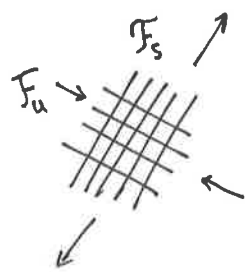
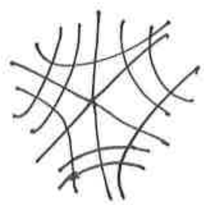
Pseudo-Anosov Mapping Classes and Train Tracks

Nielsen-Thurston Classification. Each $f \in MCG(S)$ has a rep. φ of one of these types

1. finite order $\quad \varphi^n = 1$
2. reducible $\quad \varphi(C) = C \quad C = 1\text{-subman.}$
3. pseudo-Anosov: $\exists$ transverse meas. foliations $(\mathcal{F}_u, \mu_u)$ and $(\mathcal{F}_s, \mu_s)$ s.t.
   $\varphi \cdot (\mathcal{F}_u, \mu_u) = (\mathcal{F}_u, \lambda \mu_u)$
   $\varphi \cdot (\mathcal{F}_s, \mu_s) = (\mathcal{F}_s, \frac{1}{\lambda} \mu_s)$





Analogous classification for $SL_2\mathbb{Z}$:

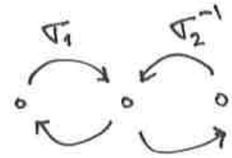
1. $|\text{trace}| = 0,1 \iff$ finite order $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
2. $|\text{trace}| = 2 \iff$ nilpotent $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$
3. $|\text{trace}| \geq 3 \iff$ Anosov $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$
   $\rightsquigarrow$ 2 real eigenvalues, measured foliations*

For T^2, the classifications are the <u>same</u>.

Some questions.
1. How to construct pAs?
2. How to algorithmically determine the NT type?
3. How do pAs act on $C(S)$?

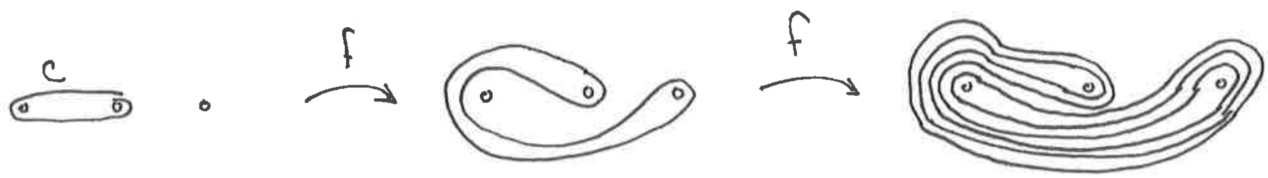
A goal: For f, h pA $\exists$ n s.t. $\langle f^n, h^n \rangle$ is either abelian or free.

Thurston's Train Tracks

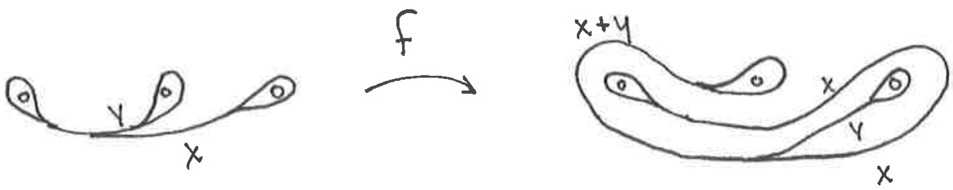
example. 

$f = \sigma_1 \sigma_2^{-1}$

Iterate f on a curve:



Replace with train track:



Transition matrix:

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \rightsquigarrow \underbrace{\lambda}_{\text{PF eigenvalue}} = \frac{3+\sqrt{5}}{2}$$

Eigenvector gives foliation: replace each edge with a foliated rectangle.

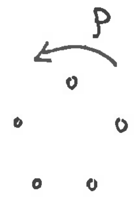
Summary:

mapping class $\overset{\text{guess}}{\rightsquigarrow}$ train track $\rightsquigarrow$ transition matrix $\rightsquigarrow$ eigenvalue/eigenvector $\rightsquigarrow$ foliations/stretch factor

Next: algorithm for finding train tracks.

