ of two copies of the real projective 3–space $\mathbb{RP}^3$, the product $\mathbb{RP}^2 \times \mathbb{S}^1$, the two $\mathbb{E}^2$–bundles over $\mathbb{S}^1$, the two $\mathbb{E}^1$–bundles over the 2–torus, or the three $\mathbb{E}^1$–bundles over the Klein bottle.*

If, in addition, the geometric structure of M has finite volume, then Case (ii) and the non-compact manifolds of Case (iii) cannot occur. In Case (i), the geometric structure induced on the base 2–orbifold Σ has finite area. In Case (iii), the geometric structure of M is necessarily modelled over $\mathbb{S}^2 \times \mathbb{E}^1$.

The conclusions of Case (i) will be more useful if we can specify which 2–orbifolds admit complete spherical, euclidean, or hyperbolic structures.

For this, it is convenient to introduce the Euler characteristic of a compact orbifold O. The proof that every differentiable manifold admits a triangulation immediately extends to show that every differentiable orbifold admits a triangulation. A *triangulation* of the orbifold O is a decomposition of the topological space underlying O into subsets of O called *orbifold simplices*, such that each point x of this underlying space has a neighborhood U which is a union of orbifold simplices, which is contained in the image of some folding chart $f_i : \widetilde{U}_i \to U_i$ of the orbifold atlas of O, and such that the decomposition of U into orbifold simplices lifts to a triangulation of $f_i^{-1}(U) \subset \widetilde{U}_i$ which is invariant under the action of the folding group G_{f_i}. In addition, we insist that the isotropy group is constant on the interior of each orbifold simplex. Then, the *orbifold Euler characteristic* of the compact orbifold O is the sum

$$\chi_{\mathrm{orb}}(O) = \sum_{\sigma} (-1)^{\dim\ \sigma} \frac{1}{|G_\sigma|} \in \mathbb{Q}$$

where the sum is over all orbifold simplices σ of the triangulation σ, and where $|G_\sigma|$ is the cardinal of the isotropy group of an arbitrary point of the interior of σ. Standard proofs that the Euler characteristic of a manifold is independent of the triangulation automatically extend to orbifolds. Note that the orbifold characteristic $\chi_{\mathrm{orb}}(O)$ is a rational number, and should not be confused with the usual Euler characteristic $\chi(|O|)$ of the topological space $|O|$ underlying the orbifold O.

We similarly define the orbifold Euler characteristic of an orbifold O of *finite type*, namely an orbifold which is isomorphic to the interior of a compact orbifold $\bar{O}$ with boundary (where orbifolds with boundary are defined by replacing manifolds by manifolds with boundary in the definition of folding maps). In this case, $\chi_{\mathrm{orb}}(O) = \chi_{\mathrm{orb}}(\bar{O})$

A fundamental property of this orbifold Euler characteristic is that it is well behaved with respect to orbifold coverings: If there is an orbifold covering $O \to O'$ of degree n, namely such that the pre-image of a regular point of O' consists of n regular points of O, then $\chi_{\mathrm{orb}}(O) = n\chi_{\mathrm{orb}}(O')$. Also, note that the orbifold Euler characteristic $\chi_{\mathrm{orb}}(O)$ coincides with the usual Euler characteristic when the orbifold O is a manifold, namely when all the isotropy groups are trivial.

When Σ is a finite type 2–orbifold of the type occurring as base orbifolds of Seifert fibrations, namely where all isotropy groups are cyclic acting by rotations or $\mathbb{Z}_2$ acting by reflection, it is immediate from definitions that $\chi_{\mathrm{orb}}(\Sigma)$ is the sum of the usual Euler characteristic of its underlying space $|\Sigma|$, of $-\frac{1}{2}$ times the number of non-compact components of the set of reflection points of Σ, and of $\sum_{i=1}^{k}(1/\alpha_i - 1)$ where Σ has exactly k isolated singular points and the isotropy group of the i–th singular point is $\mathbb{Z}_{\alpha_i}$ acting by rotation.

Proposition 2.6 (Geometrization of 2–orbifolds)**.** *Let Σ be a 2–orbifold of the type occurring as base orbifolds of Seifert fibrations, namely where all isotropy groups are cyclic acting by rotations or $\mathbb{Z}_2$ acting by reflection. Then:*

(i) *If Σ is compact, it admits a hyperbolic structure if and only if its orbifold Euler characteristic $\chi_{\mathrm{orb}}(\Sigma)$ is negative.*

(ii) *If Σ is compact, it admits a euclidean structure if and only if $\chi_{\mathrm{orb}}(\Sigma)$ is equal to 0.*

(iii) *If Σ is compact, it admits a spherical structure only if $\chi_{\mathrm{orb}}(\Sigma)$ is positive. Conversely, if $\chi_{\mathrm{orb}}(\Sigma) > 0$, either Σ admits a spherical structure, or Σ has underlying topological space a 2–sphere and exactly one singular point, or Σ has underlying topological space a 2–sphere and has exactly two singular points, of respective isotropy groups $\mathbb{Z}_p$ and $\mathbb{Z}_q$ with $p \neq q$.*

(iv) *If Σ is non compact, it always admits a complete hyperbolic structure.*

(v) *If Σ is non-compact, it admits a complete euclidean structure if and only if it falls into one of the following eight categories: the topological space underlying Σ*

is a plane and Σ has at most one singular point; the topological space underlying Σ is a half-plane, and the set of singular points of Σ forms a line; the topological space underlying Σ is a plane and Σ has exactly two singular points, with isotropy group $\mathbb{Z}_2$ acting by rotations; the topological space underlying Σ is a half-plane, and the singular set of Σ consists of one line of reflection points and of a single point with isotropy group $\mathbb{Z}_2$ acting by rotation; the topological space underlying Σ is an open annulus, with no singular point; the topological space underlying Σ is a semi-open annulus, and the singular set consists of a circle of reflection points; the topological space underlying Σ is a square with two opposite sides removed, and the singular set consists of two lines; the topological space underlying Σ is an open Möbius strip, with no singular point.

(vi) *If M is non-compact, it cannot admit a complete spherical structure.*

Here, a half-plane is the closure of one component of the complement of a line in $\mathbb{E}^2$. A *closed annulus* is a 2–manifold diffeomorphic to $\mathbb{S}^1 \times [0,1]$. If we remove 1 or 2 boundary components from a compact annulus, one gets a *semi-open* or *open* annulus. The terminology is similar for closed and open Möbius strips.

In Case (iii) of Proposition 2.6, we encounter two exceptional types of 2–orbifolds: those where Σ has underlying topological space a 2–sphere and exactly one singular point; and those where Σ has underlying topological space a 2–sphere and has exactly two singular points, of respective isotropy groups $\mathbb{Z}_p$ and $\mathbb{Z}_q$ with $p \neq q$. In either case, an easy covering argument on the complement of the singular set shows that these orbifolds are not uniformizable, namely that they are isomorphic to no orbifold M/Γ, where the group Γ acts properly discontinuously on the 2–manifold M. In particular, by Lemma 2.3, these orbifolds admit no geometric structure.

Proposition 2.6 can be straightforwardly generalized to include all 2–orbifolds, allowing dihedral isotropy groups. The list of exceptions is just a little longer. Our restriction to cyclic isotropy groups is only for the convenience of the exposition, since the only 2–orbifolds which we will encounter in this chapter mostly arise as base orbifolds of Seifert fibrations.

The proof of Proposition 2.6 is fairly elementary. The necessary conditions on the Euler characteristic in Cases (i), (ii) and (iii) follow from an immediate extension of the Gauss-Bonnet formula to 2–orbifolds. The existence part can be proved by methods similar to those used in Section 1.1 to construct geometric structures on surfaces. The analytic method generalizes to orbifolds without any major problems (Hint: First construct a 'universal orbifold covering' $\widetilde{\Sigma} \to \Sigma$ and show that $\widetilde{\Sigma}$ has no singular point unless Σ is one of the exceptional orbifolds of Case (ii); see also [84]), and it is interesting to see why it fails to provide a geometric structure on the non-uniformizable 2–orbifolds. For orbifolds of finite type, the geometric cut-and-paste method is probably easier. In particular, this is the method used in [101] where a complete proof of Proposition 2.6 for finite type 2–orbifolds is given.

After this digression on geometric 2–orbifolds, we now return to 3–manifolds.

A surprising fact is that a geometry modelled over $\mathbb{E}^3$ also leads to a Seifert fibration. Again, a detailed proof can be found in Scott [125].

Theorem 2.7. *If the manifold M admits a complete geometric structure modelled over the euclidean space $\mathbb{E}^3$, then the maximal atlas defining the geometric structure contains an atlas modelling M over $\mathbb{E}^2 \times \mathbb{E}^1 = \mathbb{E}^3$, where all changes of charts respect the splitting of $\mathbb{E}^2 \times \mathbb{E}^1$, and such that at least one of the following occurs:*

(i) *The foliation of M by the $\mathbb{E}^1$ factors defines a Seifert fibration of M; the metric of the $\mathbb{E}^2$ factor induces a euclidean structure on the base orbifold of the Seifert fibration and, if defined, the Euler number e_0 of the fibration is equal to 0.*

(ii) *The foliation of M by the $\mathbb{E}^2$ factors endows M with the structure of a locally trivial bundle over the circle $\mathbb{S}^1$ with fiber the plane $\mathbb{E}^2$; topologically, there are exactly two such bundles.*

(iii) *M is the euclidean space $\mathbb{E}^3$.*

Similarly, a geometric structure modelled over $\mathbb{S}^3$ leads to the existence of a Seifert fibration. As usual, we refer to [125] for a proof.

Theorem 2.8. *If the manifold M admits a complete geometric structure modelled over the sphere $\mathbb{S}^3$, then the maximal atlas defining this geometric structure contains an oriented atlas modelling M over $\mathbb{S}^2 \widetilde{\times} \mathbb{S}^1 = \mathbb{S}^3$, where all changes of charts respect the splitting of $\mathbb{S}^2 \widetilde{\times} \mathbb{S}^1$. In addition, the foliation of M by the $\mathbb{S}^1$ factors defines a Seifert fibration; the metric of $\mathbb{S}^2$ defines a spherical structure on the base of this fibration and the Euler number e_0 of the fibration is strictly positive. In addition, the fundamental group of M is finite.*

We should probably emphasize the coincidental nature of Theorems 2.7 and 2.8. For instance, similar statements for 3–orbifolds are false. There are compact 3–orbifolds which admit euclidean or spherical structures but which admit no fibration; see Section 3.6.

We now turn to hyperbolic structures.

Let $\mathbb{T}^2$ denote the 2–torus $\mathbb{S}^1 \times \mathbb{S}^1$. A *singular torus* in the manifold M is a continuous map $\mathbb{T}^2 \to M$; it is *incompressible* if the induced homomorphism $\pi_1\left(\mathbb{T}^2\right) \to \pi_1(M)$ is injective. Let an *end* of the 3–manifold M be the image of a proper embedding $S \times [0, \infty[\to M$ where S is a compact surface without boundary. Recall that a map is *proper* when the pre-image of every compact set is compact. The reader should beware that, what we call here an end for the sake of simplicity, is usually called a tame end neighborhood of M. A singular torus $\mathbb{T}^2 \to M$ is *essential* if it is incompressible and if it cannot be homotoped to a singular torus with image in an end of M.

Note that, if the singular torus $\mathbb{T}^2 \to M$ is incompressible and can be homotoped into an end $U \cong S \times [0, \infty[$, then the fundamental group of U must contain a subgroup isomorphic to $\mathbb{Z}^2$, and S therefore is a 2–torus or a Klein bottle. Therefore, only the *toric* ends $U \cong S \times [0, \infty[$, where S is a torus or a Klein bottle, are relevant here.

Theorem 2.9. *If the 3–manifold M admit a complete geometric structure modelled over the hyperbolic space $\mathbb{H}^3$, then M contains no essential 2–sphere, projective plane or singular torus. If, in addition, the hyperbolic structure of M has finite* *volume, then M is the union of a compact manifold with boundary and of finitely many toric ends.*

It should be observed that these topological restrictions to the existence of a hyperbolic structure are the weakest ones among those encountered in this section. This is consistent with what we already observed in Section 1.1 for the dimension 2, where all but finitely surfaces admitted a complete hyperbolic structure.

In most cases, if M admits a Seifert fibration with base orbifold Σ, then M contains an essential singular 2–torus. For instance, suppose that Σ contains a non-separating simple closed curve α which avoids the singular set, or a separating simple closed curve α avoiding the singular set such that each component of $\Sigma - \alpha$ topologically is neither a disk with 0 or 1 singular point, nor an open annulus with no singular point. Then, the fibers over α form an embedded 2–torus or Klein bottle in M, and it can be shown that the orientation cover of this surface gives an essential singular 2–torus in M. When this type of technique does not provide an essential singular 2–torus, other arguments show that the fundamental group of M is finite, or that M contains an essential 2–sphere or projective plane, all properties preventing it from admitting a complete hyperbolic structure. This yields the following property, whose proof can be found in [125, Chap. 5].

Theorem 2.10. *Let the 3–manifold M admit a Seifert fibration with base 2–orbifold Σ. Then, it admits no complete hyperbolic structure, unless Σ is non-compact and admits a euclidean structure (compare Case (v) of Proposition 2.6. In addition, M admits no finite volume complete hyperbolic structure.*

There consequently is almost no overlap between those 3–manifolds which admit a complete Seifert-type geometric structure and those which admit a complete hyperbolic geometric structure.

Finally, we now discuss the Sol geometry. We first exhibit some 3–manifolds which admit such a geometry.

Let A be an element of $\mathrm{GL}_2(\mathbb{Z})$, namely a 2×2 matrix with integer entries and determinant ± 1. The matrix A defines a linear automorphism of $\mathbb{R}^2$ which respects the lattice $\mathbb{Z}^2$, and therefore induces a linear diffeomorphism φ of the 2–torus $\mathbb{T}^2 \cong \mathbb{R}^2/\mathbb{Z}^2$. Assume in addition that the eigenvalues of A are real and distinct; this is always the case when the determinant of A is -1 and, when the determinant is $+1$, occurs exactly when the trace of A has absolute value greater than 2. Since their product is equal to ± 1, we can write these eigenvalues as $\lambda_1 = \pm e^{-t}$, $\lambda_2 = \pm e^{t}$ with $t > 0$. By definition, this property of eigenvalues means that the linear diffeomorphism φ induced by A is an *Anosov linear diffeomorphism* of $\mathbb{T}^2$.

Choose a linear isomorphism $L : \mathbb{R}^2 \to \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3 = \mathrm{Sol}$ which sends the λ_1–eigenspace to the x–axis and the λ_2 to the y–axis. Consider the group Γ_0 of isometries of Sol which consists of all horizontal translations by elements of $L\left(\mathbb{Z}^2\right) \subset \mathbb{R}^2 \times \{0\}$, and let Γ be generated by Γ_0 and by the isometry $T : (x, y, z) \mapsto (\lambda_1 x, \lambda_2 y, z + t) = (\pm e^{-t} x, \pm e^{t} y, z + t)$. It is fairly immediate that Γ acts freely and properly discontinuously on Sol, and we can consider the quotient manifold

$M = \mathrm{Sol}/\Gamma$, with the geometric structure induced by the geometric structure of Sol. Using the map L, we see that M is diffeomorphic to the *mapping torus* of the linear diffeomorphism φ, defines as the identification space $\mathbb{T}^2 \times [0,1] / \sim$, where $\sim$ identified each point $(x, 0)$ to $(\varphi(x), 1)$.

In this way, we can put a geometric structure modelled over Sol on the mapping tori of all Anosov linear diffeomorphisms of the 2–torus.

This construction actually provides all non-trivial examples of 3–manifolds with a complete geometric structure modelled over Sol, as shown by the following result.

Theorem 2.11. *If the* 3*–manifold* M *admits a complete geometric structure modelled over* Sol*, then one of the following occurs:*

(i) *The foliation of* $\mathrm{Sol} = \mathbb{R}^3$ *by horizontal planes induces a foliation of* M *whose leaves are* 2*–tori; this foliation actually defines on* M *the structure of a locally trivial bundle over* $\mathbb{S}^1$ *with fiber the* 2*–torus* $\mathbb{T}^2$*, and* M *is diffeomorphic to the mapping torus of an Anosov linear diffeomorphism of* $\mathbb{T}^2$*;*

(ii) *There is a geometry-preserving diffeomorphism between* M *and the quotient of* Sol *by a group of isometries which respects a horizontal plane; in particular,* M *is a line bundle over a* 2*–torus, a Klein bottle, an open annulus, an open Möbius strip or a plane.*

As usual, we refer to [125] for a proof of Theorem 2.11. Note that the list of 3–manifolds occurring in this context is extremely restricted.

2.5. Geometric structures with totally geodesic boundary

We now turn to geometric structures on manifolds with boundary. If we want to obtain any uniqueness properties for such geometric structures, we clearly have to impose some type of rigidity condition on the boundary. A natural condition is to require the boundary to be totally geodesic. Recall that a submanifold N of a Riemannian manifold M is *totally geodesic* if, locally, any small geodesic arc of M that joins two points of N is completely contained in N.

Let M be a manifold with boundary ∂M. Thicken M by gluing along ∂M a small collar $\partial M \times [0, 1[$, to obtain a manifold M^+ without boundary. By definition, a *geometric structure with totally geodesic boundary* on M is the restriction to M of a locally homogeneous Riemannian metric on M^+ for which the boundary ∂M is totally geodesic. Such a geometric structure is *complete* if the metric space structure induced on M is complete.

Among the eight 3–dimensional geometries, the two twisted geometries $\mathbb{H}^2 \widetilde{\times} \mathbb{E}^1$ and $\mathbb{E}^2 \widetilde{\times} \mathbb{E}^1$ locally contain no totally geodesic surface; this can easily be checked from the explicit expression of their metric given in Section 2.2.

All other six 3–dimensional geometries contain totally geodesic surfaces. For the isotropic geometries $\mathbb{S}^3$, $\mathbb{E}$, $\mathbb{H}^3$, there is such a totally geodesic surface passing through each point x and tangent to any plane in the tangent space at x. In the product geometries $\mathbb{H}^2 \times \mathbb{E}^1$, $\mathbb{S}^2 \times \mathbb{E}^1$, all totally geodesic surfaces locally are of the form $g \times \mathbb{E}^1$, $\mathbb{H}^2 \times \{x\}$ or $\mathbb{S}^2 \times \{x\}$, where g is a geodesic of $\mathbb{H}^2$ or $\mathbb{S}^2$ and

$x \in \mathbb{E}^1$. Finally, every totally geodesic surface of Sol is contained in a plane parallel to the xz– or yz–plane, in the model described in Section 2.3. In particular, these geometric models admit at most two complete totally geodesic surfaces, up to isometry. Therefore, for each geometry, there is at most two possible local models for a boundary point.

Note that, in each of the six model spaces $\mathbb{S}^3$, $\mathbb{E}^3$, $\mathbb{H}^3$, $\mathbb{H}^2 \times \mathbb{E}^1$, $\mathbb{S}^2 \times \mathbb{E}^1$, Sol admitting totally geodesic surfaces, there is an isometric involution of the model space which acts as a reflection across this surface. Conversely, it is easy to see that the fixed point set of an isometric involution is always totally geodesic. This observation provides a convenient way to analyze geometric structures with totally geodesic boundary, as follows.

Given a manifold M with boundary ∂M, let its *double* be the manifold DM without boundary obtained from the disjoint union of two copies M_1 and M_2 of M by gluing together the boundaries ∂M_1 and ∂M_2 through the natural identification maps $\partial M_1 \cong \partial M \cong \partial M_2$. This double DM comes equipped with a natural involution τ which exchanges the images of M_1 and M_2 and which fixes the image of the boundary ∂M. The differentiable structure of M gives a natural differentiable structure on DM for which the involution τ is differentiable. It then immediately follows from the above observations that the data of a geometric structure with totally geodesic boundary on M is equivalent to the data of a τ–invariant geometric structure on the double DM.

Now, the restrictions to the existence of a complete geometric structure on DM given in Section 2.4 easily translate to restrictions to the existence of complete geometric structures with totally geodesic boundary on M.

Let a *compression disk* for the boundary ∂M in M be a 2–dimensional submanifold D (with $D \cap \partial M = \partial D$) of M such that D is diffeomorphic to the disk (= 2–ball) $\mathbb{B}^2$. A compression disk is *essential* if its boundary ∂D does not bound a disk in the boundary ∂M. Similarly, a *singular compression disk* is a continuous map $\mathbb{B}^2 \to M$ sending $\mathbb{S}^1 = \partial \mathbb{B}^2$ to ∂M. Such a singular compression disk is *essential* if its restriction $\mathbb{S}^1 \to \partial M$ is not homotopic to 0 in ∂M. The celebrated Dehn's Lemma and Loop Theorem of Papakyriakopoulos [108] (see also Hempel [50]) assert that the existence of an essential singular compression disk is equivalent to that of an essential compression disk.

