MATH 8803

LOW-DIMENSIONAL TOPOLOGY AND HYPERBOLIC GEOMETRY

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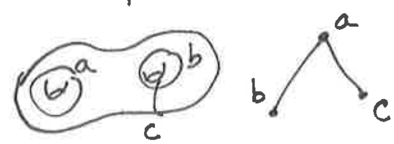
This course has two parts: I. 3-manifolds
II. Complex of curves

Topological objects studied via geometry.

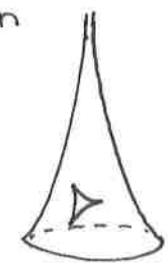


3-Manifolds, Overview

Classification of 2-manifolds mid 19^{th} cent. (closed, orient.)

χ	2	0	-2
geometry	spherical	Euclidean	hyperbolic.

regular octagon in $\mathbb{H}^2$



Gauss-Bonnet: $2\pi\chi = \int K$.

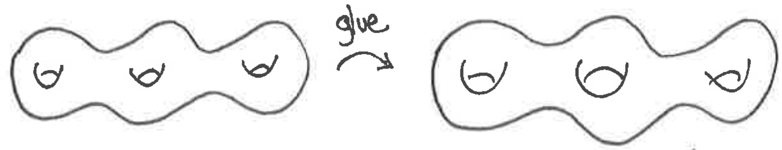
Examples of 3-manifolds

1. S^3

2. $S \times S^1$ e.g. T^3

3. $S^3 \setminus K$

4. Heegaard decompositions



all 3-mans arise this way!

5. Dehn surgery

Cut out solid torus, glue back in.

Lickorish-Wallace: all 3-mans arise from Dehn surgery on S^3.

6. Branched covers

$S^3 \setminus K \longrightarrow$ cov. space $\longrightarrow$ glue $D^2 \times S^1$ back.

Montesinos-Hilden: every 3-man is a 3-fold cover over S^3.

7. Gluing polyhedra

glue faces in pairs, delete vertices if nec.

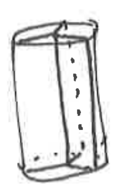
$\frac{8!}{2^4 4!} 3^4 = 8{,}505$ ways to glue faces of octahedron

surface case: $(2n)!/2^n n!$ ways to glue $2n$-gon, most are same!

Later: $S^3 \setminus \text{fig } 8 = 2$ tetrahedra

8. Seifert manifolds

Start with $S \times S^1$, twist by rat'l amount around some fibers



Classification of 3-manifolds - geometrization.

M

cut along spheres, glue balls in — Kneser 1930's

prime pieces

cut along tori — Jaco-Shalen Johannson 1970s

Seifert — complete list 1930s

atoroidal — hyperbolic

Thurston's geometrization conj. → proved for Haken 80's

proved by Perelman in general '03

hyperbolic → Mostow rigidity (60s): hyp struct. is !

Consequences:

① Poincaré conjecture: only simply conn (closed, or.) M is S^3.

Because: no counterexamples among Seifert manifolds (we have a list) or hyperbolic manifolds (π_1 infinite).

② Knot complements are Seifert, toroidal, hyperbolic according to whether the knot is torus, satellite, other.

③ Borel conjecture $n=3$: homotopy equiv $\Rightarrow$ homeomorphic.

PRIME DECOMPOSITION FOR 3-MANIFOLDS

Connect sum

M_1, M_2 closed, conn, oriented ~~3~~ n-mans

$M_i' = M_i \setminus B^n$

$$M_1 \# M_2 = M_1' \coprod_{B^3} M_2' \qquad \text{"connect sum"}$$

Properties: commutative

associative

identity: S^n.

e.g. 

Primes

M is <u>prime</u> if it cannot be written as a nontrivial connect sum ($M \# S^n$ is trivial)

eg. 

Thm (Kneser 1930s) M = closed, conn, or 3-man

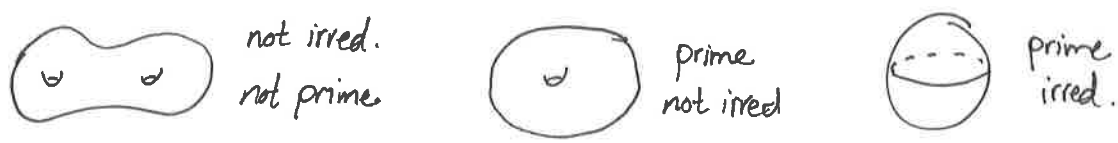
M has a unique prime decomposition.

Preliminaries

Alexander's Thm. Every smoothly embedded S^2 in $\mathbb{R}^3$ bounds a ball.

beware: horned sphere (youtube)
(there are no horned circles: Schönflies thm).

Irreducibles. M is *irreducible* if every S^{n-1} bounds a B^n.



Prop. The only orientable prime, reducible 3-man is $S^2 \times S^1$.