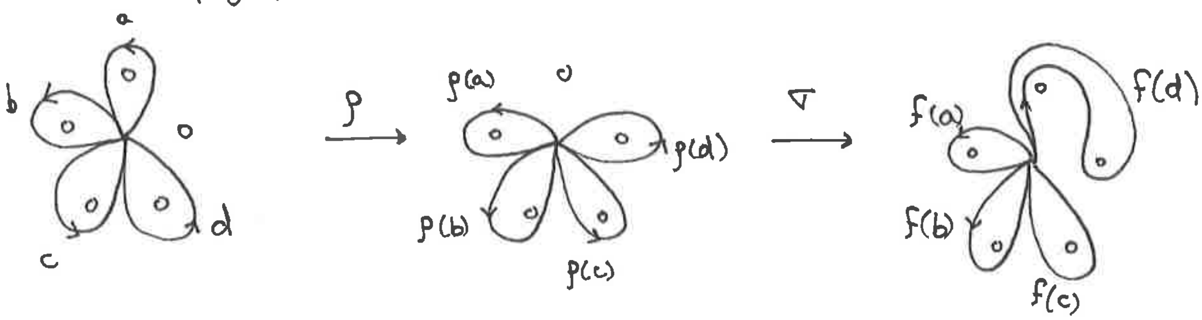
Bestvina-Handel Algorithm

Example on $S_{0,5}$ $=$ S



$f = \sigma\rho$.

Start with any graph (not smooth at vertices) that is a spine for S:



Collapse onto original graph:

$$\begin{aligned} a &\longrightarrow b \\ b &\longrightarrow c \\ c &\longrightarrow d \\ d &\longrightarrow \bar{a}\bar{d}\bar{c}\bar{b} \end{aligned}$$

Main concern: Is there an edge that backtracks under an iterate of f?

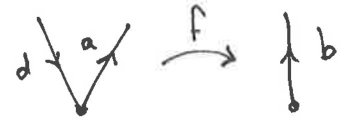
Can see $f^2(d)$ backtracks $\quad d \xrightarrow{f} \bar{a}\bar{d}\bar{c}\bar{b} \xrightarrow{f} \underline{\underline{\bar{b}(b}}cda)\bar{d}\bar{c}$

More systematically, regard half-edges as "tangent vectors"

$\rightsquigarrow$ differential Df:

$$\begin{tikzcd} a \arrow[r] & b \arrow[r] & c \arrow[r] & d \arrow[d] \\ \bar{d} \arrow[ur] & \bar{c} \arrow[l] & \bar{b} \arrow[l] & \bar{a} \arrow[l] \end{tikzcd}$$

$\rightsquigarrow$ illegal turn da (or $\bar{a}\bar{d}$) :

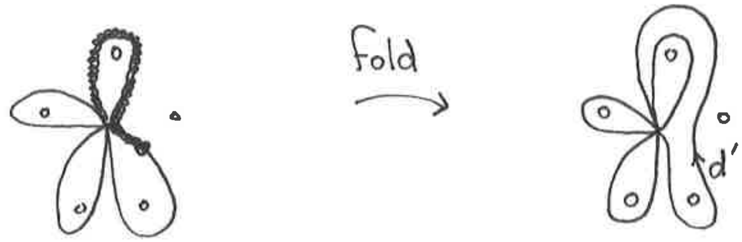


Then check if this illegal turn arises in image of f. As we said, it occurs in $f(d)$.

More generally, illegal turns are pairs of tangent vectors identified by some power of f. Suffices to look at Df.

In our example, last $1/4$ of d, all of a both map to b under f^2.

Folding. We can eliminate the problem by folding, i.e. identify the offending ~~ed~~ (partial) edges right from the start (à la Stallings).



$d = d'\bar{a}$

Get a new map of graphs using $d = d'\bar{a}$ and the fact that d' is the first $3/4$ of d:

$$
\begin{aligned}
a &\to b \\
b &\to c \\
c &\to d'a \\
d' &\to \bar{a}a\bar{d}'\bar{c} \xrightarrow{\text{tighten}} \bar{d}'\bar{c}
\end{aligned}
$$

Does the new map have any illegal turns?

$$
Df: \quad
\begin{tikzcd}
a \arrow[r] & b \arrow[r] & c \arrow[r] & d' \arrow[r] & \bar{d}' \arrow[ll, bend left] \\
 & \bar{c} \arrow[ul] & \bar{b} \arrow[l] & \bar{a} \arrow[l] &
\end{tikzcd}
$$

for instance

Yes: $\bar{b}\bar{d}'$ (and $d'b$).

But: this does not appear in the image of f

exercise: show this really ensures no folding under any iterate.

