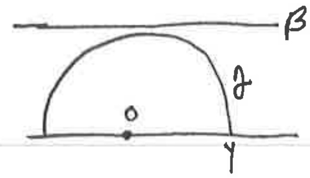
Subsurface Projections

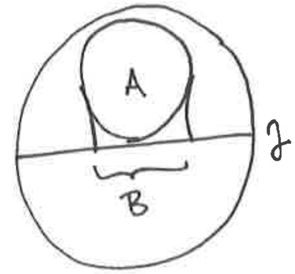
Projections in hyp space

Fact 1. $\exists\, M$ s.t. $\forall$ horocycles β, geod γ with $\beta \cap \gamma = \emptyset$
we have $\pi_\beta(\gamma) \le M$
diam

exercise: $M = 2$ for $\mathbb{H}^2$.



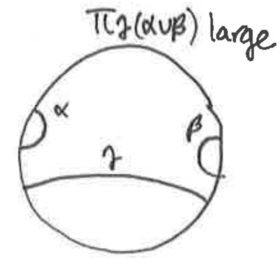
Fact 2. $\exists\, B$ s.t. $\forall$ geod γ, compact ball A with $A \cap \gamma = \emptyset$
$\operatorname{diam} \pi_\gamma(A) \le B$
"contraction property"
exercise: find B for $\mathbb{H}^2$, trees.



Masur-Minsky: If a metric space X has a coarsely transitive path family Γ with the contraction property then X is δ-hyp and elts of Γ are quasi-geodesics.

Fact 3.* $\exists\, C$ s.t. $\forall$ geod α, β, γ disjoint, at most one of
$\pi_\alpha(\beta \cup \gamma)$, $\pi_\beta(\alpha \cup \gamma)$, $\pi_\gamma(\alpha \cup \beta)$
has diam $> C$.
exercise: prove $C = 0$ for trees (see Bestvina-Bromberg-Fujiwara)

* For this fact, need to assume a discrete family of geodesics, e.g. lifts of geodesics in a hyp. surf.



Facts 3,4 work for horocycles as well.

Fact 4. Same discreteness assumption as Fact 3, same C.
For fixed α, the set of geods β with $\operatorname{diam} \pi_\alpha(\beta) > C$ is finite.

Bounded Geodesic Image Thm

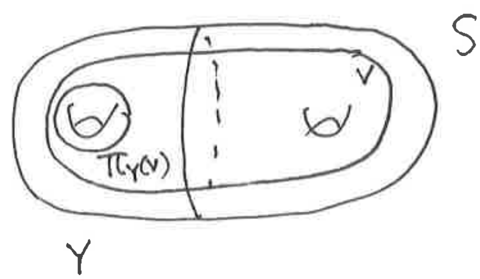
Want analogues of all of these facts. Need analogues of horocycles and projections.

Subsurface projections

S = surface
Y = subsurface
$\rightsquigarrow$ coarsely defined map

$$\pi_Y\, C(S) \longrightarrow C(Y)$$

e.g.



When Y is an annulus, need special definition.
There is a cover $S_Y \longrightarrow S$ corresponding to Y
(induces $\pi_1(S_Y) \xrightarrow{\cong} \pi_1(Y)$).
Can compactify to closed annulus $\overline{S_Y}$
$C(Y)$ has vertices for proper acs in $\overline{S_Y}$, edges for disjointness.
not discrete!
Given $v \in C(S)$ can look at preimage in S_Y hence arc in $\overline{S_Y}$.
(all such are disjoint, so lie in one simplex).
This is $\pi_Y(v)$.

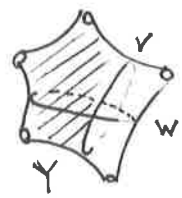
Bounded Geodesic Image Thm

this part relies on uniform hyp'ity.

<u>Thm</u> (Masur-Minsky) $\exists M$ (indep. of S) s.t. if $Y \subsetneq S$ and g is a geodesic in $C(S)$ all of whose vertices intersect Y then $\operatorname{diam} \pi_Y(g) \le M$.

Webb: $M = 100$.

<u>Applications</u> ① Consider



Let $f \in MCG(Y) \subseteq MCG(S)$ pA

Can choose n s.t.

$d_{C(Y)}(w, f^n(w)) > M$.

BGI $\Rightarrow$ every geodesic in $C(S)$ from w to $f^n(w)$ must pass through v.

(similar for v a nonsep curve in S_g).

② A construction of Aougab-Taylor.

Say $d(v_0, v_1) = 3$.