Note that a compression disk for ∂M gives a sphere in the double DM. An easy homology calculation shows that this sphere is essential if the compression disk is essential. With Theorem 2.2, this gives:

Theorem 2.12. *If the 3–manifold M admits a complete geometric structure with totally geodesic boundary modelled over $\mathbb{E}^3$, $\mathbb{S}^3$, $\mathbb{H}^3$, $\mathbb{H}^2 \times \mathbb{E}^1$ or* Sol*, then M contains no essential 2–sphere, projective plane or compression disk.*

When M admits a Seifert-type geometry with totally geodesic boundary, the fibrations of the double DM provided by Theorem 2.5 must be invariant under the double involution τ. When DM inherits a Seifert fibration from the geometric structure, this Seifert fibration can project to either a Seifert fibration of M or to a locally trivial bundle structure on M whose fibers are compact intervals.

In the first case, a Seifert fibration of a manifold with boundary is defined as in the empty boundary case: The boundary is a union of fibers, and every boundary fiber admits a neighborhood U diffeomorphic to the quotient of $\mathbb{S}^1 \times \mathbb{B}^2_+$ by a free action of a finite group respecting the product structure, in such a way that the fibers of the fibration correspond to the image of the circles $\mathbb{S}^1 \times \{z\}$ and that $U \cap \partial M$ is the image of $\mathbb{S}^1 \times \left(\mathbb{B}^2_+ \cap i\mathbb{R}\right)$. Here, $\mathbb{B}^2_+$ is the half-disk $\{z \in \mathbb{C}; |z| \leqslant 1, \mathrm{Re}\, z \geqslant 0\}$ and $i\mathbb{R}$ is the imaginary axis. By inspection, such a neighborhood U of a boundary point must be diffeomorphic, either to $\mathbb{S}^1 \times \mathbb{B}^2_+$ with the product fibration, or to $[0,1] \times \mathbb{B}^2_+ / \sim$ where $\sim$ identifies $\{0\} \times \mathbb{B}^2_+$ to $\{1\} \times \mathbb{B}^2_+$ by complex conjugation and where the fibers correspond to the sets $[0,1] \times \{z_0, \bar{z}_0\}$.

The space of leaves of a Seifert fibration is a 2–dimensional orbifold with boundary, where an orbifold with boundary is defined along the lines of the definition in the empty boundary case, allowing folding charts to originate from manifolds with boundary. The topological space underlying this base 2–orbifold is a surface with boundary, but its boundary points fall into two categories: Those which really are in the boundary of the orbifold, and correspond to fibers contained in the boundary of the 3–manifold; and those which really are in the interior of the orbifold, with isotropy group $\mathbb{Z}_2$ acting by reflection, and correspond to orientation reversing fibers in the interior of the 3–manifold.

The classification of Seifert fibered 3–manifolds with boundary is essentially the same as the one discussed in Section 2.4, and in particular Theorem 2.4. The Euler number $e_0 \in \mathbb{Q}/\mathbb{Z}$ is undefined when the boundary is non-empty.

Theorem 2.13. *Let the* 3*–manifold* M *with non-empty boundary* ∂M *admit a complete geometric structure with totally geodesic boundary, modelled over* $\mathbb{S}^2 \times \mathbb{E}^1$*,* $\mathbb{H}^2 \times \mathbb{E}^1$ *or* $\mathbb{E}^3 \cong \mathbb{E}^2 \times \mathbb{E}^1$*. Then, possibly after a change of the splitting* $\mathbb{E}^3 \cong \mathbb{E}^2 \times \mathbb{E}^1$*, one of the following occurs:*

(i) *The foliation of the model space by the* $\mathbb{E}^1$ *factors projects to a Seifert fibration on* M*. In this case, the metric of the* $\mathbb{S}^2$*,* $\mathbb{H}^2$ *or* $\mathbb{E}^2$ *factors projects to a complete spherical, hyperbolic or euclidean, respectively, geometric structure with geodesic boundary on the base orbifold of the fibration.*

(ii) *The foliation of the model space by the* $\mathbb{E}^1$ *factors projects to a locally trivial bundle structure on* M*, with base a surface with boundary and with fiber an interval. In this case, the metric of the* $\mathbb{S}^2$*,* $\mathbb{H}^2$ *or* $\mathbb{E}^2$ *factors projects to a complete spherical, hyperbolic or euclidean, respectively, geometric structure with geodesic boundary on the base surface.*

(iii) *At most one leaf of the foliation of* M *by the* $\mathbb{E}^1$ *factors is a closed subset of* M*. In this case,* M *is diffeomorphic to one of the two* $\mathbb{B}^2$*–bundles over* $\mathbb{S}^1$*, or to an interval bundle over the* 2*–torus or the Klein bottle.*

In (ii), the interval fiber of the bundle can be open, compact or semi-open. It is compact precisely when the fibration of the double DM is a Seifert fibration, and the doubling involution τ acts by reflection on each of its fibers. In (iii), the interval fibers can be compact or semi-open.

In the 3–manifold M with boundary ∂M, let a *singular annulus* be a continuous map $\mathbb{A}^2 \to M$ sending the boundary $\partial\mathbb{A}^2$ to ∂M, where $\mathbb{A}^2$ denotes the standard annulus (or cylinder) $\mathbb{A}^2 = \mathbb{S}^1 \times [0,1]$. Let an *end* of M be a subset $U \subset M$ that is properly diffeomorphic to a manifold with corners $S \times [0,\infty[$, where S is a compact surface with boundary and where $U \cap \partial M$ corresponds to $(\partial S) \times [0,\infty[$. A singular annulus $\mathbb{A}^2 \to M$ is *essential* if the induced homomorphisms $\pi_1\left(\mathbb{A}^2\right) \to \pi_1(M)$ and $\pi_1\left(\mathbb{A}^2, \partial\mathbb{A}^2\right) \to \pi_1(M, \partial M)$ are injective and if it cannot be homotoped into an end $U \cong S \times [0,\infty[$ by a homotopy keeping the image of $\partial\mathbb{A}^2$ in ∂M; note that the property involving fundamental groups is independent of the choice of base point.

As in the case of singular tori, easy homotopic considerations show that, if the induced homomorphisms $\pi_1\left(\mathbb{A}^2\right) \to \pi_1(M)$ and $\pi_1\left(\mathbb{A}^2, \partial\mathbb{A}^2\right) \to \pi_1(M, \partial M)$ are injective and if the singular annulus $\mathbb{A}^2 \to M$ can be homotoped into an end $U \cong S \times [0,\infty[$ by a homotopy keeping the image of $\partial\mathbb{A}^2$ in ∂M, then the surface S must be an annulus or a Möbius strip. In particular, only *annular* ends $U \cong S \times [0,\infty[$, namely those for which S is an annulus or a Möbius strip, are relevant here.

Doubling the annulus $\mathbb{A}^2$ along its boundary gives a 2–torus. Therefore, a singular annulus $\mathbb{A}^2 \to M$ defines a singular 2–torus $\mathbb{T}^2 \to DM$ in the double DM of M. An easy cut and paste argument shows that this singular 2–torus is essential in DM if and only if the singular annulus is essential in M.

Applying Theorem 2.9 to the double DM, one easily obtains:

Theorem 2.14. *If the 3–manifold M with boundary admits a complete hyperbolic structure with totally geodesic boundary, then M contains no essential 2–sphere, projective plane, compression disk, singular 2–torus or singular annulus. If, in addition, the hyperbolic structure of M has finite volume, then M is the union of a compact subset and of finitely many annular and toric ends.*

Similarly, an application of Theorem 2.11 gives:

Theorem 2.15. *If the 3–manifold M with non-empty boundary admits a complete geometric structure modelled over* Sol*, then M is an interval bundle over the plane, the 2–torus, the Klein bottle, a compact, semi-open or open annulus, or a compact or open Möbius strip.*

In Theorem 2.15, the interval fiber can be compact, open or semi-open.

3. Characteristic splittings

We saw that a 3–manifold M very seldom admits a geometric structure modelled over $\mathbb{S}^2 \times \mathbb{E}^1$, $\mathbb{H}^2 \times \mathbb{E}^1$, $\mathbb{H}^2 \widetilde{\times} \mathbb{E}^1$, $\mathbb{E}^2 \widetilde{\times} \mathbb{E}^1$, $\mathbb{E}^3$, $\mathbb{S}^3$ or Sol. On the other hand, the only obstructions to the existence of a hyperbolic structure which we encountered consisted of essential 2–spheres, projective planes, singular 2–tori and, in the case with boundary, compression disks and singular annuli. In this section, we will see that a general 3–manifold M splits into pieces where these topological obstructions vanish. Most of these splittings are unique up to isotopy.

Recall that two diffeomorphisms f_0, $f_1 : M \to N$ are *isotopic* if they can be connected by a family of diffeomorphisms $f_t : M \to N$ which depend differentiably on $t \in [0, 1]$. Traditionally, the term *isotopy* refers to either a family of diffeomorphisms $f_t : M \to N$, $t \in [0, 1]$, as above, or a diffeomorphism $f : M \to M$ which is isotopic to the identity. We will mostly use the second convention in this chapter.

3.1. Connected sum decompositions

First, let us focus on essential 2–spheres. Recall that a 2–sphere S embedded in the 3–manifold M is *essential* if it does not bound a 3–ball in M. The analysis of such essential spheres is equivalent to the Kneser-Milnor theory of connected sums of 3–manifolds. In addition to the original articles by Kneser [70] and Milnor [94], the book by Hempel [50] is an excellent reference for this material.

Let the 3–manifold M contain an essential 2–sphere S. If S is non-separating, there is a simple arc k embedded in $M - S$ which goes from one side of S to the other one. (Note that S always has two distinct sides, namely its normal bundle is trivial, because S is simply connected.) Thickening $S \cup k$, we obtain a 3–dimensional submanifold U of M bounded by an embedded 2–sphere S'. If S' is not essential, the closure of the complement of U in M is a 3–ball, and it is not too hard to see that M is diffeomorphic to one of the two $\mathbb{S}^2$–bundles over the circle $\mathbb{S}^1$; the topological type of the bundle so obtained depends on whether the arc k is orientation-preserving or -reversing. Therefore, if M contains an essential sphere, either it contains a separating essential sphere or it is diffeomorphic to one of the two $\mathbb{S}^2$–bundles over $\mathbb{S}^1$. This enables us to focus our attention on separating essential 2–spheres.

Now, let us introduce the additional hypothesis that the 3–manifold M is *of finite type*, namely is diffeomorphic to the interior of a compact manifold $\overline{M}$ with (possibly empty) boundary.

Theorem 3.1 (Unique decomposition along 2–spheres)**.** *Let M be a 3–manifold of finite type. Then, there is a compact 2–dimensional submanifold Σ of M such that:*

(i) *The components of Σ are separating 2–spheres in M.*

(ii) *If Σ is non-empty and if $M_0, M_1, \dots, M_n$ are the closures of the components of $M - \Sigma$, then M_0 is diffeomorphic to a 3–sphere minus n finitely many disjoint open 3–balls; for every $i \geqslant 1$, M_i contains a unique component of Σ and every separating essential sphere in M_i is parallel to this component of Σ.*

(iii) *If M is non-orientable, no M_i is diffeomorphic to $\mathbb{S}^1 \times \mathbb{S}^2$ minus an open 3–ball.*

The family Σ is empty when M is the 3–sphere $\mathbb{S}^3$, and consists of a single sphere (bounding the 3–ball M_0) when M contains no separating essential 2–sphere. If Σ has at least two components, each of its components is an essential 2–sphere.

In addition, the list of the M_i is unique up to diffeomorphism. Namely, if Σ' is another family satisfying the same conditions and we use primes to denote the data associated to Σ', there are as many $M'_{i'}$ as M_i, and we can index these pieces *so that each M_i' is diffeomorphic to the corresponding M_i. A stronger uniqueness property holds when M is orientable: If M is orientable, there exists an orientation-preserving diffeomorphism $\varphi : M \to M$ such that $\varphi(\Sigma) = \Sigma'$ and such that φ coincides with the identity outside of a certain compact subset of M.*

In (ii), two surfaces Σ_1 and Σ_2 in M are *parallel* if they are separated by a component of $M - \Sigma_1 \cup \Sigma_2$ whose closure is diffeomorphic to the product $\Sigma_1 \times [0, 1]$.

A proof of the stronger uniqueness property of Theorem 3.1 for the orientable case, which the author learned from M. Scharlemann, can be found in Bonahon [15, Appendix A]. The rest of the statement is proved in Hempel [50, Chap. 3]. The existence of Σ was originally proved by Kneser [70], and the key ingredient is that there is a number n_0, depending only on the rank of $H_2(M; \mathbb{Z}/2)$ and on the number of simplices in a triangulation of $\overline{M}$, which bounds the number of components of a family Σ_1 of disjoint essential 2–spheres in M which are pairwise non-parallel. The uniqueness of the M_i was proved by Milnor [94].

The decomposition of a 3–manifold into simpler pieces which is provided by Theorem 3.1 has an inverse construction, defined by the operation of connected sum. More precisely, in the situation of Theorem 3.1, consider each piece M_i with $i \geqslant 1$, and let $\widehat{M}_i$ be obtained from M_i by gluing a 3–ball along the 2–sphere $\Sigma \cap M_i$; since every diffeomorphism of the 2–sphere extends to the 3–ball (see for instance [131, 45]), this $\widehat{M}_i$ does not depend on the gluing map, up to diffeomorphism. Note that M_i is not a 3–ball since the 2–sphere ∂M_i is essential in M, and that $\widehat{M}_i$ consequently cannot be diffeomorphic to the 3–sphere (by Alexander's theorem [4]). Also, by the hypothesis that every essential separating 2–sphere in M_i is parallel to $\Sigma \cap M_i$, the manifold $\widehat{M}_i$ contains no separating essential 2–sphere. Therefore, Theorem 3.1 associates to any 3–manifold a finite collection of 3–manifolds $\widehat{M}_i$, $i = 1, \ldots, n$, which contain no essential separating 2–sphere and which are not diffeomorphic to the 3–sphere.

Conversely, if we are given a finite collection of 3–manifolds $\widehat{M}_1, \ldots, \widehat{M}_n$ of finite type, we can reconstruct a manifold M as follows. First remove the interior of a closed 3–ball B_i from each $\widehat{M}_i$, to obtain a manifold M_i bounded by a 2–sphere $\partial M_i = \partial B_i$; in addition, choose an orientation for each ball B_i. Then, remove from the 3–sphere the interiors of n disjoint closed balls $B_1', \ldots, B_n'$ to obtain a manifold M_0 bounded by n 2–spheres. And, finally, glue each M_i to M_0 by identifying each $\partial B_i \subset \partial M_i$ to $\partial B_i' \subset \partial M_0$ via an orientation-reversing diffeomorphism. Since any two orientation-preserving diffeomorphisms of the 2–sphere are isotopic (see for instance [131, 45]), the resulting manifold M depends only on the manifold $\widehat{M}_i$ and on the oriented 3–balls B_i and B_i'. If at least one of the $\widehat{M}_i$ is non-orientable, sliding the balls around easily shows that M actually depends only on the manifolds $\widehat{M}_i$, up to diffeomorphism; by definition, M is the *connected sum* of the manifolds $\widehat{M}_1$, $\ldots$, $\widehat{M}_n$. If all of the $\widehat{M}_i$ are orientable, some more care is required, and we need to choose an orientation for each of the $\widehat{M}_i$; then, if we insist that the balls B_i and B_i' are oriented by restriction of the orientations of $\widehat{M}_i$ and of the 3–sphere $\mathbb{S}^3$, the oriented manifold M does not depend on any other choice up to orientation-preserving diffeomorphism; by definition, the oriented manifold M is the *connected sum* of the oriented manifolds $\widehat{M}_i$. Reversing the orientation of one of the $\widehat{M}_i$ may change the diffeomorphism type of the resulting connected sum; see Hempel [50, Example 3.22].

The above operation of connected sum, combined with Theorem 3.1, provides a unique factorization of each 3–manifold of finite type as the connected sum of finitely many (oriented, if applicable) 3–manifolds $\widehat{M}_i$ which contain no essential separating 2–sphere and which are not diffeomorphic to the 3–sphere.

The classification problem is of course not the only problem topologists can be interested in. For instance, one is often led to analyze the topology of the group of all diffeomorphisms of a given manifold. We refer to Laudenbach [72] or Hendricks-Laudenbach [52] for an analysis of the diffeomorphism group of a 3–dimensional connected sum in terms of the diffeomorphism groups of its prime factors.

Having analyzed essential 2–spheres, we now turn our attention to essential projective planes. By the previous step, we can restrict the analysis to 3–manifolds that contain no essential 2–spheres. It turns out that we then have a much stronger uniqueness property for essential projective planes than for essential 2–spheres.

Theorem 3.2 (Characteristic family of 2–sided projective planes)**.** *Let M be a 3–manifold of finite type that contains no essential 2–sphere. Then, there is a compact 2–dimensional submanifold Π of M such that:*

(i) *Every component of Π is a 2–sided projective plane.*

(ii) *No two components of Π are parallel.*

(iii) *Every 2–sided projective plane in $M - \Pi$ is parallel in M to a component of Π.*

In addition, such a Π is unique up to isotopy of M.

A proof of Theorem 3.2 can be found in Negami [99].

If we split M open along the submanifold Π, we obtain a 3–manifold M' bounded by finitely many projective planes such that, by Condition (iii), every 2–sided projective plane in M' is parallel to a boundary component. However, in contrast to what we were able to do after splitting a 3–manifold along essential 2–spheres in Theorem 3.1, there is no way we can plug the boundary projective planes of M' to obtain a 3–manifold with no essential projective plane. Indeed, Poincaré duality with coefficients in $\mathbb{Z}_2$ (for instance) shows that there is no compact 3–manifold whose boundary consists of a single projective plane.

There is a way to overcome this difficulty, but it involves leaving the category of 3–manifolds and enlarging the scope of the analysis to 3–dimensional orbifolds. Although the projective plane $\mathbb{RP}^2$ does not bound any compact 3–manifold, it does bound a relatively simple orbifold. Namely considering $\mathbb{RP}^2$ as the quotient of the 2–sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ by the map $x \mapsto -x$, it bounds the 3–orbifold quotient of the 3–ball $\mathbb{B}^3 \subset \mathbb{R}^3$ by the same map $x \mapsto -x$. This orbifold has underlying topological space the cone over $\mathbb{RP}^2$, and its singular set consists of a single point, corresponding to the origin. If we plug each boundary component of M' with a copy of this 3–orbifold, the orbifold $\widehat{M'}$ obtained in this way does contain 2–sided projective planes, but these projective planes are inessential in the framework of the connected sum factorization of 3–orbifolds discussed in Section 3.6.

Therefore, Theorems 3.1 and 3.2 essentially reduce the study of finite type 3–manifolds to that of 3–orbifolds that contain no essential 2–sphere or projective plane. At this point, the switch to orbifolds is necessary only when the 3–manifold considered is non-orientable. We will first restrict attention to manifolds containing no essential 2–spheres or projective planes, and return to orbifolds in Section 3.6.

We also refer to Kalliongis-McCullough [67, 68] for an analysis of the connected components of the diffeomorphism group of a 3–manifold which contains essential projective planes, in terms of the groups similarly associated to the pieces of the decomposition of Theorem 3.2.

As an aside, we should mention that we could have discussed singular 2–spheres in 3–manifolds, in the same way as we considered essential 2–tori and annuli in Section 2.4. More precisely, a *singular* 2*–sphere* in the 3–manifold M is a continuous map $\mathbb{S}^2 \to M$, and it is *essential* precisely when it is not homotopic to a constant map. In particular, the existence of an essential singular 2–sphere is equivalent to the non-triviality of the second homotopy group $\pi_2(M)$. The reason for the omission is the celebrated Sphere Theorem of C.D. Papakyriakopoulos [108] (extended to the non-orientable case by D.B.A. Epstein [33]; see also [50, Sect. 4.12]), which states that the existence of an essential singular 2–sphere implies the existence of an essential embedded 2–sphere or projective plane.

