Complex of Curves - overview

Main object of study: $MCG(S_g) = \pi_0 \operatorname{Homeo}^+(S_g)$
$= \operatorname{Homeo}^+(S_g)/\text{homotopy}$ "mapping class group"

Motivation: ① $MCG(S_g) \cong \operatorname{Out} \pi_1(S_g)$ Dehn-Nielsen-Baer thm
$\rightsquigarrow$ $MCG(S_g)$ is analog of $GL_n\mathbb{Z} \cong \operatorname{Out} \mathbb{Z}^n$

② $MCG(S_g) \cong \pi_1^{orb}(M_g)$ M_g = moduli space of hyp. surfs

③ $MCG(S_g)$ classifies S_g-bundles

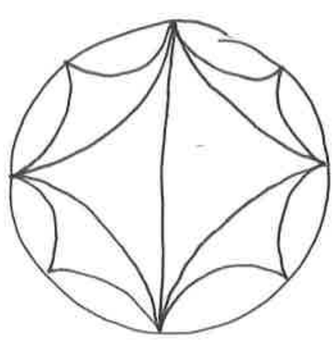
S_g-bundles over B $\leftrightarrow$ $\pi_1 B \to MCG(S_g)$
(already interesting for $B = S^1$).

Main tool: Complex of curves

$C(S_g)$ vertices: homotopy classes of essential simple closed curves in S_g
edges: disjoint representatives.

We'll see $C(S_g)$ is ① connected
② ∞-diam
③ hyperbolic
but ... ④ locally infinite.

For $g=1$ we modify the definition: disjoint $\rightsquigarrow$ minimal



"Farey graph"

HYPERBOLICITY

A geodesic metric space is δ-hyperbolic if for any geodesic Δ, the δ-nbd of any two sides contains the third.

Facts. ① $\mathbb{E}^n$ is not δ-hyp
② $\mathbb{H}^n$ is $\ln(1+\sqrt{2})$-hyp
③ Trees are 0-hyp.

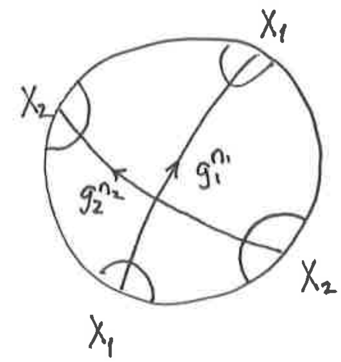
Will show $C(S_g)$ is 17-hyp (indep. of g!)
$\rightsquigarrow$ can import ideas from hyp manifolds to MCG, for instance:

Prop. M = closed hyp n-man
$g_1, g_2 \in \pi_1 M$
Then $\exists\, n_1, n_2$ s.t. $g_1^{n_1}, g_2^{n_2}$ either commute or generate F_2.

Ping Pong Lemma. X = set, $G \circlearrowright X$, $g_1, g_2 \in G$
$X_1, X_2 \neq \emptyset$, $X_1 \cap X_2 = \emptyset$
$g_1^k(X_2) \subseteq X_1$ $g_2^k(X_1) \subseteq X_2$ $\forall\, k \neq 0$.
Then $\langle g_1, g_2 \rangle \cong F_2$

Pf. w = freely red word in g_1, g_2
Say $w = g_1^7 g_2^5 g_1^{-3} g_2 g_1$
Let $x \in g_2$. Note $w(x) \in X_1 \Rightarrow w(x) \neq x \Rightarrow w \neq \text{id}$. ▨

Pf of Prop. Apply PPL to:



$\blacksquare$

This entire approach will generalize to $MCG(S_g) \curvearrowright C(S_g)$.

Curves in Surfaces

Q. How can we tell if two vertices of $C(S_g)$ have disjoint reps?

Prop (Bigon Criterion) Two transverse scc (simple closed curve) in S_g are in minimal position iff they do not form a bigon:

(minimal posn means smallest intersection number in homotopy classes).

Note: $\Rightarrow$ is easy:

Lemma. If two scc do not form a bigon then a pair of lifts to $\mathbb{H}^2$ can intersect in at most one pt.

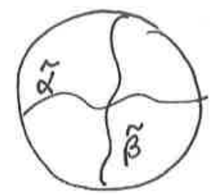
Pf. If not, an (innermost) bigon in $\mathbb{H}^2$ projects to a bigon in S_g $\blacksquare$

Pf of Bigon Criterion (sketch). Assume $\alpha, \beta \subseteq S_g$ form no bigons

Lemma $\Rightarrow$ lifts can only intersect in 1 pt.

Can argue these lifts must have distinct endpts

So:



But isotopies in S_g do not move pts at ∞

So no isotopy can reduce intersection. $\blacksquare$

Geodesics

Prop. Every scc in S_g $(g \geq 2)$ is homotopic to a unique geodesic

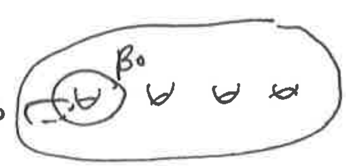
Prop. Geodesics in S_g are in minimal pos.