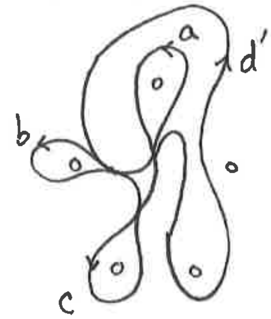
Finding the train track. Identify two tangent vectors if the are identified under some iterate of f (this is an equiv rel).

⇝ 3 equiv classes: $\{a, \bar{a}, d'\}$, $\{b, \bar{b}, \bar{d}'\}$, $\{c, \bar{c}\}$ "gates"

An illegal turn is exactly a pair from one equiv class. (in our convention reverse one of the two vectors)

But no such turn appears in f(edge).

⇝ Make a train track by squeezing together equivalence classes:



Finding the stretch factor. Transition matrix:

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Perron-Frobenius.

⇝ char poly $x^4 - x^3 - x^2 - x + 1$

⇝ PF eigenvalue ≈ 1.722

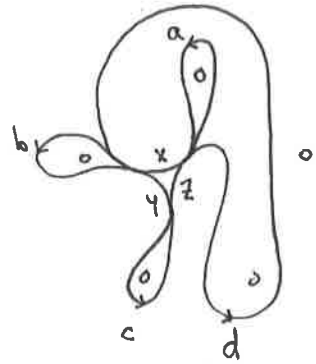
Finding the foliation. PF eigenvector (0.316, .184, .545, .755)

⇝ foliated rectangles instead of edges

⇝ foliation (collapse complementary region

Infinitesimal edges

In the above example we secretly added 3 "infinitesimal edges" $x, y,$ and z:



What Bestvina-Handel tells you to do is to blow up each vertex and add these infinitesimal edges, connecting two gates whenever some $f^n(\text{edge})$ needs to travel between those gates.

$\rightsquigarrow$ augmented graph map: $a \to b$ $\qquad d' \to \overline{d'}\,\overline{zc}$

$b \to c \qquad x \to y \to z \to x$

$c \to d' x a$

$\rightsquigarrow$ augmented matrix:

$$\left(\begin{array}{ccc|cccc} 0&0&1&0&0&1&0\\ 1&0&0&0&0&0&0\\ 0&1&0&0&0&0&1\\ \hline 0&0&0&0&0&1&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&1&0&1\\ 0&0&0&0&0&1&1 \end{array}\right)$$

5^{th} power:

$$\left(\begin{array}{ccc|cccc} 0&1&0&2&4&4&9\\ 0&0&1&0&2&4&4\\ 1&0&0&2&2&6&7\\ \hline 0&0&0&1&2&2&4\\ 0&0&0&0&1&2&2\\ 0&0&0&2&2&5&6\\ 0&0&0&2&4&6&9 \end{array}\right)$$

So each real branch eventually traverses each branch, including infinitesimals. This happens in general.

Hyperbolic Isometries and Free Groups

Goal. f_1, f_2 pA.
If $[f_1, f_2] \neq 1$ then $\exists\, n$ s.t. $\langle f_1^n, f_2^n \rangle \cong F_2$

Idea. Use $MCG(S_g) \circlearrowright C(S_g) \leftarrow \delta$-hyp

Classification of isometries of δ-hyp spaces:

1. elliptic : $\exists$ bounded orbit
2. parabolic : $\exists!$ fixed pt in ∂X
3. hyperbolic: $\exists$ two f.p. in ∂X
   $\rightsquigarrow$ invariant quasigeodesic : take one orbit and connect dots equivariantly.

Prove similarly to $\mathbb{H}^n$.

Prop. $f_1, f_2 \in \mathrm{Isom}(X)$ hyp. isoms w/ distinct fixed pts
$\exists\, n$ s.t. $\langle f_1^n, f_2^n \rangle \cong F_2$

Pf idea. A_i = quasigeodesic axis for f_i
for convenience, say $x_0 \in A_1 \cap A_2$
Take: $X_i = \{ x \in X : d(\pi_{A_i}(x), x_0) \geq M \}$ $\quad$ M large compared to δ.

(This is compatible with our pic for $\mathbb{H}^n$.)

Need to check $X_1 \cap X_2 = \emptyset$.
$f_i^n(X_j) \subseteq X_i$

Easy to see for trees. Then generalize. ∎

Conclusion: Need to show $pA \circlearrowright C(S_g)$ is hyperbolic.

NESTING LEMMA

Train track terminology. τ is <u>recurrent</u> if it has a positive measure

τ is <u>large</u> if all compl. regions are polygons or one-punctured polygons.