Theorem 3.3 (Sphere Theorem)**.** *If the* 3*–manifold* M *contains an essential singular* 2*–sphere, then it contains an essential (embedded)* 2*–sphere or a* 2*–sided projective plane.*

Note an unfortunate pitfall in this terminology if the Poincaré Conjecture does not hold: If there indeed exists a 3–manifold P which is homotopy equivalent to but not diffeomorphic to $\mathbb{S}^3$ and if M is an arbitrary 3–manifold which is not diffeomorphic to $\mathbb{S}^3$ then, in the connected sum of M and P, the connected sum 2–sphere is essential as an embedded 2–sphere but inessential as a singular 2–sphere. However, this phenomenon occurs only in this situation, and most likely never occurs in view of the strong evidence in favor of the Poincaré Conjecture.

3.2. The characteristic torus decomposition

In view of the previous section, consider now a 3–manifold M that contains no essential 2–sphere or projective plane. In Theorem 2.9, we saw that M must contain no essential singular 2–torus to admit a hyperbolic structure.

Theorem 3.4 (Characteristic torus decomposition)**.** *Let* M *be a* 3*–manifold of finite type which contains no essential sphere or projective plane. Then, up to isotopy, there is a unique compact* 2*–dimensional submanifold* T *of* M *such that:*

(i) *Every component of* T *is* 2*–sided, and is an essential* 2*–torus or Klein bottle.*

(ii) *Every component of* $M - T$ *either contains no essential embedded* 2*–torus or Klein bottle, or else admits a Seifert fibration.*

(iii) *Property (ii) fails when any component of T is removed.*

Here, a 2–torus T embedded in the 3–manifold M is *essential* if the inclusion map $T \to M$ is an essential singular 2–torus in M, in the sense of Section 2.4. A Klein bottle K embedded in M is *essential* if the composition of the orientation covering $\mathbb{T}^2 \to K$ and of the inclusion map $K \to M$ is an essential singular 2–torus.

When M is orientable, all components of the submanifold T are 2–tori, and Theorem 3.4 is therefore known as the Characteristic Torus Decomposition Theorem.

Theorem 3.4 was first announced by F. Waldhausen [147], and a complete proof was published by K. Johannson [62], W. Jaco and P. Shalen [59]; see also [57]. Actually, these authors are only considering the orientable case, but a proof of the general case (with no significant difference) can be found in Bonahon-Siebenmann [20]. The existence of the submanifold T is an easy consequence of the finiteness argument of Kneser [70] which we already encountered in Theorem 3.1; what is really important in Theorem 3.4 is its uniqueness up to isotopy. From a historical point of view, we should also mention H. Schubert's unique decomposition of a knot into its satellites [122], where a precursor of the 2–torus decomposition for knot complements first appeared.

It turns out that the Seifert pieces of the above decomposition 'absorb' all essential singular 2–tori in M, in the following sense:

Theorem 3.5 (Classification of essential singular tori)**.** *Let M and T be as in Theorem 3.4. Then, every essential singular 2–torus $\varphi : \mathbb{T}^2 \to M$ can be homotoped so that one of the following holds:*

(i) *The image of φ is contained in a Seifert fibered component of $M - T$, and φ is vertical with respect to the Seifert fibration in the sense that, at each $x \in \mathbb{T}^2$, the differential of φ at x sends the tangent plane $T_x\mathbb{T}^2$ to a plane in $T_{\varphi(x)}M$ that is tangent to the fiber passing through $\varphi(x)$; in this case, it is possible to choose a diffeomorphism between $\mathbb{T}^2$ and $\mathbb{S}^1 \times \mathbb{S}^1$ such that the restriction of φ to each circle $\{x\} \times \mathbb{S}^1$ is a covering map onto a fiber of the Seifert fibration.*

(ii) *M admits a Seifert fibration, and φ is horizontal with respect to the Seifert fibration in the sense that, at each $x \in \mathbb{T}^2$, the differential of φ at x sends the tangent plane $T_x\mathbb{T}^2$ to a plane in $T_{\varphi(x)}M$ which is transverse to the fiber passing through $\varphi(x)$. In this case, the composition of $\varphi : \mathbb{T}^2 \to M$ with the projection $M \to \Sigma$ to the base orbifold Σ of the Seifert fibration is an orbifold covering map; in particular, M is compact and Σ has orbifold Euler characteristic $\chi_{\text{orb}}(\Sigma) = 0$. In addition, the Euler number e_0 of the Seifert fibration is either undefined (when M is non-orientable) or 0.*

(iii) *The image of φ is contained in a component T_0 of T, and φ factors as the composition of a covering map $\mathbb{T}^2 \to T_0$ and of the inclusion map $T_0 \to M$.*

The prof of Theorem 3.5 is significantly more difficult than that of Theorem 3.4. In the case where M is a Haken 3–manifold (see Section 3.5 for a definition of Haken 3–manifolds), Theorem 3.5 was, again, first announced by F. Waldhausen [147], and a complete proof was published by K. Johannson [62], W. Jaco and P. Shalen [59] (see also Feustel [37] for a related result, and [57]). However, the case where M is non-Haken was settled only recently, following work of P. Scott [123, 124], G. Mess [91], D. Gabai [39], A. Casson and D. Jungreis [26].

An immediate corollary is the so-called Torus Theorem:

Corollary 3.6 (Torus Theorem)**.** *Let M be a 3–manifold of finite type that contains no essential 2–sphere or projective plane. Suppose that there exists an essential singular 2–torus $\varphi : \mathbb{T}^2 \to M$. Then, M contains an essential embedded 2–torus or Klein bottle, or else M admits a Seifert fibration (or both).*

3.3. The characteristic compression body

We now return to the case with boundary. Let M be a 3–manifold with boundary, and assume that it is *of finite type*. This now means that M is obtained from a compact manifold $\overline{M}$ with boundary by removing a 2–dimensional compact submanifold (with boundary) of ∂M. We use here the convention that a codimension 0 submanifold is any subset bounded by a codimension 1 submanifold.

The analysis of Section 3.1 carries over without modifications, and we can therefore restrict attention to the case where M contains no essential 2–sphere or projective plane.

In Section 2.5, we saw that one restriction to the existence of a hyperbolic structure with totally geodesic boundary on M is that its boundary ∂M should not admit any essential compression disk. There is a theory of connected sums along disks which is very analogous to the connected sum factorization of Theorem 3.1. However, the uniqueness part of this factorization is here much improved, because it leads to a uniqueness up to isotopy. Indeed, in close analogy with the characteristic torus decomposition of the previous section, there is a characteristic submanifold of M which absorbs all compression disks of M.

Let a *compression body* be a 3–manifold obtained from a product $V_0 = S \times [0, 1]$ by gluing 2–handles along $S \times \{1\}$, and capping off with a 3–handle some of the boundary 2–spheres which may have appeared in the process. Namely, we start from $S \times [0, 1]$, where S is a surface of finite type, not necessarily connected. Then, we glue n copies of the product $\mathbb{B}^1 \times \mathbb{B}^2$ (= 2–handles) along n disjoint embeddings of the annulus $\mathbb{B}^1 \times \partial\mathbb{B}^2$ in $S \times \{1\}$; there is a natural way to smooth the corners in this construction to obtain a differentiable 3–manifold V_1 with the same homeomorphism type. Finally, we can glue 3–balls (= 3–handles) along some boundary components of V_1 which are 2–spheres and are not contained in $S \times \{0\}$. By definition, a compression body is any 3–manifold V obtained in this way.[5]

The boundary of a compression body V can be split into two pieces: the *external boundary* $\partial_e V$, which corresponds to $S \times \{0\}$, and the *internal boundary* $\partial_i V =$

[5] The author often gets credited with the introduction of the term “compression body”. This expression was actually coined by Larry Siebenmann, as a replacement for the inelegant “product with handles” used in preliminary versions of [15]. In retrospect, the term “hollow handlebody”, reminiscent of the “hohlbretzel” already used by Waldhausen in [146], would probably be more appropriate to deal with situations where a compression body does not occur as a classifying object for compression disks.

$\partial V - \partial_e V$. It is relatively easy to see that, for a connected compression body V with non-empty internal boundary $\partial_i V \neq \emptyset$, the above description of V as a thickened surface with handles can be chosen so that it involves only 2–handles and no 3–handles.

Consider a 3–manifold M of finite type with boundary. In addition, we assume that M contains no essential 2–sphere or projective plane.

If M admits any essential compression disk for its boundary ∂M, the argument of Kneser [70] provides a compact 2–submanifold D of M, whose components are compression disks for ∂M, and such that any compression disk for ∂M that is disjoint from D must be parallel to a component of D. This submanifold D is in general far from being unique. However, if we thicken the union $\partial M \cup D$, we get a compression body $V_1 \subset M$ whose external boundary $\partial_e V_1$ is equal to ∂M and whose internal boundary $\partial_i V_1$ admits no compression disk. Some components of the internal boundary $\partial_i V_1$ may be 2–spheres, which necessarily bound components of the closure of $M - V_1$ which are 3–balls by hypothesis on M. Let V be the compression body union of V_1 and of all these 3–balls components of the closure of $M - V_1$.

Theorem 3.7 (Characteristic compression body)**.** *Let M be a 3–manifold with boundary, which is of finite type and contains no essential 2–sphere or projective plane. Then, up to isotopy, M contains a unique compression body V such that the external boundary $\partial_e V$ is equal to ∂M, such that the closure $\overline{M - V}$ contains no essential compression disk for its boundary $\partial_i V$, and such that no component of $\overline{M - V}$ is a 3–ball. In addition, any singular compression disk $\left(\mathbb{B}^2, \partial\mathbb{B}^2\right) \to (M, \partial M)$ can be homotoped inside V by a homotopy keeping the image of $\partial\mathbb{B}^2$ in ∂M.*

Theorem 3.7 is proved in Bonahon [15] in the orientable case, and the proof automatically extends to the non-orientable case. Note that, when M contains no essential compression disk and is not a 3–ball, the characteristic compression body V is just a collar neighborhood of the boundary, diffeomorphic to $\partial M \times [0, 1]$.

In the situation of Theorem 3.7, let M_0 the the closure of the complement $M - V$ in M. Then, M_0 contains no essential 2–sphere, projective plane, or compression disk for its boundary.

Conversely, let M_0 be a possibly disconnected 3–manifold of finite type that contains no essential 2–sphere, projective plane, or compression disk for its boundary. If we are given n disjoint embeddings of the two disks $\mathbb{B}^2 \times \partial\mathbb{B}^1$ in ∂M_0, we can use these pairs of disks to glue n copies of the product $\mathbb{B}^2 \times \mathbb{B}^1$ (= 1–handles) to M_0. We then obtain a new 3–manifold M. It is not very hard to check that M contains no essential 2–sphere or projective plane. Note that each 1–handle $\mathbb{B}^2 \times \mathbb{B}^1$ provides a compression disk $\mathbb{B}^2 \times \{0\}$ for ∂M. It immediately follows from the construction that, if V is the characteristic compression body associated to M by Theorem 3.7, the closure of $M - V$ is diffeomorphic to M_0. This provides an inverse construction to the splitting defined by Theorem 3.7.

3.4. The characteristic torus/annulus decomposition

The previous characteristic splittings enable us to analyze essential 2–spheres, projective planes, singular 2–tori and compression disks in 3–manifolds with boundary. It remains to consider essential singular annuli.

Theorem 3.8 (Characteristic torus/annulus decomposition)**.** *Let M be a 3–manifold of finite type with boundary, which contains no essential 2–sphere, projective plane, or compression disk for its boundary. Then, up to isotopy, there is a unique compact 2–dimensional submanifold F of M such that:*

(i) *Every component F_1 of F is 2–sided, and is an essential 2–torus, Klein bottle, annulus or Möbius strip with $F_1 \cap \partial M = \partial F_1$.*

(ii) *For every component W of $M - F$, either W contains no essential embedded 2–torus, Klein bottle, annulus or Möbius strip, or W admits a Seifert fibration for which $W \cap \partial M$ is a union of fibers, or else W admits the structure of a $\mathbb{B}^1$–bundle over a surface of finite type such that the corresponding $\partial\mathbb{B}^1$–bundle is equal to $W \cap \partial M$.*

(iii) *Property (ii) fails when any component of F is removed.*

In addition, note that the ends of a Seifert fibered component W of $M - F$ all are of toric type, and can be delimited by 2–tori and Klein bottles in W; let T_W be the union of 2–tori and Klein bottles delimiting those ends of W whose closure contain at least one annulus or Möbius strip component of F. Let T be the union of all 2–torus and Klein bottle components of F, and of all T_W as W ranges over all Seifert fibered components of $M - F$. Then, T is the characteristic 2–submanifold of the Characteristic Torus Decomposition Theorem 3.4 of the interior $\mathrm{int}\,(M) = M - \partial M$.

Again, an annulus A in M is *essential* if the inclusion map $A \to M$ is an essential singular annulus, in the sense of Section 2.5. A Möbius strip A in M is *essential* if the composition of the orientation covering $\mathbb{A}^2 \to A$ and of the inclusion map $A \to M$ is an essential singular annulus.

When M is orientable, all components of the submanifold F are tori and annuli, and the above theorem is therefore known as the Characteristic Torus/Annulus Decomposition Theorem.

As in the case of the Characteristic Torus Decomposition Theorem 3.4, Theorem 3.8 was first announced by F. Waldhausen [147], and a complete proof was published by K. Johannson [62], W. Jaco and P. Shalen [59] (and see Bonahon-Siebenmann [20] for the details of a proof in the non-orientable case, where Theorem 3.8 is interpreted as a generalization of Theorem 3.4 to certain 3–orbifolds).

The fibered parts of the characteristic torus/annulus decomposition of Theorem 3.8 absorb all essential singular annuli, as indicated by the following theorem, whose proof can be found in [62, 59, 57].

Theorem 3.9 (Classification of essential singular annuli)**.** *Let M and F by as in Theorem 3.8. Then, every essential singular annulus $\varphi : \left(\mathbb{A}^2, \partial\mathbb{A}^2\right) \to (M, \partial M)$ can be homotoped so that one of the following holds:*

(i) *The image of φ is contained in a Seifert fibered component of $M - F$, and is vertical with respect to the Seifert fibration in the sense that the restriction of φ*

to each circle $\mathbb{S}^1 \times \{y\} \subset \mathbb{S}^1 \times \mathbb{B}^1 = \mathbb{A}^2$ *is a covering map onto a generic fiber of the Seifert fibration.*

(ii) *The image of* φ *is contained in a* $\mathbb{B}^1$*–bundle component of* $M - F$*, and is vertical with respect to this bundle in the sense that the restriction of* φ *to each arc* $\{x\} \times \mathbb{B}^1 \subset \mathbb{S}^1 \times \mathbb{B}^1 = \mathbb{A}^2$ *is a diffeomorphism onto a fiber of the* $\mathbb{B}^1$*–bundle.*

(iii) *The image of* φ *is contained in an annulus or Möbius strip component* A *of* F*, and* φ *factors as the composition of a covering map* $\mathbb{A}^2 \to A$ *and of the inclusion map* $A \to M$*.*

Again a corollary of Theorem 3.9 is the following Annulus Theorem:

Corollary 3.10 (Annulus Theorem)**.** *Let* M *be a* 3*–manifold of finite type which contains no essential* 2*–sphere or projective plane. Suppose that there exists an essential singular annulus* $\varphi : \left(\mathbb{A}^2, \partial\mathbb{A}^2\right) \to (M, \partial M)$*. Then,* M *contains an essential embedded annulus or Möbius strip.*

Indeed, if M contains an essential singular annulus and if V is its characteristic compression body, an easy cut and paste argument shows that the closure of $M - V$ also contains an essential singular annulus, and enables us to reduce the analysis to the case where M contains no essential compression disk for its boundary. Then Theorem 3.9 shows that, either the characteristic torus/annulus 2–submanifold F of Theorem 3.8 has a component which is an annulus or a Möbius strip, or F is disjoint from ∂M and at least one component V of $M - F$ is a $\mathbb{B}^1$–bundle or a Seifert fibered manifold with $V \cap \partial M \neq \emptyset$. In the second case, a suitable closed curve in the base of the $\mathbb{B}^1$–bundle, or a suitable arc in the base orbifold of the Seifert fibration, provides an essential embedded annulus or Möbius strip.

3.5. Homotopy equivalences between Haken 3*–manifolds*

Originally, the characteristic torus/decomposition was not developed as an obstruction to the existence of geometric structures, but as a tool to analyze homotopy equivalences between 3–manifolds. This important topic deserves a little digression here. The survey article of Waldhausen [146] is a good reference for this material.

We begin by defining a technically important class of 3–manifolds, called Haken manifolds. Let M be a compact 3–manifold with boundary, and let F be a compact 2–submanifold of M, so that $F \cap \partial M = \partial M$. When F is a 2–sphere, a projective plane or a compression disk, we already defined what it means for F to be essential. For all other types of surfaces, we say that F is *essential* if it is 2–sided and if the homomorphisms $\pi_1(F) \to \pi_1(M)$ and $\pi_1(F, \partial F) \to \pi_1(M, \partial M)$ induced by the inclusion map are all injective, and this for all choices of base points. Note that this is consistent with the definition used for 2–tori, Klein bottles, annuli and Möbius strips in previous sections.

By definition, a *compact Haken* 3*–manifold* is a compact 3–manifold with boundary which contains no essential 2–sphere or projective plane but which contains at least one essential surface. It can be shown that the last condition is unnecessary if the boundary ∂M is non-empty: If the compact 3–manifold M has non-empty boundary, it necessarily contains an essential surface, unless M is a 3–ball; see for instance [50, Chap.13]. The crucial technical property of compact Haken 3–manifolds is that they give rise to a finite sequence $M = M_0, M_1, \ldots, M_n$ of manifolds with boundary such that each M_i is obtained by splitting M_{i-1} along an essential surface and such that the last manifold is a disjoint union of 3–balls. Such a finite sequence lends itself well to inductive procedures, which makes it a very useful technical tool.

The starting point is the following result of Waldhausen [145], extended to non-orientable 3–manifolds by W. Heil [48, 49] (see also Hempel [50]).

Theorem 3.11. *Let $f : M \to N$ be a homotopy equivalence between compact 3–manifolds with boundary such that f restricts to a homeomorphism $f_{|\partial M} : \partial M \to \partial N$. Assume that M is Haken and that N contains no essential 2–sphere or projective plane. Then, f is homotopic to a diffeomorphism, by a homotopy fixing the restriction of f to the boundary.*

When M admits no essential compression disk for its boundary and N is not a trivial interval bundle over a surface, the requirement that f restricts to a homeomorphism between boundaries can be replaced by the weaker hypothesis that $f(\partial M) \subset \partial N$; the conclusion is then also weaker, and only provides a homotopy from f to a diffeomorphism (with no control on the boundary).

We should also mention the following related result, also proved by Waldhausen in [145].

Theorem 3.12. *Let $f_0, f_1 : M \to M$ be two diffeomorphisms of a compact Haken 3–manifold which are homotopic. If M admits essential compression disks for its boundary, assume in addition that each stage of the homotopy sends ∂M to ∂M. Then, f_0 and f_1 are isotopic.*

Theorem 3.12 is the extension to Haken 3–manifolds of a celebrated result of Baer for surfaces (see for instance Epstein [33]). The combination of Theorems 3.11 and 3.12 says that, for two compact Haken 3–manifolds M and N, the space of diffeomorphisms $f : M \to N$ has the same number of connected components as the space of homotopy equivalences $f : M \to N$ (sending boundary to boundary in the presence of compression disks). This was later extended by A. Hatcher [47], who proved that these two spaces have the same homotopy type.

In Theorem 3.11, the requirement that $f(\partial M) \subset \partial N$ is crucial. Indeed, suppose that M contains an essential 2–sided annulus A with $A \cap \partial M = \partial A$. Split M along A and, in the split manifold, glue the two sides of A back together through the diffeomorphism $(x, t) \mapsto (x, 1-t)$ of $A \cong \mathbb{S}^1 \times [0, 1]$ to obtain a new 3–manifold N. The two manifolds M and N are both homotopy equivalent to the space obtained from M by collapsing each arc $\{x\} \times [0, 1]$ of $A \cong \mathbb{S}^1 \times [0, 1]$ to a point. However, M and N are in general not diffeomorphic; for instance, if the two boundary components of A lie in different boundary components of M, the two manifolds M and N will have a different number of boundary components. Note that this construction is analogous to the homotopy equivalence between the annulus and the Möbius strip.