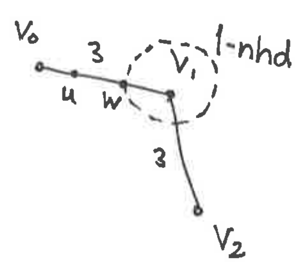
Let $v_2 = T_{v_1}^{M+1}(v_0)$.

Claim: $d(v_0, v_2) = 4$.

Pf: To see ≥ 4 use BGI: any geod $v_2 \longrightarrow v_0$ must pass through 1 nbd of v_1.

To see ≤ 4 find a path:

$v_0,\ u,\ w,\ T_{v_1}^{M+1}(u),\ v_2$



Can keep going: $v_3 = T_{v_2}^{M+1}(v_0)$.

Get distances $6, 10, 18, 34, \ldots$

LEASURE'S QUASIGEODESICS

Problem: compute distance in $C(S)$.

If $C(S)$ were locally finite could do a brute force search for geodesics.

Assume $d(v,w) \geq 3$. Will find a <u>nice</u> $(2,2)$ quasigeodesic $v \rightsquigarrow w$.

Note $v \cup w$ cuts S_g into a union of disks.
A vw-cycle is a loop that intersects each disk in at most one arc
Take a geodesic $v = v_0, \dots, v_n = w$
Truncate each v_i to a vw-cycle v_i' : follow v_i (starting anywhere) and when you return to the same disk twice, do a surgery.

Observation: $i(v_i', v_{i+1}') = 2$
Pf: only intersections are in disks where we did surgery and only one arc of each curve in such a disk.

$\Rightarrow$ $d(v_i', v_j') \leq 2|i-j|$

If $d(v_i', v_j') < |i-j|$, choose a geodesic $v_i' \longrightarrow v_j'$ and convert to vw-cycles again.
At end: $(2,2)$-quasigeodesic.

$\leftarrow$ can get scrunching of more than $1/2$ if you don't do this.

Moral: can approximate distance with uncomplicated curves.
Will do this with BGI.

Proof of Bounded Geodesic Image Theorem (Webb)

$AC(Y)$ = arc and curve complex of Y
qi to $C(Y)$.
$\pi_Y : C^\circ(S) \to P(AC^\circ(Y))$ subsurface proj.

Thm $\exists M$ s.t. if $Y \subseteq S$
$g = (u_i)$ = geod in $C(S)$
with $\pi_Y(u_i) \neq \emptyset \ \forall i$
then $\operatorname{diam} \pi_Y(g) \leq M$

Proof idea: simplify g wrt Y à la Leasure.

vw-loops

$u, v, w \in C(S)$.
Say u is a vw-loop if for each arc $\alpha \subseteq w \setminus v$ either have
① $|u \cap \alpha| \leq 1$
② $|u \cap \alpha| = 2$ and signs of intersection are opposite.

Will apply to $v = \partial Y$, $w = u_i$

To show: Given any $g = (u_i)$, v, w
can replace u_i with u_i' to get
quasigeod $g' = (u_i')$. (like Leasure).

Recipe for vw-loop conversion $u \rightsquigarrow u'$

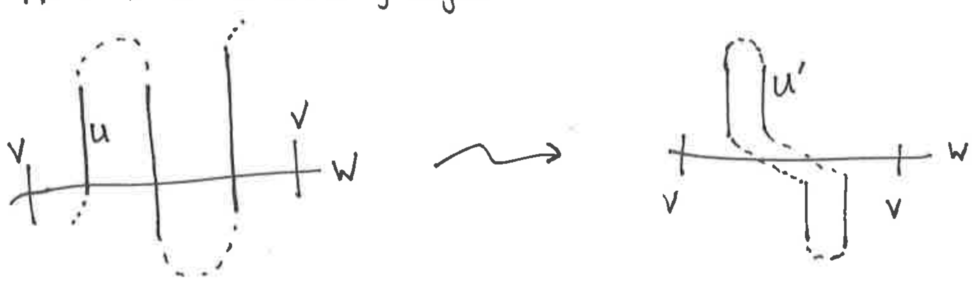
If u already a vw-loop, $u' = u$.
Otherwise, let β = a minimal arc of u failing the defn
note $\partial\beta \subseteq \alpha$ where $\alpha \subset w \setminus v$ is the arc where the failure happens.

Case ① $|\beta \cap \alpha| = 2$, signs of int are same.

Case ② $|\beta \cap \alpha| = 3$, nonalternating signs.