A diagonal extension of τ is a track obtained by adding edges with endpts in cusps of τ

$E(\tau)$ = set of diag. ext. of τ.

$P(\tau)$ = polyhedron of non-neg measures

$$PE(\tau) = \bigcup_{\sigma \in E(\tau)} P(\sigma)$$

$\operatorname{int} P(\tau) \subseteq P(\tau)$ all measures strictly pos.

<u>Nesting Lemma</u>. τ = large, recurrent train track.

$$N_1(\operatorname{int}(PE(\tau)) \subseteq PE(\tau)$$

N_1 = 1-nbd in $C(S_g)$.

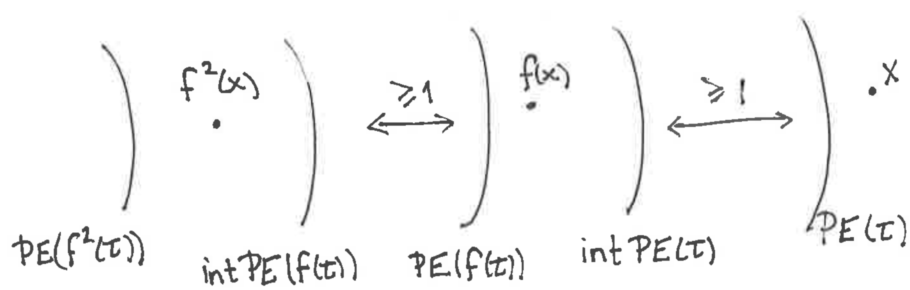
i.e. α carried by diag. ext. of τ,
α passes through each branch of τ
β disj. from α
$\Rightarrow$ β carried by some diag ext. of τ.

(on first pass, can pretend τ is maximal, i.e. $E(\tau) = \tau$; our example has this).

Here is how we apply this: τ = train track for f.

① $f^n(PE(\tau)) \subset \operatorname{int} PE(\tau)$ $\quad$ $n=5$ in above example.

② $N_1(\operatorname{int} PE$



$\Rightarrow$ f acts hyperbolically!

Proof of Nesting Lemma

Let $\alpha \in \operatorname{int} PE(\tau)$

σ = smallest diag ext. of τ carrying α

$\rightsquigarrow \alpha \in \operatorname{int} P(\sigma)$

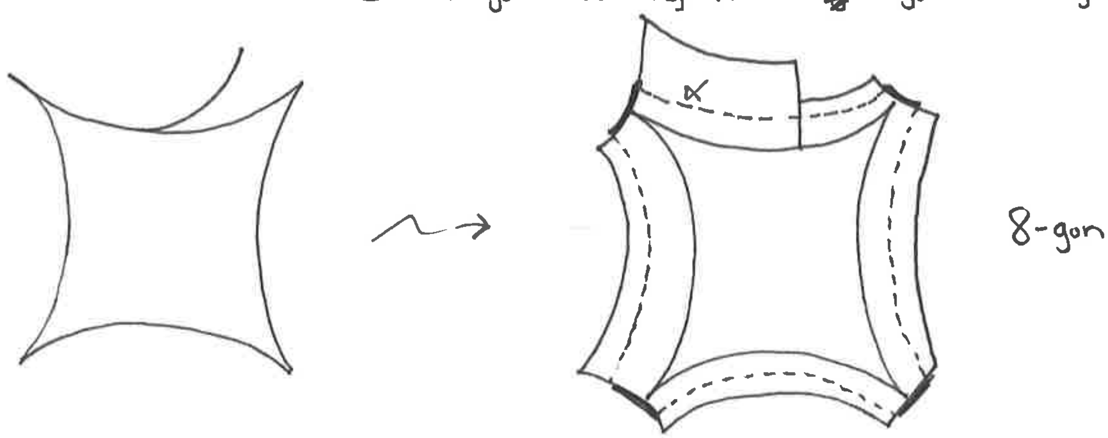
Suffices to show that if $\alpha \cap \beta = \emptyset$ then $\beta \in PE(\sigma)$.

Fatten branches of σ to rectangles; widths given by α.

Cut S_g along α and vertical sides of rectangles.

$\rightsquigarrow$ two kinds of pieces: ① rectangles inside the above rectangles

② $2k$-gons coming from k-gons in $S_g \setminus \sigma$



If $\beta \cap \alpha = \emptyset$ β has no choice but to follow along rectangles as in ① and/or cut across the $2k$-gons. ∎