This construction can be slightly generalized in the following way. Let A be a 2–sided 2–submanifold of M such that each component of A is an essential annulus or Möbius strip, and such that the closure V of some component of $M - A$ is homeomorphic to a $\mathbb{B}^2$–bundle over $\mathbb{S}^1$. Let N be obtained by replacing V in M by another $\mathbb{B}^2$–bundle W over $\mathbb{S}^1$, such that the components of A wrap around the $\mathbb{S}^1$ factor of the bundle the same number of times as in V. Then, M and N are both homotopy equivalent to the space obtained from M by collapsing each fiber of the $\mathbb{B}^2$–bundle structure of V to a point, but M and N are in general not diffeomorphic. Call such a homotopy equivalence $M \to N$ a *flip homotopy equivalence*. Johannson [62] proved that this is essentially the only counter-example:

Theorem 3.13 (Homotopy equivalences between Haken 3–manifolds)**.** *Let M and M' be two compact Haken 3–manifolds which admit no essential compression disk for their boundaries, and let $f : M \to M'$ be a homotopy equivalence. Consider the characteristic decomposition of M along a family F of 2–tori, Klein bottles, annuli and Möbius strips, as in Theorem 3.8, and let W be the union of a small tubular neighborhood of the annulus and Möbius strip components of F and of those components of $M - F$ which touch ∂M and admit the structure of a Seifert fibration or a $\mathbb{B}^1$–bundle. Let F' and W' be similarly defined in M'. Then, f can be homotoped so that:*

(i) *$f^{-1}(W') = W$ and $f^{-1}(M' - W') = M - W$;*
(ii) *f induces a homeomorphism from $M - W$ to $M' - W'$.*
(iii) *f induces a homotopy equivalence from W to W';*

In addition, f is homotopic to a product of flip homotopy equivalences $M = M_0 \to M_1 \to \ldots \to M_n = M'$ as above.

An immediate corollary is the following.

Corollary 3.14. *If $f : M \to N$ is a homotopy equivalence between compact Haken 3–manifolds and if M contains no essential disk, annulus or Möbius strip, then f is homotopic to a homeomorphism.*

Finally, to completely analyze homotopy equivalences between compact Haken 3–manifolds, we need to understand what happens in the presence of essential compression disks.

First, note that the fundamental group $\pi_1(M)$ is the only non-trivial homotopy group of a Haken manifold M. Indeed, since M contains no essential 2–sphere or projective plane, the Sphere Theorem 3.3 shows that $\pi_2(M) = 0$. Since M contains an essential surface, its fundamental group is infinite, and its universal covering $\widetilde{M}$ is a non-compact 3–manifold with $\pi_1(\widetilde{M}) = 0$ and $\pi_2(\widetilde{M}) = \pi_2(M) = 0$. This implies that $\widetilde{M}$ is contractible by the Hurewicz Theorem, and that $\pi_n(M) = \pi_n(\widetilde{M}) = 0$ for every n.

In particular, the data of a homotopy equivalence $f : M \to M'$ between Haken 3–manifolds is equivalent to the data of the induced isomorphism $f_* : \pi_1(M) \to$

$\pi_1(M')$.

If M is a compact Haken 3–manifold which contains essential compressions disks and if V is its characteristic compression body, $\pi_1(M)$ is isomorphic to the free product of the fundamental groups of the components of $\overline{M-V}$ and of some infinite cyclic groups. Note that each component of $\overline{M-V}$ is a compact Haken 3–manifold which admits no essential compression disk. Also, no component of $\overline{M-V}$ can have an infinite cyclic fundamental group by, for instance, Theorem 3.11.

As a consequence, it follows from Kurosh's theorem on the uniqueness of free product decompositions (see [80]) that a homotopy equivalence $f: M \to M'$ between compact Haken 3–manifolds, with respective characteristic compression disks V and V', induces a homotopy equivalence $g: \overline{M-V} \to \overline{M'-V'}$. The homotopy equivalence $g: \overline{M-V} \to \overline{M'-V'}$ is analyzed by Theorem 3.13. Then, understanding f is essentially a matter of comparing the way the handles of V and V' fit with respect to the components of $\overline{M-V}$ and $\overline{M'-V'}$.

A typical example is the following. Let S be a compact surface without boundary, not a 2–sphere or a projective plane, and let M be a 3–manifold with boundary obtained from the disjoint union of $S \times [0,1]$ and of a 1–handle $\mathbb{B}^1 \times [0,1]$ by identifying $\mathbb{B}^1 \times \{0,1\}$ to two disjoint disks in $S \times \{0,1\}$. Up to diffeomorphism, there are four manifolds M which are obtained in this way, according to whether M is orientable or not and to whether ∂M is connected or not. However, these four Haken 3–manifolds have the same homotopy type.

These homotopy equivalences between compact Haken 3–manifolds with essential compression disks or, more generally, homotopy equivalences of connected sums of such 3–manifolds, are analyzed in detail in Kalliongis-McCullough [65, 66].

3.6. Characteristic splittings of 3–orbifolds

There is, for 3–dimensional orbifolds, a theory of characteristic splittings which closely parallels the one for 3–manifolds which we described in the preceding sections. There are several motivations for such an extension.

In Section 3.1, we already mentioned that 3–orbifolds constitute the natural framework for a theory of connected sums along projective planes. Namely, after splitting a non-orientable 3–manifold along characteristic 2–sided projective planes as in Theorem 3.2, we can plug the boundary projective planes so obtained with copies of the 3–orbifold $\mathbb{B}^3/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts by the antipodal map $x \mapsto x$ to obtain a 3–orbifold N. The methods of this section will then enable us to obtain for N characteristic splittings analogous to those of Sections 3.2, 3.3 and 3.4, which enable us to analyze essential 2–tori, compression disks and annuli in N and M.

Another reason is that we will see in Section 4.5 that there is an existence theorem for geometric structures on 3–orbifolds which is somewhat stronger than the current existence theorems for 3–manifolds. This existence theorem for 3–orbifolds requires the vanishing of certain topological obstructions, analogous to those of Section 2.4. As in the case of manifolds, when these topological obstructions do not vanish, they are best understood in terms of characteristic splittings similar to those of the preceding sections.

Also, we will see that the switch to orbifolds enables us to unify the cases with and without boundary, and in particular to consider the Torus/Annulus Decomposition of Section 3.4 as a special case of the simpler Torus decomposition of Section 3.2.

However, the main benefit of these characteristic splittings of 3–orbifolds, and of the subsequent existence theorems for geometric structures, is that they provide more insight on properly discontinuous group actions on 3–manifolds.

The splitting theorems for 3–orbifolds are usually obtained by a simple word-by-word translation of the corresponding statements for manifolds. The guiding principle in establishing the dictionary is the following. The orbifold equivalent of an object of type T is such that, in an orbifold M/Γ which is the quotient of a 3–manifold M by a finite group Γ, an orbifold object of type T in the orbifold M/Γ is exactly the image of a Γ–invariant family of disjoint objects of type T in the manifold M. This principle will probably become clearer to the reader after we put it in practice.

Let a *sphere* 2*–orbifold* be the orbifold quotient of the unit 2–sphere $\mathbb{S}^2$ by a finite group of diffeomorphisms. By inspection or by using Proposition 2.6 on the geometrization of 2–orbifolds, this finite subgroup of diffeomorphisms is conjugate to a subgroup of the orthogonal group $O(3)$. It is an easy exercise to list all such sphere 2–orbifolds. For instance, exactly two of them have no singular points, namely the 2–sphere $\mathbb{S}^2$ and the projective plane $\mathbb{RP}^2$. The quotient of $\mathbb{S}^2$ by $\mathbb{Z}_p$ acting by rotations gives the orbifold whose underlying topological space is a 2–sphere, and whose singular set consist of two points each with isotropy group $\mathbb{Z}_p$ acting by rotations. Another example includes the quotient of $\mathbb{S}^2$ by the full symmetry group of the regular dodecahedron; its underlying topological space is a disk, its singular set is the boundary of this disk, and the non-trivial isotropy groups are all $\mathbb{Z}_2$ acting by reflection, except for exactly three singular points where the isotropy groups are the dihedral groups of respective orders 4, 6 and 10, acting in the standard way.

Similarly, a *ball* 3*–orbifold* is the orbifold quotient of the closed unit ball $\mathbb{B}^3$ by a finite group of diffeomorphisms. It follows from a deep theorem, the proof of the Smith conjecture [133] (see in particular [90]) that this finite group of diffeomorphisms is conjugate to a finite subgroup of the orthogonal group $O(3)$. As a consequence every ball 3–orbifold is, in a suitably defined sense, a cone over its boundary, which is a sphere 2–orbifold.

Recall that an orbifold is *uniformizable* if it is isomorphic to the orbifold quotient of a manifold by a properly discontinuous group action. Otherwise, it is *non-uniformizable*. In Lemma 2.3, we saw that an orbifold which admits a complete geometric structure is necessarily uniformizable.

By an easy covering space argument (and compare Proposition 2.6), the only non-uniformizable 2–orbifolds Σ are those of the following list:

(i) the underlying space of Σ is a 2–sphere, and its singular set consists of a single point, where the isotropy group is $\mathbb{Z}_p$ acting by rotations with $p \geqslant 2$;

(ii) the underlying space of Σ is a 2–sphere, its singular set consists of two points, and their isotropy groups are $\mathbb{Z}_p$ and $\mathbb{Z}_q$ acting by rotations, with $p, q \geqslant 2$ distinct;

(iii) the underlying space of Σ is a 2–ball, the singular set is the boundary of this 2–ball, and all non-trivial isotropy groups are $\mathbb{Z}_2$ acting by reflection except at one point, where the isotropy group is the dihedral group of order $2p$ acting in the usual way, with $p \geqslant 2$;

(iv) the underlying space of Σ is a 2–ball, the singular set is the boundary of this 2–ball, and all non-trivial isotropy groups are $\mathbb{Z}_2$ acting by reflection except at two points, where the isotropy groups are the dihedral groups of order $2p$ and $2q$ acting in the usual way, with p, $q \geqslant 2$ distinct.

The existence of a geometric structure on the 3–orbifold M imposes restrictions on the 2–suborbifolds of M, where suborbifolds are defined in the obvious way: A *suborbifold* of the orbifold O is a subset S of the topological space underlying O such that, for every folding chart $f : \widetilde{U} \to U$ of the orbifold atlas O, $f^{-1}\big(\widetilde{U} \cap S\big)$ is a submanifold of U. Note that S inherits an orbifold structure by restriction of the charts of O.

By analogy with the case of 3–manifolds, a sphere 2–suborbifold S of M (namely a 2–dimensional suborbifold of M which is a sphere 2–orbifold) is *essential* if it is 2–sided and if it does not bound a ball 3–orbifold in M. Here, S is 2–*sided* if it admits a neighborhood that is isomorphic to the orbifold $S \times \mathbb{B}^1$ or, equivalently, if its (suitably defined) orbifold normal bundle is trivial.

Theorem 3.15. *If the* 3*–orbifold M admit a complete geometric structure, then every* 2*–suborbifold of M is uniformizable. If in addition, the geometric structure is modelled over any of the eight geometries of Section 2 except* $\mathbb{S}^1 \times \mathbb{S}^2$*, then M contains no essential sphere* 2*–suborbifold.*

The first statement comes from Lemma 2.3. Indeed, this result asserts that M is uniformizable, and any uniformization for M (namely an isomorphism between M and the orbifold X/Γ where the group Γ act properly discontinuously on the manifold X) provides a uniformization for any suborbifold of M.

The second statement is analogous to Theorem 2.2 and follows from the fact that the remaining model spaces X are irreducible. Indeed, Lemma 2.3 shows that M is isomorphic to the quotient orbifold X/Γ of the model space X by a properly discontinuous action of a group Γ of isometries. If S is a 2–sided sphere 2–suborbifold S of $M = X/\Gamma$, its pre-image in X is a Γ–invariant family of disjoint 2–spheres S_i in X. Since X is diffeomorphic to $\mathbb{R}^3$ or $\mathbb{S}^3$, an arbitrary component S_1 of this pre-image bounds a 3–ball B_1 in X. Then, if Γ_1 denotes the stabilizer of B_1 in Γ, the 2–suborbifold S bounds the ball 3–suborbifold B_1/Γ_1 in M. This shows that every 2–sided sphere 2–suborbifold of M bounds a ball 3–orbifold in M, and completes the proof of Theorem 3.15.

The requirement that the 3–orbifold M contains no essential sphere 2–suborbifold is stronger than might appear at first glance. When M is a manifold, namely when its singular set is empty, this condition holds if and only if it contains no essential 2–sphere and no 2–sided projective plane. Another fundamental case is when M is the mirror orbifold $DN/\mathbb{Z}_2$ associated to a 3–manifold N with boundary where, as in Section 2.5, DN is the double obtained by gluing two copies of N along their boundary and where $\mathbb{Z}_2$ acts on DN by exchange of these two copies. Then, the orbifold $M = DN/\mathbb{Z}_2$ satisfies this condition if and only if N admits no essential compression disks, 2–spheres, or projective planes.

There is no convenient characteristic splitting which would reduce the analysis of 3–orbifolds to those which contain no non-uniformizable 2–suborbifolds. Consequently, we have to introduce this hypothesis right away, and exclude from the analysis those 3–orbifolds which contain non-uniformizable 2–suborbifolds.

Let M be a 3–orbifold of finite type which contains no non-uniformizable 2–suborbifolds. Then M has a natural splitting as a connected sum of 3–orbifolds without essential sphere 2–suborbifolds, which closely parallels the splitting discussed in Section 3.1. Indeed, we can consider in M a finite family Σ of 2–sided sphere 2–suborbifolds which are pairwise disjoint, do not bound any ball 3–orbifold in M, and are pairwise not parallel in the sense that no two components of Σ are separated by a component of $M - \Sigma$ which is (orbifold) isomorphic to the product of a sphere 2–orbifold and of an interval. If the orbifold M is of finite type, namely is isomorphic to the interior of a compact 3–orbifold with boundary, the argument of Kneser again shows that there exists such a finite family Σ which is maximal for these properties. Then, cut M open along Σ, and glue a ball 3–orbifold B over each boundary component S of the 3–orbifold so obtained; namely B is a cone over S. By construction the (possibly disconnected) 3–orbifold $\widehat{M}$ so obtained contains no essential sphere 2–orbifold. The proof of Theorem 3.1 immediately generalizes to show:

Theorem 3.16. *If M is a 3–orbifold of finite type which does not contain any non-uniformizable 2–suborbifold, the irreducible 3–orbifold $\widehat{M}$ associated to M by the above construction (and containing no essential sphere 2–suborbifold) is independent of the choice of Σ, up to orbifold isomorphism.*

(As indicated in Section 3.1, if we apply this splitting-gluing process to a 3–manifold M that contains 2–sided projective planes, the orbifold $\widehat{M}$ provided by Theorem 3.16 will have singular points, precisely two for each projective plane.)

Conversely, it is possible to reconstruct M from the orbifold $\widehat{M}$ through connected sum operations, although the situation is slightly more complicated than for manifolds. The connected sum of the 3–orbifolds M_1 and M_2 is defined as soon as we are given ball 3–suborbifolds $B_1 \subset M_1$, $B_2 \subset M_2$ and an isomorphism between B_1 and B_2. Then the connected sum $M_1 \# M_2$ is defined by gluing the orbifolds $M_i' = M_i - \operatorname{int}(B_i)$ along their boundaries, using the restriction of the isomorphism $B_1 \cong B_2$. In the case of manifolds, we only had to worry about orientations. The situation is somewhat more complex for orbifolds, because a finite type 3–orbifold M can contain several ball 3–suborbifolds which are isomorphic, but not equivalent by an ambient orbifold isomorphism of M, and because there usually are more isotopy classes of isomorphisms between ball 3–orbifolds than between ball 3–manifolds. However, there are only finitely many such ambient isomorphism types of ball 3–orbifolds in M, and finitely many isotopy classes of isomorphisms between them. Therefore, there are only finitely many possible connected sums $M_1 \# M_2$ of the finite type 3–orbifolds M_1 and M_2, and the combinatorics of these finitely many possibilities are easy to analyze. We also have to consider self-connected sums where $M_1 = M_2$, for instance to deal with a manifold M that contains a non-separating 2–sided projective plane, but this presents no significant difficulty. In particular, in Theorem 3.16, the 3–orbifold M can be recovered from $\widehat{M}$ up to a finite number of ambiguities, which are easily analyzed.

Because of this, we can now restrict attention to 3–orbifolds which contain no non-uniformizable 2–suborbifolds and no essential sphere 2–suborbifolds.

In Section 2.4, we saw that a geometric structure of Seifert type on a manifold often leads to Seifert fibration on the manifold. A similar phenomenon occurs for orbifolds. Actually, the situation is much simpler in the orbifold framework, because Seifert fibrations just correspond to locally trivial $\mathbb{S}^1$–bundles in the category of orbifolds, as we now explain.

Let F be a manifold (to simplify; a similar definition could be made where F is an orbifold). A (locally trivial) orbifold F*–bundle* consists of two orbifolds M and B and of a continuous map $p : M \to B$ between their underlying topological spaces such that, for every $x \in B$, there exists a neighborhood U of x in the topological space underlying B, a folding chart $f : \widetilde{U} \to U$ of B, and a folding chart $g : \widetilde{U} \times F \to p^{-1}(U)$ of M for which: the folding group G_g of g respects the product structure of $\widetilde{U} \times F$; the folding group G_f of f consists of those automorphisms of $\widetilde{U}$ which are induced by elements of G_g; the map p coincides with the map from $p^{-1}(U) = \widetilde{U} \times F/G_g$ to $U = \widetilde{U}/G_f$ that is induced by the projection $\widetilde{U} \times F \to \widetilde{U}$. Note that the folding groups G_f and G_g may be different since some elements of G_g may act by the identity on $\widetilde{U}$.

For such an orbifold F–bundle, the pre-images $p^{-1}(x)$ are the *fibers* of the bundle. Note that each fiber has a natural orbifold structure, and is orbifold covered by the manifold F. If x is a regular point of B, namely if the isotropy group of x is trivial, the fibers above nearby points are all orbifold isomorphic to $p^{-1}(x)$. If B is connected, the set of its regular points is connected, and we conclude that all fibers over regular points are isomorphic. By definition, the orbifold $p^{-1}(x)$ with x regular is the *generic fiber* of the bundle.

In particular, if we go back to the definition of a Seifert fibration, we see that a Seifert fibration on a 3–manifold M gives an orbifold $\mathbb{S}^1$–bundle over the base 2–orbifold B of the Seifert fibration. Conversely, by inspection of all possible local types, one easily sees that an orbifold $\mathbb{S}^1$–bundle $p : M \to B$ where M is a 3–manifold (with all isotropy groups trivial) defines a Seifert fibration on M.

Another important example occurs when we have a 3–manifold M with boundary and we consider the orbifold $DM/\mathbb{Z}_2$, where DM is the double obtained by gluing two copies of M along their boundary and where $\mathbb{Z}_2$ acts by exchange of the two copies. If this 3–orbifold $DM/\mathbb{Z}_2$ is endowed with the structure of an $\mathbb{S}^1$–bundle, then inspection now shows that there are two cases: Either the generic fiber is the manifold $\mathbb{S}^1$, and the bundle structure induces a Seifert fibration on M for which ∂M is a union of fibers and for which the base orbifold is a 2–orbifold B with boundary; in this case, the base orbifold of the $\mathbb{S}^1$–bundle is the quotient orbifold

$DB/\mathbb{Z}_2$ where DB is the double obtained by gluing two copies of B along their boundary and $\mathbb{Z}_2$ acts by exchange of the two copies. Or the generic fiber is the 1–orbifold $\mathbb{S}^1/\mathbb{Z}_2$ where $\mathbb{Z}_2$ acts on $\mathbb{S}^1$ by reflection, the base orbifold B of the $\mathbb{S}^1$–bundle is a manifold, and the bundle structure induces a locally trivial (manifold) bundle on M with base the 2–manifold B and with fiber the interval $\mathbb{B}^1$, in such a way that ∂M corresponds to the $\partial\mathbb{B}^1$–sub-bundle.

The orbifold $\mathbb{S}^1$–bundles over a given 2–orbifold B are classified in Bonahon-Siebenmann [19]. The classification is very much in the spirit of Seifert's classification of Seifert fibrations [127]. It involves the consideration of the possible local types for such a bundle, plus certain global and semi-global invariants. The classification and its proof are not intrinsically more difficult than in the case of Seifert fibrations, but they are considerably more tedious because of the large number of possible local types.

As in Section 2.4, the existence of a complete Seifert-type geometry on a 3–orbifold M usually leads to a fibration on M. For simplicity, we restrict attention to finite volume structures.