Change of Coordinates Principle

Configurations of curves can often be put into a standard picture via homeo of S_g.

examples ① If $\alpha \subseteq S_g$ is a nonsep scc in S_g, $\exists\ h \in \mathrm{Homeo}(S_g)$ s.t $h(\alpha) = \alpha_0$

② If $\alpha, \beta \subseteq S_g$ have $i(\alpha, \beta) = 1$ (geometric int num) then $\exists\ h \in \mathrm{Homeo}(S_g)$ s.t. $h(\alpha, \beta) = (\alpha_0, \beta_0)$



Proofs use classification of surfaces.

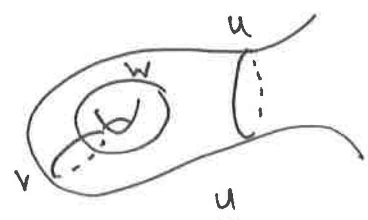
CONNECTIVITY

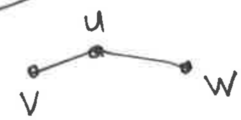
<u>Thm</u> $C(S_g)$ is connected, $g \geq 2$.

Pf. Induction on $i(v,w)$.

For $i(v,w)=0$, nothing to do.

For $i(v,w)=1$, use change of coords:





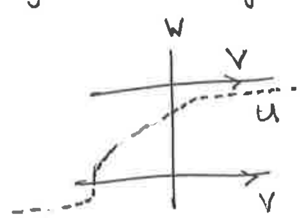
Now assume $i(v,w) \geq 2$.

Orient the curves v, w. and assume minimal pos.

Look at two consecutive intersections along w.

Orientations can agree or disagree.

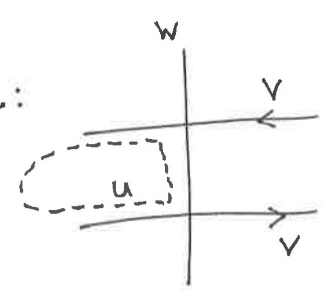
If they agree:



Note u is essential since $i(u,v)=1$.

By induction u connected to v and w.

If they don't agree:



u is essential because otherwise v, w not in min pos.

By induction u conn. to v, w.

∎

HYPERBOLICITY

<u>Thm</u> (Masur-Minsky). $C(S_g)$ is δ-hyp.

We'll show δ can be taken indep of g (Hensel-Przytycki-Webb and others)
Proof from Sisto's blog.

<u>Guessing</u> <u>geodesics</u> <u>lemma</u> (Masur-Schleimer) X = metric graph.
X is δ-hyp iff $\exists\, D$ and $\forall\, x,y \in X^{(0)}$ $\exists$ connected subgraph $A(x,y)$ s.t.

① $d(x,y) \leq 1 \Rightarrow \operatorname{diam} A(x,y) \leq D$.

② $A(x,y) \subseteq N_D\left(A(x,z) \cup A(z,y)\right)$ $\forall\, x,y,z$.

<u>Note</u>. $\Rightarrow$ easy: $A(x,y)$ is any geodesic
$D = \max(\delta, 1)$.

We will replace $C(S_g)$ with $C'(S_g)$. The latter has extra edges, namely, add edges between vertices a,b with $i(a,b) = 1$.

To check: ① $C'(S_g)$ is quasi-isometric to $C(S_g)$
(and constants do not depend on g)

② If X is δ-hyp, Y qi to X then Y is δ'-hyp
(δ' depends only on δ & qi constants).

Note: We need the guessing geodesics lemma precisely because we don't know how to find geodesics. And so it is hard to check δ-hyp'ity directly.

Thm. $C'(S_g)$ is δ-hyp.

Pf. First: $A(a,b) = \{$vertices of $C'(S_g)$ formed from one arc of a, one arc of $b\} \cup \{a,b\}$

each arcs should have distinct endpts

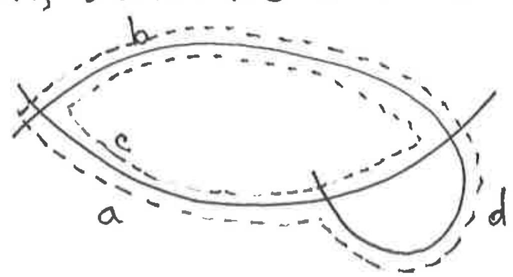
Claim. $A(a,b)$ connected

Pf. Define a partial order $c < d$ if b-arc of d contains the b-arc of c (so d is closer to being b)

this isn't quite a partial order as stated since two curves can have same b-arc but opposite a-arcs

Want for all $c \in A(a,b)$ a $d \in A(a,b)$ s.t. $d > c$ and $\overset{c}{\bullet}\!\!-\!\!\!-\!\!\overset{d}{\bullet}$

To find d, prolong one side of the b-arc of c until it hits a again, shorten the a-arc of c:



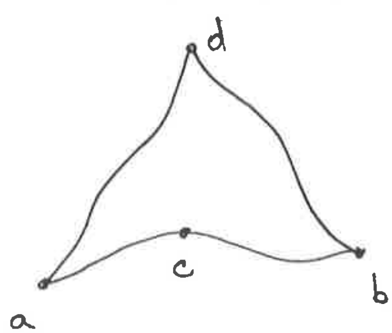
By defn, $d > c$. To see $i(c,d) \leq 1$ note the worst that can happen is the prolonged arc ends up on the wrong side of c.