Pf. M prime, reducible
$\longrightarrow$ M has nonseparating sphere S.
Let α = arc in M connecting two sides of S.
$\rightsquigarrow$ $N(S \cup \alpha) \cong (S^2 \times S^1) \setminus B^3$
M prime $\Longrightarrow M = S^2 \times S^1$.

Still need: $S^2 \times S^1$ is prime. Any separating sphere S lifts to $\widetilde{S^2 \times S^1} \cong \mathbb{R}^3 \setminus \{0\}$. By Alexander, the lift bounds a ball. One side of S, ~~[illegible]~~, is simply conn (since $\pi_1(S^2 \times S^1) = \mathbb{Z}$) so it lifts to $\widetilde{S^2 \times S^1}$. This lift is the ball we found. So one side of S is a ball.

EXISTENCE OF PRIME DECOMP.

Step 1. Eliminate $S^2 \times S^1$ summands

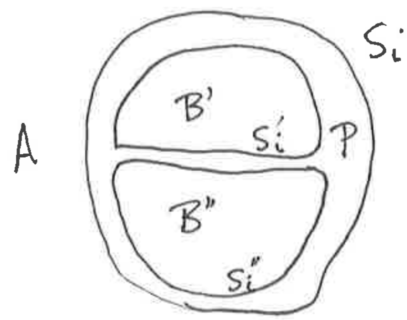
- If M has any nonsep. S^2 then as above there is an $S^2 \times S^1$ summand.
- At most finitely many for homological reasons:

$$H_1(\# M_i) = \oplus H_1(M_i)$$
$$\& \quad H_1(S^2 \times S^1) = \mathbb{Z}.$$

Step 2. $\{S_i\}$ = collection of disjoint spheres with no punctured sphere complementary regions.

D = disk, $D \cap \{S_i\} = \partial D \subseteq S_i$.

S_i', S_i'' obtained from S_i by surgery along D:



Can replace S_i with S_i' or S_i'' to get collection of disjoint spheres with no punc. sphere regions.

Indeed:
- If B', B'' both punc. spheres then S_i bounds a punc. sphere. Say B' not a punc. sphere.
- Then $A \cup B'' \cup P$ also not a punc. sphere. Because $B'' \cup P$ is one, so this means A was a punc. sphere.

Step 3. There is a bound on the # of S_i so $\{S_i\}$ is a collection of disjoint spheres with no punc. sphere regions.

- $\mathcal{T}$ = smooth triangulation of M, say, N simplices.
- Make the S_i transverse to every simplex (induct on skeleta).

Eliminate:

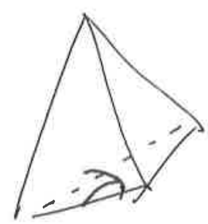
(i) spheres entirely in 3-cell

Alexander thm.

(ii) circles in 2-cell not bounding disk in 3-cell

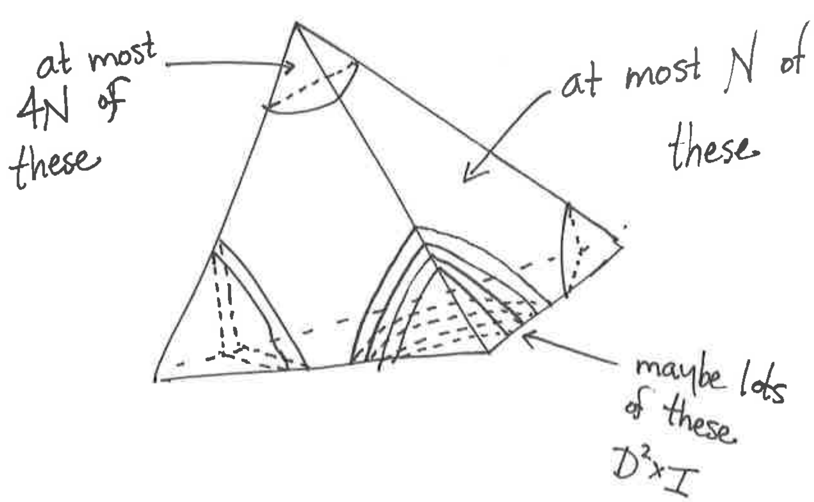
Step 2.

(iii) arcs in 2-cell connecting edge to self



Isotopy.

Now intersections look like:



We'll show the complementary regions containing these $D^2 \times I$ each contribute a $\mathbb{Z}_2$ to $H_1(M)$, so there are finitely many.

Each such region is an I-bundle over a surface with boundary a union of at most 2 spheres.

Two possibilities: ① $S^2 \times I$ = punc. sphere   ruled out!

② Mapping cylinder of $S^2 \longrightarrow \mathbb{RP}^2$ (collapsing I to $\{0\}$ is covering map) $= \mathbb{RP}^3 \setminus B^3$

Since $H_1(\mathbb{RP}^3) = \mathbb{Z}_2$ we are done.

Uniqueness of Prime Decomp.

Idea. Given two sphere systems giving two decomps, use surgery a la Step 2 to make them disjoint. At this point the sphere systems must be parallel.