Theorem 3.17. *If the* 3*–orbifold M admits a complete geometric structure of finite volume modelled over $\mathbb{S}^2\times\mathbb{E}^1$, $\mathbb{H}^2\times\mathbb{E}^1$, $\mathbb{H}^2\widetilde{\times}\mathbb{E}^1$ or $\mathbb{E}^2\widetilde{\times}\mathbb{E}^1$, then at least one of the following occurs:*

(i) *The foliation of M by the $\mathbb{E}^1$ factors defines an orbifold $\mathbb{S}^1$–bundle structure on M with base orbifold a* 2*–orbifold B. In this case, the metric of the $\mathbb{S}^2$, $\mathbb{H}^2$ or $\mathbb{E}^2$ factors projects to a complete spherical, hyperbolic or euclidean structure of finite area on the base orbifold B.*

(ii) *The model space is $\mathbb{S}^2\times\mathbb{E}^1$, and the $\mathbb{S}^2$–factors define an orbifold $\mathbb{S}^2$–bundle structure on M with base orbifold a compact* 1*–orbifold B.*

The proof is identical to that of the similar statement for manifolds, namely Theorem 2.5 in Section 2.4. The reader may want to back-track to that statement, and see how the list of compact exceptions in (iii) of Theorem 2.5 coincides with the list of all manifolds that correspond to the orbifold bundles of (ii) in the above Theorem 3.17.

The lucky coincidence which occurred for manifolds does not repeat for orbifolds: A geometric structure modelled on $\mathbb{E}^3$ or $\mathbb{S}^3$ for a 3–orbifold M does not necessarily produce a fibration. For instance, if Γ is the group consisting of all isometries of $\mathbb{E}^3$ that respect the lattice $\mathbb{Z}^3\subset\mathbb{R}^3=\mathbb{E}^3$, there is no identification of $\mathbb{E}^3$ with $\mathbb{E}^1\times\mathbb{E}^2$ for which the $\mathbb{E}^1$ factors induce a fibration of the orbifold $\mathbb{E}^3/\Gamma$.

However, in the case of $\mathbb{E}^3$, the 3–dimensional crystallographic groups, namely the groups of isometries of $\mathbb{E}^3$ that act properly discontinuously and with compact quotient, were classified in the XIX-th century; see for instance Janssen [60] or Opechovski [102]. This classification is equivalent to the classification of all compact 3–orbifolds which admit a geometric structure modelled on $\mathbb{E}^3$. Up to orbifold isomorphism, there are 219 such compact euclidean 3–orbifolds (230 if we fix an orientation on orientable orbifolds). For non-compact euclidean 3–orbifold, one can rely on the celebrated theorem of Bieberbach [11, 9] which asserts that, for every properly discontinuous group Γ of isometries of $\mathbb{E}^n$, there is an isometric splitting $\mathbb{E}^n \cong \mathbb{E}^p \times \mathbb{E}^q$ such that Γ respects some slice $\mathbb{E}^p \times \{x_0\}$ and $\mathbb{E}^p \times \{x_0\}/\Gamma$ is compact. Therefore, every non-compact euclidean 3–orbifold is an orbifold $\mathbb{E}^1$– of $\mathbb{E}^2$–bundle over a compact euclidean 2– or 1–orbifold. Since the 17 compact euclidean 2–orbifolds have been known for centuries (see for instance Montesinos [96]) and since there are only two compact euclidean 1–orbifolds ($\mathbb{S}^1$ and $\mathbb{S}^1/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts by reflection), this makes it an easy exercise to list all non-compact euclidean 3–orbifolds.

There is a similar classification for finite groups of isometries of $\mathbb{S}^3$; see Goursat [44], Threlfall-Seifert [137] and Du Val [31]. This is again equivalent to the classification of all spherical 3–orbifolds. For most of these spherical 3–orbifolds, the splitting $\mathbb{S}^3 = \mathbb{S}^2 \widetilde{\times} \mathbb{S}^1$ induces an orbifold $\mathbb{S}^1$–bundle on this 3–orbifold, with basis a spherical 2–orbifold. However, several spherical 3–orbifolds admit no $\mathbb{S}^1$–bundle structure; see Dunbar [30].

Theorem 3.18. *If the* 3*–orbifold M admits a complete geometric structure modelled over* $\mathbb{H}^3$*, it contains no essential* 2*–sided torus* 2*–suborbifold. In addition, it admits no structure as an orbifold* $\mathbb{S}^1$*–bundle, except in the case where the base* 2*–orbifold of the bundle is an open disk* 2*–orbifold or an open annulus* 2*–orbifold; this case cannot occur if the hyperbolic structure has finite volume.*

We have here used the automatic translation convention: A torus 2–suborbifold of M is a suborbifold which is isomorphic to the quotient of a 2–torus by a finite group action. Such a torus 2–suborbifold T is *incompressible* if, for every disk 2-suborbifold D (namely isomorphic to the quotient of a disk by a finite group action) in M with ∂D contained in T, there is a disk 2–suborbifold D' of T such that $\partial D' = \partial D$. A torus 2–suborbifold T of M is *essential* is it is incompressible and if it does not bound any end of M, namely if the closure of no component of $M - T$ is isomorphic to the orbifold $T \times [0, \infty[$. Incidentally, when M is a manifold, these conditions may seem weaker than the ones we considered in Section 2.4. However, for a 2–torus embedded in the 3–manifold M, they are actually equivalent by the Loop Theorem [108], Waldhausen's classification of incompressible surfaces in interval bundles [145, Proposition 3.1], and another lemma of Waldhausen [145, Proposition 5.4] which says that two disjoint incompressible surfaces are homotopic if and only if they are separated by the product of a surface with the interval.

For the Sol geometry, the same proof as in the manifold case of Theorem 2.11 yields the following result. See Scott [125] or Dunbar [29].

Theorem 3.19. *If the* 3*–orbifold M admits a finite volume complete geometric structure modelled over* Sol*, it admits an orbifold fibration over the manifold* $\mathbb{S}^1$ *with generic fiber the* 2*–torus* $\mathbb{T}^2$ *or the orbifold* $\mathbb{T}^2/\mathbb{Z}_2$ *where* $\mathbb{Z}_2$ *acts on* $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ *by reflection on both* $\mathbb{S}^1$*–factors.*

In Theorem 3.18, we saw that essential sphere and torus 2–orbifolds are topological obstructions to the existence of hyperbolic structures on an orbifold. In this context, a quasi-automatic translation of the Torus Decomposition Theorem 3.4, proved in Bonahon-Siebenmann [20], gives:

Theorem 3.20 (Characteristic torus decomposition for orbifolds)**.** *Let M be a 3–orbifold of finite type which contains no non-uniformizable 2–suborbifold and no essential sphere 2–suborbifold. Then, up to orbifold isomorphism, there is a unique compact 2–dimensional suborbifold T of M such that:*

(i) *Every component of T is 2–sided, and is an essential torus 2–suborbifold.*

(ii) *Every component of $M-T$ either contains no essential torus 2–suborbifold, or admits an $\mathbb{S}^1$–bundle structure.*

(iii) *Property (ii) fails when any component of T is removed.*

As indicated earlier, in the case of a manifold M with boundary, applying this theorem to the double orbifold $DM/\mathbb{Z}_2$ subsumes both the Torus Decomposition Theorem 3.4 and the Torus/Annulus Decomposition Theorem 3.8 of Sections 3.2 and 3.4.

The same doubling trick allows us to avoid the study of 3–orbifolds with boundary.

Many other properties of 3–manifolds can be generalized to 3–orbifolds. For instance, see [136] for a generalization of Theorem 3.11 (on homotopy equivalences between Haken 3–manifolds) to a certain class of 3–orbifolds.

4. Existence properties for geometric structures

4.1. The Geometrization Conjecture

The central conjecture is that, for a 3–manifold of finite type, the topological obstructions considered in Sections 2.4 and 2.5 are the only obstructions to the existence of a complete geometric structure. If we combine this with the characteristic splittings of Section 3, this gives:

Conjecture 4.1 (*Geometrization Conjecture for 3–manifolds*)**.** Let M be a 3–manifold of finite type with boundary which contains no essential 2–sphere, projective plane or compression disk. Let F be the 2–submanifold provided by the Characteristic Torus/Annulus Decomposition Theorem 3.8. Then, every component of $M-F$ admits a complete geometric structure with totally geodesic boundary.

If, in addition, M consists of a compact part and of finitely many toric or annular ends, then the geometric structures of the components of $M-F$ can be chosen to have finite volume, except in the following cases:

(i) $F=\emptyset$, M is non-compact, and M is diffeomorphic to an (open, closed or semi-open) interval bundle over a plane, an open annulus, an open Möbius strip, a 2–torus or a Klein bottle;

(ii) F consists of a single 2–torus, and $M-F$ is diffeomorphic to $F\times\mathbb{E}^1$; in this case, M is compact and admits a geometry modelled over Sol.

We will see in later sections that this conjecture is now proved in many important cases.

However, we should probably mention that the situation is still very unclear for 3–manifolds whose topological type is not finite. Indeed, some new topological obstructions then occur. For instance, the Whitehead manifold, a contractible 3–manifold which is not homeomorphic to $\mathbb{E}^3$ (see for instance Rolfsen [115, Sect. 3.I]), cannot admit a complete geometric structure since it is simply connected but is not homeomorphic to any of the model spaces. Some more topological obstructions related to finite topological type are discussed in Section 6.3. For 3–manifolds with finitely generated fundamental groups, it seems reasonable to conjecture that they can admit a complete geometric structure only if they have finite topological type (the so-called Marden Conjecture), which reduces the question to Conjecture 4.1. However, there is no clear conjecture for the manifolds with infinitely generated fundamental groups. This is in contrast to the case of surfaces, where complex analysis always provided a complete geometric structure.

4.2. Seifert manifolds and interval bundles

For the fibered pieces of the torus/annulus decomposition, the conclusions of the Geometrization Conjecture 4.1 can be proved by a relatively explicit construction; see Scott [125] or Kojima [71]. The proof is fairly simple for a Seifert fibration where the Euler number e_0 is undefined or 0, and requires just a little more care when $e_0 \neq 0$. It is convenient to consider, in addition to the four Seifert type geometries of Section 2.2, the non-maximal geometries of $\mathbb{E}^2 \times \mathbb{E}^1$ (contained in the geometry of $\mathbb{E}^3$) and $\mathbb{S}^2 \widetilde{\times} \mathbb{S}^1$ (contained in the geometry of $\mathbb{S}^3$).

Theorem 4.1 (Geometrization of Seifert fibered 3–manifolds)**.** *Let the* 3*–manifold* M *with boundary admit a Seifert fibration with base* 2*–orbifold* Σ*. Let the orbifold* Σ *be endowed with a complete geometric structure. Let* $l > 0$ *be equal to* $|e_0|\,\mathrm{area}\,(\Sigma)$ *if the Euler number* e_0 *is defined (modulo a choice of orientation) and non-zero, and let* $l > 0$ *be arbitrary otherwise. Then,* M *admits a complete geometric structure modelled over* $\mathbb{S}^2 \times \mathbb{E}^1$, $\mathbb{S}^2 \widetilde{\times} \mathbb{S}^1$, $\mathbb{E}^2 \times \mathbb{E}^1$, $\mathbb{E}^2 \widetilde{\times} \mathbb{E}^1$, $\mathbb{H}^2 \times \mathbb{E}^1$ *or* $\mathbb{H}^2 \widetilde{\times} \mathbb{E}^1$*, with totally geodesic boundary, in such a way that the* $\mathbb{E}^1$ *or* $\mathbb{S}^1$ *factors correspond to the fibers of the Seifert fibration, such that all generic fibers have length* l*, and such that the other factors project to the original geometric structure on the base orbifold* Σ*. In addition, if* M *is compact and oriented with* $e_0 \neq 0$*, the geometry is necessarily twisted. If* M *is compact and oriented with* $e_0 = 0$*, or compact and non-orientable, then the geometry is necessarily untwisted. If* M *is non-compact, the geometry can arbitrarily be twisted or untwisted.*

Note that the hypotheses of Theorem 4.1 are also necessary by Theorem 2.5.

Topologically, one might think that some Seifert fibered 3–manifolds are missing, namely those where the base 2–orbifold Σ admits no complete geometric structure. However, by Proposition 2.6, Σ then has underlying topological space a 2–sphere with 1 or 2 singular points. In this case, M is a lens space, and consequently admits a geometric structure modelled over $\mathbb{S}^2 \widetilde{\times} \mathbb{S}^1$ for a different Seifert fibration; see

[125, 127, 103, 96].

Therefore, for the torus/annulus decomposition F of a finite type manifold M, any Seifert fibered component of $M - F$ really admits a complete geometric structure.

The proof of the similar statement for interval bundles is much simpler.

Theorem 4.2 (Geometrization of 3–dimensional interval bundle)**.** *Let the 3–manifold M with boundary admit the structure of an interval bundle over a surface S. Let Σ be endowed with a complete geometric structure. Then, M admits a complete geometric structure modelled over $\mathbb{S}^2 \times \mathbb{E}^1$, $\mathbb{E}^2 \times \mathbb{E}^1$ or $\mathbb{H}^2 \times \mathbb{E}^1$, with totally geodesic boundary, in such a way that the $\mathbb{E}^1$ factors correspond to the fibers of the bundle, and such that the other factors project to the original geometric structure on the base orbifold Σ.*

4.3. The Hyperbolization Theorem for Haken 3–manifolds

The most important theorem of this chapter certainly is the following existence theorem for hyperbolic structures on 3–manifolds.

In Section 3.5, we encountered the notion of compact Haken manifold. More generally, a *Haken manifold* is a 3–manifold which is obtained from a compact Haken manifold $\overline{M}$ by removing a compact 2–submanifold from $\partial \overline{M}$. In particular, a Haken manifold is a 3–manifold of finite type with boundary.

Theorem 4.3 (Hyperbolization Theorem)**.** *Let M be a Haken manifold which contains no essential 2–sphere, projective plane, 2–torus, compression disk or annulus. Then, M admits a complete hyperbolic structure with totally geodesic boundary.*

If, in addition, every end of M is toric or annular, then the complete hyperbolic structure has finite volume, unless M is diffeomorphic to an (open, closed or semi-open) interval bundle over a plane, an open annulus, an open Möbius strip, a 2–torus or a Klein bottle.

In other words, for Haken 3–manifolds, the conditions of Theorems 2.9 and 2.14 are necessary and sufficient for the existence of a complete hyperbolic structure with totally geodesic boundary (and possibly with finite volume). Combined with Theorems 4.1 and 4.2, Theorem 4.3 provides a proof of the Geometrization Conjecture 4.1 for all Haken 3–manifolds.

This theorem was first announced by W. Thurston around 1977; see [139, 140]. The proof is very complex and, for a while, was not available in written form, although partial expositions such as those by Thurston [138, 141, 142], Morgan [97] or Sullivan [135] have been very influential in the further development of the field. Some detailed expositions of the proof of Theorem 4.3 are now beginning to become available. Technically, given an incompressible surface S in the Haken manifold M, the proof splits into two very distinct cases, according to whether or not a finite cover of M admits a structure of bundle over the circle $\mathbb{S}^1$ for which the pre-image of S is a union of fibers. The case of bundles over the circle is developed in detail in the monograph by Otal [105]. In the paper [106], Otal also gives a detailed proof of the non-bundle case, using a simplification of Thurston's original argument developed by C. McMullen [75, 76]. The monograph by Kapovich [69] provides an exposition of the non-bundle case, following Thurston's original approach.

These proofs of the hyperbolization theorem are based on the idea, going back to Haken and Waldhausen, of successively cutting the 3–manifold along incompressible surfaces until one reaches a polyhedral ball. The characterization by Andreev [6, 7] of the topological type of acute angle polyhedra in $\mathbb{H}^3$ enables one to put a hyperbolic structure on this polyhedral ball. The core of the proof is a difficult gluing process which, when the polyhedral ball is glued back together to reconstruct the original 3–manifold, progressively reconstructs a hyperbolic structure on the 3–manifold. Altogether, this approach is very reminiscent of the second method we used in Section 1.1 to construct hyperbolic structures on surfaces of finite type. A more analytic approach to the geometrization conjecture of 3–manifolds, such as the one proposed in [5] and in spirit closer to the first method of geometrization of surfaces discussed in Section 1.1, would certainly be more attractive but does not seem to be within reach at this point.

4.4. Hyperbolic Dehn filling

Hyperbolic Dehn filling is a method of constructing many hyperbolic manifolds by deformation of the structure of a complete hyperbolic 3–manifold with cusps. In addition to the original lecture notes by Thurston [138], the book by Benedetti and Petronio [10] is a good reference for the material in this section.

We begin with some topological preliminaries. Let M be a 3–manifold which is the interior of a compact 3–manifold $\overline{M}$ whose boundary $\partial\overline{M}$ consists of finitely many 2–tori. Let M' be a 3–manifold without boundary obtained as follows: Glue copies of the solid torus $\mathbb{S}^1 \times \mathbb{B}^2$ along some of the components of $\partial\overline{M}$, and remove the other components of $\partial\overline{M}$. Such a manifold M' is said to be obtained from M by *Dehn filling.*

For instance, a celebrated result of Lickorish and Wallace [73, 151] says that every compact orientable 3–manifold can be obtained by Dehn filling along the complement of a link (=1–submanifold) in the 3–sphere $\mathbb{S}^3$. See also the chapter by S. Boyer [21] for a more extensive discussion of Dehn fillings.

There are many possible ways of gluing a copy of the solid torus $\mathbb{S}^1 \times \mathbb{B}^2$ along a 2–torus component T of $\partial\overline{M}$. However, one easily sees that, up to diffeomorphism inducing the identity on $\overline{M}$, the resulting manifold is completely determined by the isotopy class of the simple closed curve $\{*\} \times \partial\mathbb{B}^2$ in T. In addition, this isotopy class is completely determined by the class of $H_1(T;\mathbb{Z})\,/\pm 1$ defined by $\{*\} \times \partial\mathbb{B}^2$, where ± 1 acts by multiplication on the homology group $H_1(T;\mathbb{Z})$ and where the ambiguity comes from the fact that the curve $\{*\} \times \partial\mathbb{B}^2$ is not oriented. See for instance Rolfsen's book [115, Chap. 9].

To specify a Dehn filling, one considers the boundary components $T_1, \dots, T_n$ of $\overline{M}$. The Dehn filling is then determined by the data of the *Dehn filling invariants* associated to these boundary 2–tori as follows: If a solid torus $\mathbb{S}^1 \times \mathbb{B}^2$ is glued along the 2–torus T_i, the Dehn filling invariant is the element $\delta_i \in H_1(T_i;\mathbb{Z})/\pm 1$ represented by the curve $\{*\} \times \partial\mathbb{B}^2$; if no solid torus is glued along T_i, the corresponding Dehn filling invariant is $\delta_i = \infty$, by definition. The motivation for this convention will become clear when we discuss the Hyperbolic Dehn Filling Theorem 4.4 below. Note that every list of indivisible elements of $\{\infty\} \cup H_1(T_i;\mathbb{Z})/\pm 1$ can be the list of Dehn filling invariants of some Dehn filling (an element of $\delta \in H_1(T_i;\mathbb{Z})/\pm 1$ is *indivisible* if it cannot be written as $p\delta'$ for some integer $p \geqslant 2$ and $\delta' \in H_1(T_i;\mathbb{Z})/\pm 1$, and ∞ is indivisible by convention).

After these topological preliminaries, consider an orientable 3–manifold M which admits a complete hyperbolic metric of finite volume. By Theorem 2.9, it is diffeomorphic to the interior of a compact 3–manifold $\overline{M}$ whose boundary consists of 2–tori. By Lemma 2.3, the hyperbolic 3–manifold M is isometric to a quotient $\mathbb{H}^3/\Gamma$, where the group Γ acts properly discontinuously and by fixed point free isometries on the hyperbolic 3–space $\mathbb{H}^3$. By the theory of covering spaces, the group Γ is isometric to the fundamental group $\pi_1(M)$, and we therefore have an injective homomorphism $\rho_0 : \pi_1(M) \to \mathrm{Isom}^+(\mathbb{H}^3)$ whose image is discrete, where $\mathrm{Isom}^+(\mathbb{H}^3)$ is the group of orientation-preserving isometries of $\mathbb{H}^3$.