Similar to Case ①

Case ③ $|\beta \cap \alpha| = 3$ alternating signs



Can show u' is ① essential
② in min pos with v, w
③ a vw-loop.

Claim: If we apply this recipe to a geod $g = (u_i)$ we get a path $g' = (u_i')$ that is a $(4,0)$-quasi-geod.

Pf: Same as Leasure. Use $i(u_i', u_{i+1}') \leq 4$. ▨

Now for the magic:

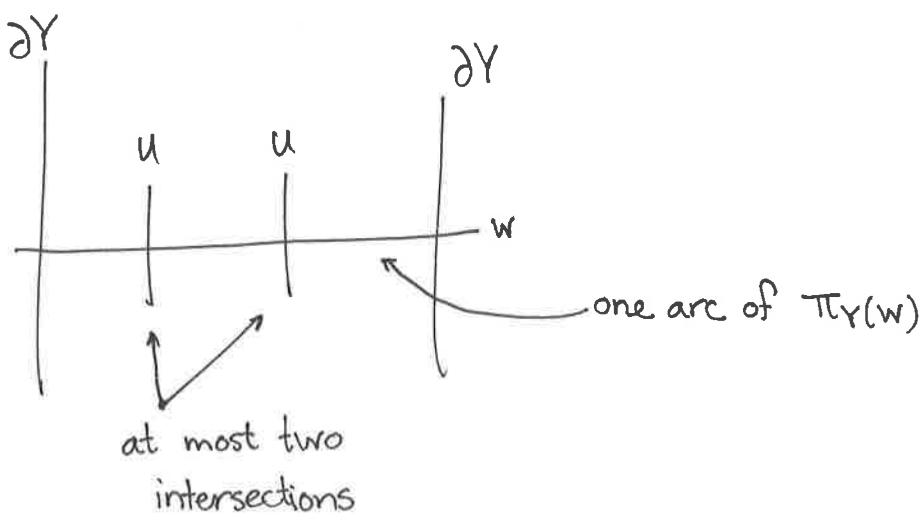
<u>Lemma</u>. $Y \subseteq S$. Say $v \in \partial Y$, w fill* S i.e. $d(v,w) \geq 3$.

$u = vw$-loop, $i(u,v) \neq 0$ i.e. $d(u,v) \geq 2$

Then: ① $d_Y(u,w) \leq 2$ Y nonannular

② $d_Y(u,v) \leq 5$ Y annular.

<u>Pf of ①</u>.



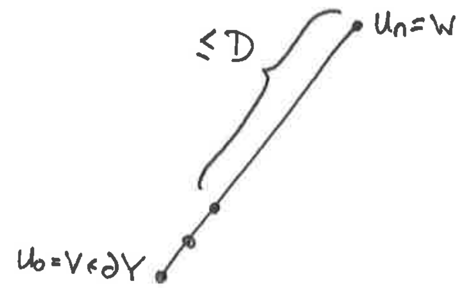
Arcs/curves with at most two intersections cannot fill i.e. cannot have distance 3. ▨

* Webb requires $d \geq 3$ in the claim and the Lemma.

Lemma. $\exists D$ s.t. $\forall\, Y \subseteq S$ $\forall\, v \in \partial Y$

$\forall$ geod $v = u_0, \dots, u_n = w$ $\quad n \geq 3$

have: $d_Y(u_i, u_n) \leq D \quad i \geq 2$.



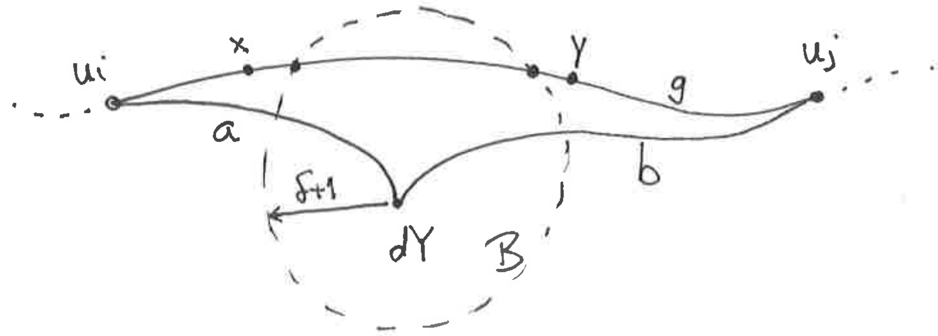
Pf. Replace $g = (u_i)$ with $g' = (u_i')$ a $(4,0)$-quasigeod.