Notice the $A(a,b)$ satisfy ① since $A(a,b) = \{a,b\}$ when $\overset{a}{\bullet}\!\!-\!\!\!-\!\!\overset{b}{\bullet}$

Claim. The $A(a,b)$ form thin triangles as in ②

Pf. Fix a,b and $c \in A(a,b)$ and d.

Need $e \in A(a,d) \cup A(d,b)$ close to c.

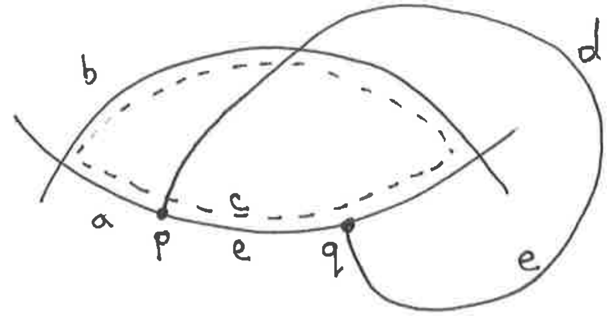


To find e: Consider 3 consec. intersections of d with c

(if fewer than 3, d is already close to c, so e = d).

Say 2 of these intersections are on the a-arc.

call them p, q:



Form e from the arc of d shown and the arc of $c \subseteq a$ as shown.

Note $i(c,e) \leq 2 \Rightarrow d(c,e) \leq 2$. ∎

Guessing Geodesics

see Bowditch "Uniform hyp" Prop 3.1 for a proof of the stronger one.

We'll prove something a little weaker than the lemma used above.

<u>Lemma</u>. (Hamenstädt) X = metric space. Suppose $\exists D$ s.t. $\forall x,y \in X$ there is a path $p(x,y)$ connecting them and so:

1. $\text{diam } p(x,y) \leq D$ if $d(x,y) \leq 1$
2. $\forall x,y$ and $x',y' \in p(x,y)$, $d_{Haus}(p(x',y'),$ subpath of $p(x,y)$ from x' to $y') \leq D$
3. $p(x,y) \subseteq N_D(p(x,z) \cup p(z,y))$ $\forall x,y,z$.

Then X is δ-hyp.

So to prove the theorem, need to either prove the stronger lemma (i.e. eliminate ② above) or check ② for $C'(S_g)$.

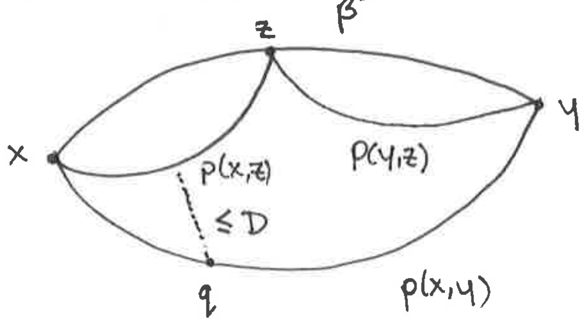
Idea: show the $p(x,y)$ are (close to) geodesics

Pf. Two steps.

Step 1. If β is any path $x \longrightarrow y$ then $p(x,y) \subseteq N_R(\beta)$ where $R \sim \log(\text{length } \beta)$.

recall: in $\mathbb{H}^n$ if a path leaves the R nbd of a geodesic its length is $\sim e^R$.

To prove this, let $q \in p(x,y)$ and split β in half, draw the p paths. Note q is close to one using condition ③.

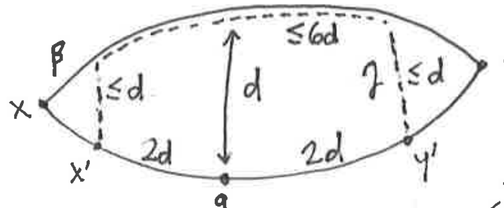


Induct. Base case given by condition ①.

Step 2. Improve this when β is geodesic: $p(x,y)$ is close to β.

Let q = furthest pt on $p(x,y)$ from β.
say $d(q,\beta) = d$.
Pick $x', y' \in p(x,y)$ before/after q at distance $2d$ — actual dist, not dist along $p(x,y)$

Have:



$\ell(\gamma) \le 8d$

$\rightsquigarrow \quad d \le d(q,\gamma) \le O(\log d) \Rightarrow d$ bounded above.

this fn only depends on the constants.

look at pic.   by Step 1 and ② applied to x', y'.

Step 3. β close to $p(x,y)$ (similar)