We can actually put this in a more general framework. Consider on M a hyperbolic structure (or, more generally, an (X,G)–structure) which is not necessarily complete. Lift this structure to the universal covering $\widetilde{M}$ of M. Then, by following paths in $\widetilde{M}$, a relatively easy argument shows that every isometry from a small open subset of $\widetilde{M}$ to an open subset of $\mathbb{H}^3$ uniquely extends to a locally isometric map $D : \widetilde{M} \to \mathbb{H}^3$; see Thurston [138] or Benedetti-Petronio [10], and compare Singer [130]. This map D is a global isometry if and only if the hyperbolic metric is complete. From the uniqueness of the extension, we see that there is a homomorphism $\rho : \pi_1(M) \to \mathrm{Isom}^+(\mathbb{H}^3)$ such that $D(\gamma x) = \rho(\gamma) D(x)$ for every $x \in \widetilde{M}$ and $\gamma \in \pi_1(M)$. The map D is a *developing map* for the hyperbolic structure considered, and the homomorphism ρ is its *holonomy*.

Thurston observed that, if we consider the holonomy $\rho_0 : \pi_1(M) \to \mathrm{Isom}^+(\mathbb{H}^3)$ of a complete hyperbolic structure, any homomorphism $\rho : \pi_1(M) \to \mathrm{Isom}^+(\mathbb{H}^3)$ that is sufficiently close to ρ_0 is the holonomy of a usually incomplete hyperbolic structure on M, which is of a very specific type near the ends of M. When this geometric structure is incomplete, its completion (as a metric space) is usually not a manifold. However, for some representations ρ near ρ_0, the completion of the geometric structure on M is a manifold, and, topologically, is obtained from M by Dehn filling. This enabled Thurston to prove the following theorem.

Theorem 4.4 (Hyperbolic Dehn Filling Theorem). *Let the orientable* 3*–manifold* M *admit a finite volume complete hyperbolic structure. By Theorem 2.9, we know that there exists a compact* 3*–manifold* $\overline{M}$ *such that* M *is diffeomorphic to the interior of* $\overline{M}$ *and such that the boundary of* $\partial\overline{M}$ *consists of finitely many* 2*–tori* $T_1, \ldots, T_n$. *Then, there is a finite subset* X_i *of each* $H_1(T_i;\mathbb{Z})/\pm 1$ *such that the following holds: If the manifold* N *is obtained from* M *by Dehn filling in such a way that the Dehn filling invariant* $\delta_i \in \{\infty\} \cup H_1(T_i;\mathbb{Z})/\pm 1$ *associated to each* 2*–torus*

T_i is not in X_i, then N admits a finite volume complete hyperbolic structure.

As indicated above, the proof of Theorem 4.4 can be found in [138] or [10].

The proof of Theorem 4.4 also provides additional geometric information on the hyperbolic structure of the 3–manifold N. Then, for every compact subset K of M and every $\varepsilon > 0$, the hyperbolic metric of N can be chosen so that it is ε–close to the metric of M over K provided the Dehn filling invariants $\delta_i \in \{\infty\} \cup H_1(T_i;\mathbb{Z}) / \pm 1$ are all sufficiently close to ∞. In addition, when $\delta_i \neq \infty$, we can arrange that the core $\mathbb{S}^1 \times \{0\}$ of the solid torus $\mathbb{S}^1 \times \mathbb{B}^2$ glued along the 2–torus T_i is a closed geodesic of N whose length tends to 0 as δ_i tends to ∞ in $\{\infty\} \cup H_1(T_i;\mathbb{Z}) / \pm 1$ (for the topology of the 1–point compactification, for which the neighborhoods of ∞ are the complements of finite subsets of $H_1(T_i;\mathbb{Z}) / \pm 1$).

The requirement that M is orientable is not crucial in Theorem 4.4. When M is non-orientable and admits a finite volume complete hyperbolic metric, we know from Theorem 2.9 that M is the union of a compact part and of finitely many ends, each diffeomorphic to $\mathbb{T}^2 \times [0,\infty[$ or $\mathbb{K}^2 \times [0,\infty[$, where $\mathbb{T}^2$ and $\mathbb{K}^2$ respectively denote the 2–torus and the Klein bottle. Again, any homomorphism $\rho : \pi_1(M) \to \mathrm{Isom}\left(\mathbb{H}^3\right)$ near the holonomy of the complete hyperbolic structure of M is the holonomy of a possibly incomplete hyperbolic structure on M, and we can consider the completion of this metric. However, this completion can almost never be a hyperbolic manifold near the Klein bottle ends of M. Nevertheless, this method provides a complete finite volume metric on any 3–manifold N obtained by sufficiently complicated Dehn filling on toric ends of M, leaving the Klein bottle ends topologically untouched.

There is also a version of Theorem 4.4 where we allow the complete hyperbolic metric of M to have infinite volume, provided we require the metric of M to be "geometrically finite"; see Comar [27] or Bonahon-Otal [18]. Again, it enables one to construct a complete hyperbolic metric on any 3–manifold N obtained by sufficiently complicated Dehn filling on the toric ends of M.

One of the drawbacks of Theorem 4.4 is that it is not explicit. Namely, it does not provide a method to determine the exceptional sets X_i, and not even an estimate on their sizes. There is strong experimental evidence that these X_i should be relatively small.

In some cases, it is possible to carry out explicitly the procedure of hyperbolic Dehn filling. A celebrated example is that of the complement of the figure eight knot in $\mathbb{S}^3$, investigated by Thurston in [139]. In this example, Thurston was able to analyze a large portion of the space of homomorphisms $\rho : \pi_1(M) \to \mathrm{Isom}\left(\mathbb{H}^3\right)$, and to show that certain 'integer points' in this space defined hyperbolic 3–manifolds obtained Dehn filling on the figure eight knot complement. He also observed that, as one approached some points of the boundary of this domain, the hyperbolic structures degenerated to Seifert-type geometric structures. The analysis of this example was instrumental in the development of the Geometrization Conjecture 4.1. A similar procedure is implemented for many link complements, as well as punctured 2–torus bundles over the circle, in the software SnapPea discussed in Section 6.1, and gives explicit upper bounds for the exceptional sets X_i in the examples considered.

There is also a more theoretical evidence for the X_i to be small, obtained by leaving the world of geometric structures. Indeed, instead of trying to put a hyperbolic structure on the manifold N obtained from M by Dehn filling, we can just try to endow it with a complete metric of negative curvature. M. Gromov and W. Thurston provided a technique to achieve this for most Dehn fillings.

More precisely, let M be an orientable 3–manifold with a finite volume complete hyperbolic structure. Then, each end of M has a neighborhood U which is isometric to a model H/Γ_0 where, in the hyperbolic 3–space $\mathbb{H}^3 = \left\{(u,v,w) \in \mathbb{R}^3; w > 0\right\}$, H is a *horoball* of the form $H = \left\{(u,v,w) \in \mathbb{R}^3; w \geqslant w_0\right\}$ for some positive constant w_0, and where Γ_0 is a group of horizontal translations in $\mathbb{H}^3 \subset \mathbb{R}^3$ which is isomorphic to $\mathbb{Z}^2$; see for instance [138, Chap. 5] or [10, Sect. D.3]. The 2–torus ∂U then is isometric to the quotient under Γ_0 of the plane of equation $w = w_0$. The hyperbolic metric of $\mathbb{H}^3$ induces a euclidean metric on this plane, and therefore on the 2–torus ∂U. Let $\overline{M}$ be the compact manifold with boundary obtained by removing from M all the neighborhoods $U_1, \ldots, U_n$ so associated to the ends of M. Note that M is diffeomorphic to the interior of $\overline{M}$, so that we can use $\overline{M}$ to define Dehn fillings along the ends of M. Let T_i denote the component $T_i = \partial U_i$ of $\partial \overline{M}$.

Note that, in the euclidean 2–torus T_i, every non-trivial homology class in $H_1(T_i;\mathbb{Z})$ can be realized by a closed geodesic and that all closed geodesics in the same homology class have the same length.

Theorem 4.5 (2π–theorem)**.** *For M, $\overline{M}$ and $\partial\overline{M} = \bigcup_{i=1}^n T_i$ as above, let N be obtained from M by Dehn filling with Dehn filling invariants $\delta_i \in \{\infty\} \cup H_1(T_i;\mathbb{Z}) / \pm 1$. Suppose that, whenever the invariant δ_i is not ∞, the class $\delta_i \in H_1(T_i;\mathbb{Z}) / \pm 1$ can be realized in the euclidean 2–torus T_i by a closed geodesic of length strictly greater than 2π. Then, the 3–manifold N obtained by Dehn filling M according to the Dehn filling invariants δ_i admits a complete Riemannian metric whose curvature is bounded between two negative constants.*

The proof of Gromov and Thurston [46] is based on an explicit construction which extends the hyperbolic metric on $\overline{M} \subset M$ to a negatively curved metric on N.

A packing argument shows that the neighborhoods U_i of the ends of M can be chosen so that no closed geodesic of $T_i = \partial U_i$ has length less than 1. An elementary argument then bounds the number of homology classes of $H_1(T_i;\mathbb{Z})$ whose closed geodesic representatives have length at most 2π. This argument eventually gives the following corollary of Theorem 4.5, whose proof can be found in Bleiler-Hodgson [12].

Theorem 4.6. *Let M be any orientable 3–manifold with a complete hyperbolic structure of finite volume, diffeomorphic to the interior of a compact manifold $\overline{M}$ and let the 2–tori $T_1, \ldots, T_n$ be the boundary components of $\overline{M}$. Then, there are finite subsets $X_i \subset H_1(T_i;\mathbb{Z}) / \pm 1$, each with at most 48 elements, such that any manifold obtained from M by Dehn filling whose Dehn filling invariants are not in the X_i admits a complete Riemannian metric of negative curvature. When the* *boundary of $\overline{M}$ is connected, the exceptional set X_1 can be chosen to have at most 24 elements.*

The proof of Theorem 4.6 gives some additional information on the shape of the exceptional sets X_i; see Bleiler-Hodgson [12].

The main interest of Theorems 4.5 and 4.6 is that if a 3–manifold N admits a complete Riemannian metric whose curvature is bounded between two negative constants, then N must satisfy the same topological restrictions as those given in Theorem 2.9 for the existence of a complete hyperbolic structure. Therefore, if the Geometrization Conjecture is true, any 3–manifold obtained by Dehn surgery as in Theorems 4.5 and 4.6 will also admit a complete hyperbolic metric. This gives a conjectural estimate on the size of the exceptional sets X_i in the Hyperbolic Dehn Filling Theorem 4.4.

There is evidence that the exceptional sets should actually be smaller than predicted by Theorem 4.6. See the chapter by S. Boyer [21] for a summary of what is currently known in this direction.

4.5. Geometrization of 3–orbifolds

In 1982, Thurston announced a proof of the following result.

Theorem 4.7 (Geometrization Theorem for 3–orbifolds)**.** *Let M be a 3–orbifold of finite type, which contains no non-uniformizable 2–suborbifold and no essential sphere 2–suborbifold, and let T be the characteristic torus 2–suborbifold provided by Theorem 3.20. Assume in addition that the singular set of M is non-empty and has dimension at least 1. Then, every component of $M-T$ admits a complete geometric structure.*

Corollary 4.8. *Let M be a 3–orbifold of finite type which contains no non-uniformizable 2–suborbifold, and no essential sphere or torus 2–suborbifold. Assume in addition that the singular set of M is non-empty and has dimension at least 1. Then, M admits a complete geometric structure, modelled over one of the eight 3–dimensional geometries of Section 2.*

Expositions of Theorem 4.7 have only begun to appear in recent months. A complete exposition can be found in Cooper-Hodgson-Kerckhoff [28], while Boileau-Porti [14] is restricted to the important case where the singular locus consists of disjoint circles. Earlier partial results can be found in [54], [134], [156]; see also [86] for the case where the orbifold admits a finite orbifold covering which is a Seifert fibered manifold.

Theorem 4.7 has the following important corollary.

Corollary 4.9 (Geometrization of 3–manifolds with symmetries)**.** *Let M be a 3–manifold of finite type which contains no essential 2–sphere, projective plane or 2–torus. Suppose that there exists a periodic diffeomorphism $f : M \to M$ whose fixed point set has dimension at least 1. Then, M admits a complete geometric structure.*

Corollary 4.9 follows by application of Theorem 4.7 to the orbifold $M/\mathbb{Z}_n$, where the action of the cyclic group $\mathbb{Z}_n$ is generated by f.

5. Uniqueness properties for geometric structures

5.1. Mostow's rigidity

The most important feature of hyperbolic structures is their uniqueness properties, which follows from Mostow's Rigidity Theorem [98]. This result deals with uniform lattices of $\mathbb{H}^n$ (or, more generally, of any rank 1 homogeneous space), namely discrete groups Γ of isometries of $\mathbb{H}^n$ such that the quotient $\mathbb{H}^n/\Gamma$ has finite volume. The Rigidity Theorem of Mostow asserts that, for every group isomorphism $\varphi : \Gamma_1 \to \Gamma_2$ between two such uniform lattices Γ_1 and Γ_2, there is an isometry $F : \mathbb{H}^n \to \mathbb{H}^n$ such that $\varphi(\gamma_1) = F\gamma_1 F^{-1}$ for every $\gamma_1 \in \Gamma_1 \subset \mathrm{Isom}(\mathbb{H}^n)$. A proof of this deep result can be found in the monograph by Mostow [98]. Other proofs appear in Thurston [138] or Benedetti-Petronio [10].

We can apply this result to the case of a complete hyperbolic 3–manifold M with totally geodesic boundary and with finite volume. As in Section 2.5, the hyperbolic structure of M gives a hyperbolic structure on the double DM, obtained by gluing two copies of M along their boundary. In particular, DM is isometric to the quotient of $\mathbb{H}^3$ by the properly discontinuous action of a group Γ' of isometries. We can then consider the uniform lattice Γ generated by Γ' and by any lift of the isometric involution which exchanges the two copies of M in $DM \cong \mathbb{H}^3/\Gamma'$. By construction, M is isometric to the quotient $\mathbb{H}^3/\Gamma$.

When we have two such 3–manifolds $M_1 \cong \mathbb{H}^3/\Gamma_1$ and $M_2 \cong \mathbb{H}^3/\Gamma_2$, any diffeomorphism $\varphi : M_1 \to M_2$ lifts to a diffeomorphism $\Phi : \mathbb{H}^3 \to \mathbb{H}^3$ which conjugates the action of Γ_1 to the action of Γ_2. Mostow's Rigidity Theorem provides an isometry $F : \mathbb{H}^3 \to \mathbb{H}^3$ which also conjugates Γ_1 to Γ_2 and induces the same isomorphism $\Gamma_1 \to \Gamma_2$. In particular, F induces an isometry $f : M_1 \to M_2$. If we identify M_1 to one of the two halves of the double DM_1, the fact that F and Φ act similarly on the corresponding subgroup $\pi_1(M_1) \subset \Gamma_1$ shows that f is homotopic to φ. This proves:

Theorem 5.1 (Hyperbolic Rigidity Theorem)**.** *Let M_1 and M_2 be two complete hyperbolic 3–manifolds with totally geodesic boundary and with finite volume. Then, every diffeomorphism $\varphi : M_1 \to M_2$ is homotopic to an isometry f.*

We can complement this theorem by adding that f and φ are actually isotopic. This follows from Waldhausen's Isotopy Theorem 3.12 when M is Haken, and in particular when M is non-compact or has non-empty boundary, and from the recent work of D. Gabai and collaborators which will be discussed in Section 6.4 in the general case.

An important practical corollary of Theorem 5.1 is that, if the 3–manifold M admits a finite volume complete hyperbolic metric with totally geodesic boundary, any geometric invariant of this hyperbolic metric is actually a topological invariant of M. Simple examples of such geometric invariants include the volume, or the (locally finite) set of lengths of the closed geodesics of the metric. More elaborate examples involve the Chern-Simons invariant [92], its refinement the eta-invariant [155, 93, 107], or the Ford domain discussed in Section 6.1.

5.2. Seifert geometries and Sol

We now consider the Seifert geometries. Given a 3–manifold M, we would like to classify the complete geometric structures modelled over $\mathbb{S}^2 \times \mathbb{E}^1$, $\mathbb{H}^2 \times \mathbb{E}^1$, $\mathbb{H}^2 \widetilde{\times} \mathbb{E}^1$ or $\mathbb{E}^2 \widetilde{\times} \mathbb{E}^1$ with which M can be endowed, up to isotopy. Namely, such a geometric structure is identified to an atlas which locally models M over the model space, and is maximal among all atlases with this property. We identify two such geometrical structures when the corresponding maximal atlases differ only by composition with a diffeomorphism of M which is isotopic to the identity.

We first restrict attention to the cases where, as in Theorem 2.5, the $\mathbb{E}^1$ factors of these geometries induce a Seifert fibration or a locally trivial bundle with fiber $\mathbb{E}^1$. We then split the problem in two parts: Classify all such fibrations of M, up to isotopy, and then, for a given fibration, classify the geometric structures which give this fibration. We can also add the model spaces $\mathbb{E}^2 \times \mathbb{E}^1$ and $\mathbb{S}^2 \widetilde{\times} \mathbb{S}^1$ to the geometries considered since, by Theorems 2.7 and 2.8, such geometric structures usually arise from geometric structures modelled over $\mathbb{E}^3$ and $\mathbb{S}^3$.

When a 3–manifold admits a Seifert fibration, this fibration is usually unique.

Theorem 5.2 (Topological uniqueness of Seifert fibrations)**.** *Let the finite type* 3*–manifold M admit a Seifert fibration with base* 2*–orbifold* Σ. *Suppose that the orbifold Euler characteristic* $\chi_{\mathrm{orb}}(\Sigma)$ *of* Σ *is non-positive; when* $\chi_{\mathrm{orb}}(\Sigma) = 0$*, suppose in addition that the manifold M is compact and orientable, and that the Euler number $e_0 \in \mathbb{Q}$ is non-trivial. Then, the Seifert fibration of M is unique up to isotopy.*

Theorem 5.2 was proved by Waldhausen [144] for Haken manifolds, by Scott [126] for most non-Haken manifolds and by Boileau-Otal [13] for the remaining cases.

The hypotheses that $\chi_{\mathrm{orb}}(\Sigma) \leqslant 0$ and $e_0 \neq 0$ when $\chi_{\mathrm{orb}}(\Sigma) = 0$ in Theorem 5.2 are necessary because the corresponding 3–manifolds may admit several non-isotopic Seifert fibrations. For instance, when the base 2–orbifold Σ has underlying topological space the 2–sphere $\mathbb{S}^2$ with $\leqslant 2$ singular points (in which case $\chi_{\mathrm{orb}}(\Sigma) > 0$), the manifold M is a lens space and admits many Seifert fibrations of the same type. Similarly, when Σ is the 2–torus manifold $\mathbb{T}^2$ with no singular point (in which case $\chi_{\mathrm{orb}}(\Sigma) = 0$) and $e_0 = 0$, M is the 3–torus $\mathbb{T}^3 = \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$, which admits many non-isotopic fibrations as a locally trivial $\mathbb{S}^1$–bundle. A more exotic example occurs when Σ has underlying topological space $\mathbb{S}^2$ with 3 singular points, two of which have isotropy group $\mathbb{Z}_2$; in this case M admits another Seifert fibration, whose base 2–orbifold is the projective plane with 0 or 1 singular point.

However, in the cases where Theorem 5.2 does not apply, Orlik-Vogt-Zieschang [104] proved a classification of the Seifert fibrations of M up to diffeomorphism of M that is homotopic to the identity; see also Orlik [103]. The work of various authors

[8, 13, 16, 55, 56, 117, 119] (see Boileau-Otal [13] for a historical guide through these references) later proved that every diffeomorphism of M that is homotopic to the identity is actually isotopic to the identity when M is orientable; when M is non-orientable, then it contains a 2–sided essential 2–sphere, projective 2–plane, 2–torus or Klein bottle, and the machinery used in the proofs of Theorems 3.1, 3.2 and 3.4 easily proves the same result in this case. Therefore, the classification of [104] is actually a classification of Seifert fibrations up to isotopy. This classification has too many cases to be listed here, and we can only refer the reader to [104] and [103] for precise statements.

The situation is simpler for $\mathbb{E}^1$–fibrations. The only minor complication occurs when the base of the fibration is non-compact. The example to keep in mind here is that the trivial $\mathbb{E}^1$–bundles over the 2–torus minus 1 point and over the 2–sphere minus 3 points have diffeomorphic underlying spaces. To deal with this problem, it is convenient to consider the compact interval $\widehat{\mathbb{E}}^1$ obtained by adding an end point to each of the two ends of $\mathbb{E}^1$. Then, if M admits a (locally trivial) $\mathbb{E}^1$–fibration with base a surface S, this fibration canonically extends to an $\widehat{\mathbb{E}}^1$–fibration with the same base S, whose underlying space $\widehat{M}$ is the union of M and of a 2–fold covering of S.