Each u_i is D'-close to g' $\qquad D' = $ fn of $4, \delta$.

So: u_i' close to u_n' in Y by prev. lemma

u_i close to some u_j' (quasigeods are unif close to geods) $\blacksquare$

Proof of Thm. Let $B = (\delta+1)$-ball around ∂Y:



i, j arbitrary

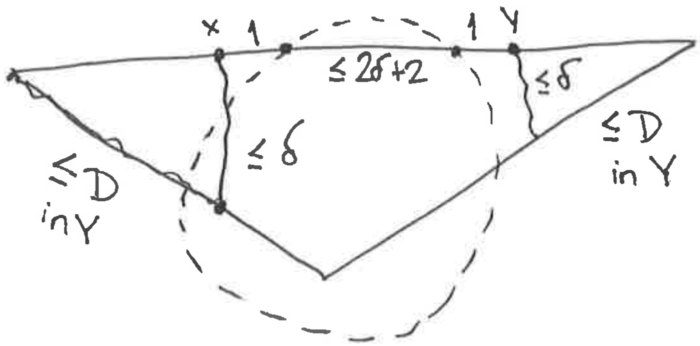
a, b = other two sides of $u_i, u_j, \partial Y$ triangle

x / y = vertices right before/after g passes thru B. (otherwise $x = u_i$, $y = u_j$)

Key: x, y have distance $\delta+2$ from ∂Y so any path of length δ has all vertices intersecting ∂Y.

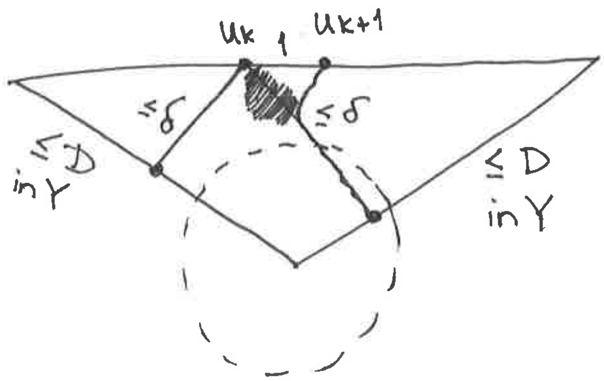
Now, the points of $(u_i, \dots, u_j)$ are within δ of $a \cup b$.
At some point they switch from close-to-a to close-to-b
That can happen in B or out of B.

Case ① x within δ of a
y within δ of b.



Get a path of length $\leq 2D + 4\delta + 4$ in Y.

Case ② $\exists\ u_k, u_{k+1}$ outside B with u_k δ-close to a
u_{k+1} δ-close to b



Get path of length $\leq 2D + 2\delta + 1$. $\blacksquare$

Behrstock Lemma

$\xi(S)$ = complexity = $3g-3+n$ = $\dim C(S)+1$.

__Lemma.__ $Y, Z \subseteq S$ overlapping
$\xi(Y), \xi(Z) \geq 4$.
x = curve with $\pi_Y(x), \pi_Z(x) \neq \emptyset$.
Then $\quad d_Y(x, \partial Z) \geq 10 \Rightarrow d_Z(x, \partial Y) \leq 4$

i.e. can't both be large.
This is analogous to Fact 3 above. (think of x as ∂X).

__Facts.__ Let $U \subseteq S$ $\quad \xi(U), \xi(S) \geq 4$.
$u, v \in C(S)$
a_u, a_v projection arcs in U
$\pi_U(u), \pi_U(v)$ projection curves.

1. $i(a_u, a_v) = 0 \Rightarrow d_U(u, v) \leq 4$
2. $i(u,v) > 0 \Rightarrow i(u,v) \geq 2^{(d_U(u,v)-2)/2}$
3. $i(u,v) \leq 2 + 4 \cdot i(a_u, a_v)$.

Pf of Lemma (Leininger). $d_Y(x, \partial Z) \geq 10 > 2 \Rightarrow$ distance realized by curves $u \in \pi_Y(x)$, $v \in \pi_Y(\partial Z)$ s.t. $i(u,v) \geq 2^4 = 16$ (Fact 2). Now, u & v come from arcs a_u, a_v with $i(a_u, a_v) \geq (16-2)/4 > 3$ (Fact 3). Note $a_u \subseteq x$, $a_v \subseteq \partial Z$. One arc of a_u b/w pts of intersection with a_v lies in Z. This arc is disjoint from x-arcs in Z, so $d_Z(x, \partial Y) \leq 4$ (Fact 1). ▨