If the 3–manifold M has finite type, it is the interior of a compact 3–manifold $\overline{M}$ with boundary. It is an easy consequence of Waldhausen's collar lemma [145, Lemma 3.5] that this compactification $\overline{M}$ is unique up to diffeomorphism whose restriction to M is isotopic to the identity. Since $\pi_1(S) = \pi_1(M)$ is finitely generated, S is also of finite type. Considering a compactification of S, we conclude from the uniqueness of $\overline{M}$ that we can isotop the $\mathbb{E}^1$–fibration so that the associated $\widehat{\mathbb{E}}^1$–fibration space $\widehat{M}$ is obtained from $\overline{M}$ by removing disjoint annuli and Möbius strips from the boundary $\partial\overline{M}$, one for each end of S. Note that $\widehat{M} = \overline{M}$ when S is compact.

Waldhausen [145, Lemma 3.5] proved the following uniqueness result.

Theorem 5.3 (Topological uniqueness of $\mathbb{E}^1$–fibrations)**.** *Let the 3–manifold M be the interior of a compact 3–manifold $\overline{M}$ with boundary, and and consider two $\mathbb{E}^1$–fibrations whose associated $\widehat{\mathbb{E}}^1$–fibration spaces $\widehat{M}$, $\widehat{M}' \subset \overline{M}$ are isotopic in $\overline{M}$. Then, these two $\mathbb{E}^1$–fibrations of M are isotopic.*

In particular, if the 3–manifold M admits an $\mathbb{E}^1$–fibration over a compact surface S, this fibration is unique up to isotopy.

As a summary, if the 3–manifold M admits a Seifert fibration or an $\mathbb{E}^1$–fibration, this fibration is usually unique up to isotopy, except in a few cases which are well understood.

Having analyzed the topological aspects of these fibrations, we can now investigate the geometries corresponding to a given structure. Namely, for a fixed Seifert fibration or $\mathbb{E}^1$–fibration of M, we want to analyze the complete geometric structures on M modelled over $\mathbb{S}^2 \times \mathbb{E}^1$, $\mathbb{H}^2 \times \mathbb{E}^1$, $\mathbb{H}^2 \widetilde{\times} \mathbb{E}^1$, $\mathbb{E}^2 \widetilde{\times} \mathbb{E}^1$, $\mathbb{E}^2 \times \mathbb{E}^1$ or $\mathbb{S}^2 \widetilde{\times} \mathbb{S}^1$ for which the $\mathbb{E}^1$– or $\mathbb{S}^1$–factors give the fibration considered. We consider these geometric structures modulo the natural equivalence relation of fibration-preserving isotopy, namely we identify two such geometric structure when one is the image of the other by a fibration-preserving diffeomorphism (namely a diffeomorhism sending fiber to fiber) of M which is isotopic to the identity through a family of fibration-preserving diffeomorphisms. If M is such a fibered 3–manifold, let $\mathcal{G}_{\mathrm{f}}(M;X)$ be the moduli spaces of such fibered equivalence classes of complete geometric structures on M, with model space X and whose associated fibration is the fibration considered.

Consider the case of a Seifert fibration on M, with base 2–orbifold Σ. Let M be endowed with a geometric structure modelled over $X \times \mathbb{E}^1$ or $X \widetilde{\times} \mathbb{E}^1$, with $X = \mathbb{S}^2$, $\mathbb{E}^2$ or $\mathbb{H}^2$, and such that the Seifert fibration is defined by the $\mathbb{E}^1$–factors. To unify the notation, we write here $\mathbb{S}^2 \widetilde{\times} \mathbb{S}^1 = \mathbb{S}^2 \widetilde{\times} \mathbb{E}^1$, which is consistent since $\mathbb{S}^2 \widetilde{\times} \mathbb{S}^1$ is *locally* a twisted product of $\mathbb{S}^2$ and $\mathbb{E}^1$ although there is no globally defined space $\mathbb{S}^2 \widetilde{\times} \mathbb{E}^1$. The geometry of M then projects to a complete geometric structure over the orbifold Σ, modelled over the space X. Changing the geometric structure of M by a fibration-preserving isotopy only modifies the geometric structure of Σ by an (orbifold) isotopy. This defines a natural map from $\mathcal{G}_{\mathrm{f}}\left(M; X \times \mathbb{E}^1\right)$ or $\mathcal{G}_{\mathrm{f}}\left(M; X \widetilde{\times} \mathbb{E}^1\right)$ to the space $\mathcal{G}(\Sigma; X)$ of isotopy classes of complete geometric structures on Σ, modelled over X.

The moduli spaces $\mathcal{G}(\Sigma; X)$ are easy to determine. One way to do this is to describe a geometric structure on Σ by gluing of elementary pieces, as in the second construction of geometric structures on surfaces of finite type in Section 1.1, and to keep track of the parameters involved. When Σ is a manifold (with no singular point), this approach goes back to Fricke and Klein, and easily extends to the framework of 2–orbifolds. In particular, this analysis is carefully worked out in Ohshika [101] for compact 2–orbifolds Σ, and the analysis easily extends to all 2–orbifolds of finite type. If the base 2–orbifold Σ of a Seifert fibration has finite type, recall that its underlying topological space $|\Sigma|$ is a surface with (possibly empty) boundary, where boundary points correspond to those points where the isotropy group is $\mathbb{Z}_2$ acting by reflection. The orbifold Σ may also have s isolated singular points, where the isotropy group is cyclic acting by rotation, c ends of 'cylindrical type', isomorphic to the manifold $\mathbb{S}^1 \times [0, \infty[$, and r ends of 'rectangular type', isomorphic to $\left(\mathbb{S}^1/\mathbb{Z}_2\right) \times [0, \infty[$ where $\mathbb{S}^1/\mathbb{Z}_2$ denotes the orbifold quotient of $\mathbb{S}^1$ by $\mathbb{Z}_2$ acting by reflection. Let $\chi(|\Sigma|)$ denote the Euler characteristic of the topological space $|\Sigma|$ underlying Σ, which should not be confused with the orbifold Euler characteristic $\chi_{\mathrm{orb}}(\Sigma)$ of Σ which we encountered in Section 2.4.

Then, when $X = \mathbb{H}^2$ and the orbifold Euler characteristic $\chi_{\mathrm{orb}}(\Sigma)$ is negative, $\mathcal{G}\left(\Sigma; \mathbb{H}^2\right)$ is homeomorphic to $\mathbb{R}^{-3\chi(|\Sigma|)-c+r+2s} \times [0, \infty[^{c+r}$, where $\chi(|\Sigma|)$, c, r and s are as above. The element of $[0, \infty[$ associated to an end of Σ which is isomorphic to $\mathbb{S}^1 \times [0, \infty[$ or $\left(\mathbb{S}^1/\mathbb{Z}_2\right) \times [0, \infty[$ is the infimum of the lengths of all 1–suborbifolds of Σ that are isotopic to $\mathbb{S}^1 \times \{0\}$ or $\left(\mathbb{S}^1/\mathbb{Z}_2\right) \times \{0\}$, respectively; this infimum is 0 exactly when the end is a cusp of finite area. In the relatively degenerate cases where the orbifold Euler characteristic of Σ is non-negative, the moduli space $\mathcal{G}\left(\Sigma; \mathbb{H}^2\right)$ is homeomorphic to the empty set, $\{0\}$, $\mathbb{R}$ or $[0, \infty[$.

Similarly, when $X = \mathbb{E}^2$, $\mathcal{G}\left(\Sigma; \mathbb{E}^2\right)$ is homeomorphic to some $\mathbb{R}^n$ with $n \leqslant 3$ or is empty. For instance, $\mathcal{G}\left(\Sigma; \mathbb{E}^2\right)$ is homeomorphic to $\mathbb{R}^3$ when Σ is the 2–torus $\mathbb{T}^2$, to $\mathbb{R}^2$ for the Klein bottle, and to $\mathbb{R}$ when Σ is the 2–orbifold whose underlying topological space is an open disk and whose singular set consists of two points with isotropy group $\mathbb{Z}_2$.

When $X = \mathbb{S}^2$, the situation is much simpler since the moduli spaces $\mathcal{G}\left(\Sigma; \mathbb{S}^2\right)$ consist of at most one point.

Again, we refer to [101] for the details of this analysis of the moduli spaces $\mathcal{G}(\Sigma; X)$ for geometric structures on base orbifolds of Seifert fibrations.

Let us return to our original problem. We have associated an element of $\mathcal{G}(\Sigma; X)$ to each element of $\mathcal{G}_{\mathrm{f}}\left(M; X \times \mathbb{E}^1\right)$ or $\mathcal{G}_{\mathrm{f}}\left(M; X \widetilde{\times} \mathbb{E}^1\right)$. We also have another invariant of the geometric structures considered, namely the length l of a generic fiber of the Seifert fibration. By Theorems 2.5 and 4.1, this length l can take any positive value, except when M is compact and orientable and the Euler number $e_0 \in \mathbb{Q}$ is non-zero, in which case l is necessarily equal to $-\left|e_0\right| \operatorname{area}(\Sigma)$.

If two geometric structures m, $m' \in \mathcal{G}_{\mathrm{f}}\left(M; X \times \mathbb{E}^1\right)$ or $\mathcal{G}_{\mathrm{f}}\left(M; X \widetilde{\times} \mathbb{E}^1\right)$ have the same generic fiber length l one can easily arrange, by an isotopy of m respecting each fiber, that m and m' induce the same metric on each fiber of the Seifert fibration. The key point is that the space of oriented diffeomorphisms of the circle has the homotopy type of the circle.

One could think that, if two geometric structures m, $m' \in \mathcal{G}_{\mathrm{f}}\left(M; X \times \mathbb{E}^1\right)$ or $\mathcal{G}_{\mathrm{f}}\left(M; X \widetilde{\times} \mathbb{E}^1\right)$ induce the same geometric structure on the base 2–orbifold Σ and the same metric on each fiber, then m and m' coincide. However, there is an additional invariant, which is best understood when the Seifert fibration is *oriented*, namely when we can and do choose an orientation of each fiber of the fibration which varies continuously with the fiber.

Consequently, suppose that the Seifert fibration of M is oriented, and let m, $m' \in \mathcal{G}_{\mathrm{f}}\left(M; X \times \mathbb{E}^1\right)$ or $\mathcal{G}_{\mathrm{f}}\left(M; X \widetilde{\times} \mathbb{E}^1\right)$ be two geometric structures which induce the same metrics on the base Σ and on each fiber of the fibration. Note that, in this situation, the metrics m and m' coincide exactly when, at each point of M, the planes orthogonal to the fiber for m and, respectively, m' coincide. We can measure how far we are from this situation as follows. Let Σ_0 be the 2–manifold consisting of the regular points of the orbifold Σ. We then define a differential form $\omega_{m,m'} \in \Omega^1(\Sigma_0)$ of degree 1 on Σ_0 by the following property: if v is a vector in Σ_0, lift it to a vector $\widetilde{v}$ in M which is m–orthogonal to the fiber; then, $\omega_{m,m'}(v)$ is the m'–scalar product of $\widetilde{v}$ and of the unit vector tangent to the oriented fiber; it easily follows from the fact that m and m' induce the same metric on all fibers that this is independent of the choice of the lift $\widetilde{v}$. More geometrically, if α is an arc in Σ_0 and if we lift it to two arcs $\widetilde{\alpha}$ and $\widetilde{\alpha}'$ in M which are respectively m– and m'–orthogonal to the fibers and which have the same starting point, the integral of $\omega_{m,m'}$ over α is equal to the signed distance from the end point of $\widetilde{\alpha}$ to the end point of $\widetilde{\alpha}'$, for the metric induced by m and m' on the oriented fiber corresponding to the end point of α.

If α is a closed loop which is homotopic to 0 in Σ_0 and if we are considering an untwisted geometry $X \times \mathbb{E}^1$, the integral of $\omega_{m,m'}$ over α is 0 since the end points of $\widetilde{\alpha}$ and $\widetilde{\alpha}'$ are both equal to their common starting point. For a twisted geometry

$X\widetilde{\times}\mathbb{E}^1$ and if α is again a closed loop homotopic to 0, we saw in Section 2.2 that the end points of $\widetilde{\alpha}$ and $\widetilde{\alpha}'$ are both obtained by shifting their starting point by an amount of $-A$, where A is the signed area enclosed by α; as a consequence, these end points coincide and it again follows that the integral of $\omega_{m,m'}$ over α is equal to 0. In both cases, it follows that the differential form $\omega_{m,m'}$ is closed. In particular, it defines a cohomology class in $H^1(\Sigma_0;\mathbb{R})$. The integral of $\omega_{m,m'}$ over a loop in Σ_0 which goes around an isolated singular point of Σ is trivial; it follows that this element of $H^1(\Sigma_0;\mathbb{R})$ comes from a cohomology class of $H^1(|\Sigma|;\mathbb{R})$, where $|\Sigma|$ denotes the topological space underlying the orbifold Σ.

If m'' is another metric which is isotopic to m' by an isotopy respecting each fiber and which induces the same metric as m' on each fiber, one easily sees that $\omega_{m,m''} = \omega_{m,m'} + df$ where the function $f : \Sigma_0 \to \mathbb{R}$ measures the amount of rotation of the isotopy on each fiber. It follows that the element of $H^1(|\Sigma|;\mathbb{R})$ that is represented by $\omega_{m,m'}$ depends only on the classes of m and m' in $\mathcal{G}_{\mathrm{f}}\left(M; X\times\mathbb{E}^1\right)$ or $\mathcal{G}_{\mathrm{f}}\left(M; X\widetilde{\times}\mathbb{E}^1\right)$.

The metric m' can easily be recovered from the closed differential form $\omega_{m,m'}$. Therefore, the space of elements $m'\in\mathcal{G}_{\mathrm{f}}\left(M; X\times\mathbb{E}^1\right)$ or $\mathcal{G}_{\mathrm{f}}\left(M; X\widetilde{\times}\mathbb{E}^1\right)$ which have the same generic fiber length and the same image in $\mathcal{G}(\Sigma;X)$ as m is naturally identified to $H^1(|\Sigma|;\mathbb{R})$.

This analysis works when the Seifert fibration of M is oriented. However, it easily extends to the general case, provided we replace $H^1(|\Sigma|;\mathbb{R})$ by the cohomology group $H^1\big(|\Sigma|;\widehat{\mathbb{R}}\big)$ with coefficients twisted by the orientation cocycle of the Seifert fibration. More precisely, consider the space $\widehat{M}$ of pairs (x,o) where $x\in M$ and o is a local orientation of the fiber of the Seifert fibration at x, with the natural topology. This manifold $\widehat{M}$ is a 2–fold covering of M, and the Seifert fibration of M lifts to a Seifert fibration of $\widehat{M}$ which is canonically oriented by choosing the orientation o at each $(x,o)\in\widehat{M}$. The 2–fold covering $\widehat{M}\to M$ descends to a 2–fold covering $\big|\widehat{\Sigma}\big|\to|\Sigma|$ between the spaces underlying their base 2–orbifolds. The twisted cohomology group $H^1\big(|\Sigma|;\widehat{\mathbb{R}}\big)$ is defined by consideration of cochains on $\big|\widehat{\Sigma}\big|$ which are anti-equivariant with respect to the covering automorphism $\tau:\big|\widehat{\Sigma}\big|\to\big|\widehat{\Sigma}\big|$ that exchanges the two sheets of the covering $\big|\widehat{\Sigma}\big|\to|\Sigma|$, namely of cochains c such that $\tau^*(c) = -c$. As indicated, the extension of the above analysis to this twisted context is automatic.

A careful consideration of the argument actually shows:

Theorem 5.4. *Let the 3–manifold M be endowed with a Seifert fibration with base 2–orbifold Σ. For $X=\mathbb{S}^2$, $\mathbb{E}^2$ or $\mathbb{H}^2$, let $\mathcal{G}_{\mathrm{f}}\left(M; X\times\mathbb{E}^1\right)$ and $\mathcal{G}_{\mathrm{f}}\left(M; X\widetilde{\times}\mathbb{E}^1\right)$ be the space of complete geometric structures modelled over the spaces indicated, where the $\mathbb{E}^1$–factors correspond to the fibers of the Seifert fibration, where these geometric structures are considered up to fibration-preserving isotopy. Let $\mathcal{G}(\Sigma;X)$ denote the space of isotopy classes of complete geometric structures on the orbifold Σ modelled over X. Then, if the spaces $\mathcal{G}_{\mathrm{f}}\left(M; X\times\mathbb{E}^1\right)$ or $\mathcal{G}_{\mathrm{f}}\left(M; X\widetilde{\times}\mathbb{E}^1\right)$ are non-empty (and compare Theorems 2.5(i) and 4.1 for this), the natural maps $\mathcal{G}_{\mathrm{f}}\left(M; X\times\mathbb{E}^1\right)\to\mathcal{G}(\Sigma;X)$ and $\mathcal{G}_{\mathrm{f}}\left(M; X\widetilde{\times}\mathbb{E}^1\right)\to\mathcal{G}(\Sigma;X)$ are trivial bundles with fiber $H^1\big(|\Sigma|;\widehat{\mathbb{R}}\big)$*

or $H^1\left(|\Sigma|\,;\widehat{\mathbb{R}}\right) \times]0,\infty[$*, where* $H^1\left(|\Sigma|\,;\widehat{\mathbb{R}}\right)$ *is the twisted cohomology group defined above, and where the factor* $]0,\infty[$ *corresponds to the length of the generic fiber and occurs in all cases unless when* M *is compact orientable and the Seifert fibration has non-trivial Euler number* $e_0 \in \mathbb{Q}$*.*

For interval bundles, the same argument proves:

Theorem 5.5. *Let the* 3*–manifold* M *be endowed with an (open) interval bundle structure with base surface* S*. For* $X = \mathbb{S}^2$*,* $\mathbb{E}^2$ *or* $\mathbb{H}^2$*, let* $\mathcal{G}_{\mathrm{f}}\left(M; X \times \mathbb{E}^1\right)$ *be the space of complete geometric structures modelled over the spaces indicated, where the* $\mathbb{E}^1$*–factors correspond to the fibers of the interval bundle, where these geometric structures are considered up to fibration-preserving isotopy. Let* $\mathcal{G}\left(\Sigma; X\right)$ *denote the space of isotopy classes of complete geometric structures on the orbifold* Σ *modelled over* X*. Then, the natural map* $\mathcal{G}_{\mathrm{f}}\left(M; X \times \mathbb{E}^1\right) \to \mathcal{G}\left(\Sigma; X\right)$ *is a trivial bundle with fiber* $H^1\left(|\Sigma|\,;\widehat{\mathbb{R}}\right)$*, where* $H^1\left(|\Sigma|\,;\widehat{\mathbb{R}}\right)$ *is the twisted cohomology group defined above.*

6. Applications of 3–dimensional geometric structures

We conclude with a discussion of a few purely topological applications of the use of geometric (mostly, hyperbolic) structures on 3–manifolds. This selection is only intended to give a sample of such applications. It clearly reflects the personal tastes of the author, and is by no means intended to be exhaustive.

6.1. Knot theory

The area where the use of geometric structures, essentially hyperbolic geometry, has had the greatest practical impact is probably knot theory. Let L be a *link* in the 3–sphere $\mathbb{S}^3$, meaning that L is a 1–dimensional submanifold of $\mathbb{S}^3$. A connected link is also called a *knot*. Knot theory aims at classifying all such links up to diffeomorphism of $\mathbb{S}^3$; see for instance the standard references [115, 22]. To show that two links are different modulo diffeomorphism of $\mathbb{S}^3$, the traditional method is to use algebraic topology to extract some algebraic invariants of these links; if the invariants computed happen to be different, this shows that the links are different.

The consideration of hyperbolic structures provides a completely new type of invariants. Indeed, the Hyperbolization Theorem 4.3 shows that the complement $\mathbb{S}^3 - L$ of a link $L \subset \mathbb{S}^3$ admits a finite volume complete hyperbolic structure unless one of the following holds:

(i) $\mathbb{S}^3 - L$ contains an embedded essential 2–torus.

(ii) $\mathbb{S}^3 - L$ admits a Seifert fibration.

In Case (ii), the Seifert fibration of $\mathbb{S}^3 - L$ can be chosen so that it extends to a Seifert fibration of $\mathbb{S}^3$, for which L consists of finitely many fibers of this fibration. Such links are called *torus links*. Since the Seifert fibrations $\mathbb{S}^3$ are easily classified [127, 103], torus links are easily classified.

In Case (i), L is said to be a *satellite link*. The Characteristic Torus Decomposition of Theorem 3.4 provides a canonical factorization of a satellite link into links which are, either torus links, or non-satellite links.

This reduces the analysis of all links to those whose complement admits a finite volume complete hyperbolic structure. As observed in Section 5.1, Mostow's Rigidity Theorem 5.1 implies that any geometric invariant of this hyperbolic structure is a topological invariant of the link complement $\mathbb{S}^3 - L$, and therefore of the link $L \subset \mathbb{S}^3$. Among such geometric invariants, we already mentioned the volume of the hyperbolic structure. A more powerful invariant of the hyperbolic structure is its Ford domain, which we now briefly describe; see for instance Maskit's book [82, Chap. IV.F] for details.

Let M be a non-compact orientable hyperbolic 3–manifold of finite volume. We saw in Theorem 2.9 and in our discussion of Theorem 4.5 that M has finitely many ends and that each end e has a neighborhood U_e isometric to a model H_e/Γ_e, where $H_e \subset \mathbb{H}^3$ is a horoball $\left\{(u,v,w) \in \mathbb{R}^3; w \geqslant w_e\right\}$ and where the group $\Gamma_e \cong \mathbb{Z}^2$ acts by horizontal translations. Adjusting the constants $w_e > 0$, and in particular choosing them large enough, we can arrange that these neighborhoods U_e are pairwise disjoint and have the same volume. For every $x \in M$, consider those arcs which join x to the union of the neighborhoods U_e and, among those arcs, consider those which are shortest. The *Ford domain* of M consists of those x for which there is a unique such shortest arc joining x to the U_e. One easily sees that the Ford domain is independent of the cusp neighborhoods U_e, provided they are chosen sufficiently small and of equal volumes.

To each end e of M is associated a component of the Ford domain, consisting of those points which are closer to U_e than to any other $U_{e'}$. This component is isometric to $\operatorname{int}(P_e)/\Gamma_e$ where P_e is a locally finite convex polyhedron in $\mathbb{H}^3$ which is invariant under the horizontal translation group $\Gamma_e \cong \mathbb{Z}^2$. Here, a *locally finite convex polyhedron* in $\mathbb{H}^3$ is the intersection P of a family of closed half-spaces bounded by totally geodesic planes in $\mathbb{H}^3$, such that every point of P has a neighborhood which meets only finitely many of the boundaries of these half-spaces. The reader should beware of a competing terminology, used by many authors, where the Ford domain is defined as the collection of the polyhedra P_e.

By construction, the polyhedra P_e, endowed with the action of the group $\Gamma_e \cong \mathbb{Z}^2$, are uniquely determined modulo isometry of $\mathbb{H}^3 = \left\{(u,v,w) \in \mathbb{R}^3; w > 0\right\}$ respecting ∞, namely modulo homothety and euclidean isometry of $\mathbb{R}^3$ respecting $\mathbb{H}^3$. In particular these polyhedra P_e, endowed with their action of Γ_e, are geometric invariants of the hyperbolic metric of M, and therefore are topological invariants of M by Mostow's Rigidity Theorem 5.1. In particular, an invariant extracted from the Ford domain is an invariant of M; simple examples include the number of vertices, edges and faces of $\partial P_e/\Gamma_e$, or the way these faces fit together (namely the combinatorial structure of the polyhedral decomposition of $\partial P_e/\Gamma_e$), or the geometry of these faces.

The Ford domain is such a powerful invariant that it is possible to reconstruct M from it. Indeed, it comes equipped with an isometric pairing of its faces. The manifold M is then obtained from the disjoint union of P_e/Γ_e by gluing its faces through this pairing. Actually, in the case where M is the complement $\mathbb{S}^3 - L$ of a link L, reconstructing M from the Ford domain is not sufficient to characterize the link L, since there are different links which have homeomorphic complements; see for instance [115, Sect. 9.H]. However, L is completely determined if we specify the meridians of the components of L, an information which is easily encoded in the groups Γ_e associated to the ends of M; again, see [115, Chap. 9]. For knots, this meridian information is in fact unnecessary by a deep theorem of Gordon-Luecke [43].

The problem is of course to be able to compute this invariant in practice. The first problem is that the Hyperbolization Theorem 4.3 is only an abstract existence theorem, and that the proofs available are non-constructive. The second problem is that, even if we are given a hyperbolic structure on M, for instance under the form of a free isometric properly discontinuous action of a group Γ on $\mathbb{H}^3$ such that $\mathbb{H}^3/\Gamma$ is diffeomorphic to M, it may be hard to explicitly determine the corresponding Ford domain.

The pioneering work in this area was developed by R. Riley [113, 114]. For certain links L in $\mathbb{S}^3$, he used a computer to find finitely many isometries $A_1, \dots, A_n$ of $\mathbb{H}^3$ such that the group Γ generated by the A_i acts freely and properly discontinuously on $\mathbb{H}^3$ and such that Γ is abstractly isomorphic to the fundamental group of $\mathbb{S}^3 - L$; Waldhausen's Theorem 3.11 on homotopy equivalences of Haken 3–manifolds then guarantees that M is diffeomorphic to $\mathbb{H}^3/\Gamma$. Riley also determined the Ford domains of these hyperbolic manifolds. However, the use of a computer raises the question of rounding errors: For instance, if we compute the isometry corresponding to a word in the A_i and if the value provided by the computer is the identity, does this mean that this isometry is really the identity (which is what we need to make sure that the algebraic structure of Γ is the one expected), or does this just mean that this isometry is very close to the identity (which could have dire consequences for the proper discontinuity of the action)? In these examples, once the first set of computations by the computer had provided him with appropriate conjectures on what the generators A_i and the shape of the Ford domain should be, Riley was able to justify these computations *a posteriori* by exact arithmetic computations in a number field. Namely, he then rigorously proved that the A_i provided by the computer could be approximated by isometries generating a group Γ with the required properties, and that the Ford domain determined by the computer was indeed an approximation of the exact Ford domain of Γ.

Riley's group theoretic approach is unfortunately not very efficient from a computational point of view. The software SnapPea [152], later developed by J. Weeks (with collaborators for some additional features), uses a more geometric approach and works incredibly well in practice. Given a link L in $\mathbb{S}^3$, SnapPea computes a hyperbolic structure on the complement $\mathbb{S}^3 - L$, if it exists, and describes its Ford domain. SnapPea also computes various invariants of this hyperbolic structure, as well as hyperbolic structures on 3–manifolds obtained by 'sufficiently complicated' Dehn filling, as in the Hyperbolic Dehn Filling Theorem 4.4.

In practice, SnapPea works very fast for links with a reasonable number of crossings. However, there is a drawback which has to do with rounding errors. Since SnapPea works only with finite precision, its outputs can mathematically only be considered as conjectural approximations to the exact situation. In theory, it is possible to justify these guesses *a posteriori* by using exact arithmetic as in [113, 114], but this is often not workable in practice. In any case, ever since the first versions of SnapPea started circulating, it has established itself as an invaluable tool to study examples, and make and disprove conjectures, in hyperbolic geometry and knot theory. In particular, it has been extensively used to establish useful tables of links and hyperbolic 3–manifolds, with the *caveat* about the theoretical reliability of the output due to rounding errors; see for instance [1, 53, 153].

There is something interesting about the algorithm used by SnapPea to find a hyperbolic structure on a link complement. It is a variation of the famous method used by Thurston in [138] to construct a hyperbolic structure on the complement of the figure eight knot. Namely it decomposes the link complement into finitely many 'ideal simplices', with all vertices at infinity, and tries to put a hyperbolic metric on each of these ideal simplices, so that the metrics fit nicely along the faces and edges of the decomposition. When SnapPea fails to find such hyperbolic structures on the ideal simplices, it uses various combinatorial schemes to modify the decomposition into ideal simplices until it reaches a solution. What is remarkable is that, although this algorithm works extremely well in practice, there is, at this point, no general proof of the Hyperbolization Theorem 4.3 for link complements which is based on this strategy. Conversely, the proofs of Theorem 4.3 mentioned in Section 4.3 are usually non-constructive.

6.2. Symmetries of 3–manifolds

One of the early successes in the use of hyperbolic geometry to study the topology of 3–manifolds was the proof of the following conjecture of P.A. Smith.

Theorem 6.1 (Smith (ex-)Conjecture)**.** *Let $f : \mathbb{S}^3 \to \mathbb{S}^3$ be an orientation-preserving periodic diffeomorphism of the 3–sphere whose fixed point set is non-empty. Then f is conjugate to a rotation of $\mathbb{S}^3 = \mathbb{R}^3 \cup \{\infty\}$ by a diffeomorphism of $\mathbb{S}^3$.*

The original proof, expounded in [133], is a combination of various ingredients, coming from different branches of mathematics. The main idea is to consider the fixed point set L of f, which is a knot in $\mathbb{S}^3$ (the connectedness of L was Smith's original result in [132]). Various minimal surface arguments reduce the problem to the case where the complement $\mathbb{S}^3 - L$ contains no f–invariant essential surface. If there is no such essential surface, the Hyperbolization Theorem 4.3 provides either a Seifert fibration or a finite volume hyperbolic structure on $\mathbb{S}^3 - L$. An easy fundamental group computation in the case of a Seifert fibration, and the use of much more subtle algebraic and number theoretic properties of subgroups of $\mathrm{PSL}_2(\mathbb{C})$ in the case of a hyperbolic structure, then enable one to complete the proof.

This original proof of the Smith Conjecture is now superseded by the Orbifold Geometrization Theorem 4.7.

Other topological applications involve the *symmetry group* of a 3–manifold M with (possibly empty) boundary, defined as the group $\pi_0 \mathrm{Diff}\,(M)$ of isotopy classes of diffeomorphisms of M.

If M is endowed with complete hyperbolic metric with finite volume and with totally geodesic boundary, Mostow's Rigidity Theorem 5.1 says that every diffeomorphism f of M is homotopic to an isometry of the metric. Theorem 3.12 for the case when M is Haken, and Theorem 6.11 of the next section for the general case, show that f is actually isotopic to an isometry. In addition, it is not hard to see that two distinct isometries of M cannot be homotopic. This proves that, if $\mathrm{Isom}\,(M)$ is the isometry group of M, the natural map from $\mathrm{Isom}\,(M)$ to $\pi_0 \mathrm{Diff}\,(M)$ is a bijection.

Theorem 6.2. *Let the 3–manifold M admit a complete hyperbolic structure with finite volume and with totally geodesic boundary (compare Theorems 2.14 and 4.3). Then, there is a finite group G acting on M such that the natural map from G to $\pi_0 \mathrm{Diff}\,(M)$ is a bijection. In particular, the group $\pi_0 \mathrm{Diff}\,(M)$ is finite.*

This improves a result of K. Johannson, who had proved in [63] that $\pi_0 \mathrm{Diff}\,(M)$ is finite when the compact 3–manifold M with boundary is Haken and contains no essential disk, 2–torus or annulus. When M contains an embedded essential disk, 2–torus or annulus, the consideration of Dehn twists along this surface usually implies that $\pi_0 \mathrm{Diff}\,(M)$ is infinite.

In Theorem 6.2, the fact that $\pi_0 \mathrm{Diff}\,(M)$ can be realized by the action of the finite group is a powerful tool. See [15] or [38] for a few applications to problems in classical topology.

Another important property comes from the Orbifold Geometrization Theorem 4.7. Let the 3–manifold M admit a finite volume hyperbolic structure, and let G be a finite group acting on M. We can then consider the 3–orbifold M/G. Suppose that the fixed point set of some non-trivial element of G has dimension at least 1, namely that the singular set of the orbifold M/G is at least 1–dimensional.

The orbifold M/G then satisfies the hypotheses of the Orbifold Geometrization Theorem 4.7. Indeed, every 2–suborbifold of M/G is uniformized by its pre-image in M, and is therefore uniformizable. An essential sphere or torus 2–suborbifold would lift to a 2–sphere, projective plane, 2–torus or Klein bottle in M, which would have to be essential by the fact that all finite group actions on $\mathbb{B}^3$ are standard [90] or by the Equivariant Dehn Lemma [88, 89]; but this would contradict the existence of the hyperbolic structure of M, by Theorem 2.9. See for instance [20, Sect. D] for details.

At this point, Theorem 4.7 asserts that the orbifold M/G admits a finite volume geometric structure. This geometric structure of M/G lifts to a G–invariant geometric structure on the manifold M. Since we already know that M admits a finite volume hyperbolic structure, this G–invariant geometric structure is necessarily hyperbolic by Theorem 2.10. In addition, Mostow's Rigidity Theorem 5.1, together with Theorems 3.12 and 6.11, show that this G–invariant hyperbolic structure is isotopic to the original one. If we use the isotopy to conjugate the group action instead of changing one hyperbolic metric to the other, this proves:

Theorem 6.3. *Let the finite group G act on a 3–manifold M which admits a finite volume hyperbolic structure with totally geodesic boundary. Then the action is conjugate to an isometric action by a diffeomorphism of M which is isotopic to the identity.*

Actually, we only discussed the case without boundary. But, as usual, the case with boundary is easily deduced from this one by consideration of the double manifold DM with the action of the group $G \oplus \mathbb{Z}/2$, where the factor $\mathbb{Z}/2$ acts by exchange of the two halves of DM.

6.3. Covering properties

A classical problem in geometric topology is to decide if a non-compact manifold has finite topological type. A particularly interesting source of non-compact manifolds is provided by (connected) coverings $\widetilde{M} \to M$, where the manifold M has finite topological type. It turns out that the use of hyperbolic geometry can provide some answers to this type of problems. We give here a few samples of the type of results which can be obtained through this approach.

An immediate corollary of the existence theorems for hyperbolic structures is the following:

Theorem 6.4. *Let M be a 3–manifold which admits a complete hyperbolic structure or, more generally, a complete metric of non-positive curvature (compare Theorems 4.3, 4.4, 4.5 or 4.7). Then the universal cover of M is homeomorphic to the euclidean space $\mathbb{E}^3$. More generally, for every connected covering $\widetilde{M} \to M$ where the fundamental group $\pi_1(\widetilde{M})$ is abelian, $\widetilde{M}$ has finite topological type.*

The first statement is a celebrated theorem of Hadamard (see for instance do Carmo [25, Chap. 7] or Eberlein [32, Sect. 1.4], and compare Sections 1.2 and 2.1). The second one follows from the fact that isometric actions of abelian groups on a simply connected manifold of non-positive curvature are easily classified; see [32, Sect. 1.9].

A more subtle geometric argument leads to the following purely topological result.

Theorem 6.5. *Let M be the interior of a compact 3–manifold $\overline{M}$ with boundary which contains no essential 2–sphere, projective plane, 2–torus or annulus, and such that at least one component of $\partial\overline{M}$ has negative Euler characteristic. If $\widetilde{M}$ is a connected covering of M whose fundamental group is finitely generated, then $\widetilde{M}$ has finite topological type.*

A proof can be found in Morgan [97, Proposition 7.1]. The idea is to use the Hyperbolization Theorem 4.3 to endow M with a complete hyperbolic metric. The metric provided by the proof of Theorem 4.3 is of a certain type, called "geometrically finite". A relatively simple observation then shows that the lift of this metric to $\widetilde{M}$ is also geometrically finite (this is where we need the hypothesis that at least one component of $\partial\overline{M}$ has negative Euler characteristic), from which it follows that $\widetilde{M}$ has finite topological type. Presumably, the hypothesis that $\overline{M}$ contains no essential 2–sphere, projective plane, 2–torus or annulus is unnecessary, as the general case should follow from the above one and from the use of the characteristic splittings of Section 3.

Similar results can be obtained from relatively deep results on the geometry of ends of hyperbolic 3–manifolds, obtained by Thurston [139] and the author [17]. Indeed, one of the topological consequences of this analysis is the following result:

Theorem 6.6. *Let the* 3*–manifold M admit a complete hyperbolic metric. Assume that the fundamental group $\pi_1(M)$ is finitely generated and does not (algebraically) split as a non-trivial free product of two groups. Then M has finite topological type.*

This immediately gives the following corollary:

Corollary 6.7. *Let the* 3*–manifold M admit a hyperbolic metric (compare Theorems 4.3, 4.4, 4.5 or 4.7), and consider a covering $\widetilde{M} \to M$ with $\widetilde{M}$ connected. If the fundamental group $\pi_1(\widetilde{M})$ is finitely generated and does not split as a non-trivial free product, then $\widetilde{M}$ has finite topological type.*

We should also include in this section the important residual finiteness property for fundamental groups of 3–manifolds.

Recall that a group G is *residually finite* if, for every non-trivial $g \in G$, there exists a finite index subgroup of G which does not contain g. This property has important algebraic consequences for the group G; see for instance Magnus [79]. However, what is more of interest to topologists is that, when G is the fundamental group $\pi_1(M)$ of a manifold M, the residual finiteness of $\pi_1(M)$ is equivalent to the following topological property: For every compact subset K of the universal covering $\widetilde{M}$, there is a connected *finite index* covering $\widetilde{M}_0$ of M such that the natural projection $\widetilde{M} \to \widetilde{M}_0$ is injective on K. The equivalence between these two properties is an easy exercise, and the topological property is very useful in practice.

The connection with geometric structures is a theorem of Mal'cev [81] which asserts that every finitely generated group of matrices (with entries in an arbitrary commutative field) is residually finite. Since all isometry groups of the 3-dimensional geometries embed in matrix groups, a geometric structure on the 3-manifold M embeds $\pi_1(M)$ in such a matrix group. This proves:

Theorem 6.8. *Let the finite type* 3*–manifold M admit a geometric structure. Then its fundamental group $\pi_1(M)$ is residually finite.*

Let M be a compact Haken 3–manifold. For the characteristic torus decomposition T of Section 3.2, the geometrization results of Sections 4.2 and 4.3 provide geometric structures on the components of $M - T \cup \partial M$, and in particular show that the fundamental groups of these components are residually finite. It is non-trivial to conclude from this that the fundamental group of M itself is residually finite, but this is indeed a result of Hempel [51]:

Theorem 6.9. *Let M be a Haken 3–manifold. Then the fundamental group $\pi_1(M)$ is residually finite.*

Note that residual finiteness is preserved under finite index extensions. Therefore, Theorem 6.9 holds under the slightly weaker hypothesis that the manifold M is *virtually Haken*, namely admits a finite cover which is Haken.

6.4. Topological rigidity of hyperbolic 3–manifolds

Finally, we mention some recent results which prove for hyperbolic 3–manifolds the results obtained by Waldhausen for homotopy equivalences and isotopies of Haken manifolds, as discussed in Section 3.5.

Theorem 6.10. *Let $f : M \to N$ be a homotopy equivalence between a compact hyperbolic 3–manifold M and a 3–manifold N which contains no essential 2–sphere. Then f is homotopic to a diffeomorphism.*

The requirement that the 3–manifold N contains no essential 2–sphere is here only to circumvent any possible counter-example to the Poincaré conjecture. Indeed, it easily follows from the hyperbolic structure of M that every decomposition of the fundamental groups $\pi_1(M) \cong \pi_1(N)$ as a free product must be trivial. Therefore, the hypothesis that N is homotopy equivalent to M already implies that every 2–sphere embedded in N must bound in N a homotopy 3–ball, namely a contractible 3–submanifold of N.

Theorem 6.11. *Let $f_0,\ f_1 : M \to M$ be two diffeomorphisms of a compact hyperbolic 3–manifold M. If f_0 and f_1 are homotopic, then they are isotopic.*

Theorems 6.10 and 6.11 were proved in three steps. First, by an elegant but comparatively simple argument, Gabai proved in [40] that the conclusion of these theorems holds in finite covers of M and N; the main ingredients of this part of the proof are the techniques developed by Waldhausen in [145]. Then, by more elaborate arguments which use in a crucial way the geometry at infinity of hyperbolic 3–space, Gabai was able to prove Theorems 6.10 and 6.11 under the additional assumption that the hyperbolic 3–manifold M satisfies a certain “insulator condition” [41]. He also conjectured that any compact hyperbolic 3–manifold satisfies this insulator condition. This easily translates to a similar conjecture for discrete 2–generator subgroups of the isometry group $\mathrm{Isom}^+\left(\mathbb{H}^3\right) = \mathrm{PSL}_2(\mathbb{C})$. Now, a 2–generator subgroup of $\mathrm{PSL}_2(\mathbb{C})$ is determined by its generators, and therefore by a finite number of complex parameters. It quickly became clear that any counterexample to the conjecture would provide one in a certain compact portion of the corresponding parameter space. Gabai, R. Meyerhoff and N. Thurston then scanned this compact region of the parameter space, and, through a careful control of rounding errors, were able to rigorously prove that every compact hyperbolic 3–manifold M satisfies the insulator condition. The details of this computer-assisted part of the proof of Theorems 6.10 and 6.11 can be found in [42].

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