Tight Geodesics

Problems with $C(S)$: ① not locally finite ⇝ hard to do algorithms

② MCG action not prop disc ⇝ hard to glean info about MCG.

Will remedy this somewhat.

Tight geodesics

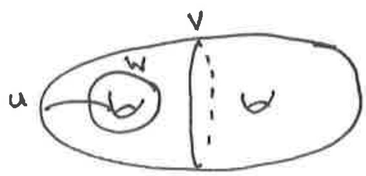
A tight geodesic from v to w is a seq. of simplices

$$v = \sigma_0, \ldots, \sigma_n = w$$

s.t. ① $\sigma_i = \partial F(\sigma_{i-1}, \sigma_{i+1})$ $\quad F =$ span of $\sigma_{i-1}, \sigma_{i+1}$ = smallest subsurface containing both

② $d(v_i, v_j) = |i-j| \quad \forall\, v_i \in \sigma_i, v_j \in \sigma_j \quad i \neq j$.

example.



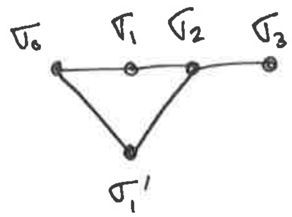
v is the canonical choice to get from u to w.

Tightening

Given a geodesic $v_0, \ldots, v_n$ can tighten at v_i : replace v_i by $\partial F(v_{i-1}, v_{i+1})$

Prop. If we tighten at v_i then tighten at v_{i-1}, result is still tight at v_i. In particular, tight geodesics exist.

Pf. Say $\sigma_0, \sigma_1, \sigma_2, \sigma_3$ already tight at σ_2 and we tighten at σ_1:



New path is still geodesic (it has same length as a geodesic).
$\Rightarrow$ all components of σ_1' & σ_3 intersect
$\Rightarrow$ $F(\sigma_1', \sigma_3)$ connected.

$i(\sigma_1', \sigma_2) = 0 \Rightarrow \sigma_1' \subseteq F(\sigma_1, \sigma_3)$ since $\sigma_2 = \partial F(\sigma_1, \sigma_3)$
$\Rightarrow F(\sigma_1', \sigma_3) \subseteq F(\sigma_1, \sigma_3)$ (use connectedness).

Need: σ_1', σ_3 fill $F(\sigma_1, \sigma_3)$.
So let $\alpha \subseteq F(\sigma_1, \sigma_3)$ and say $i(\alpha, \sigma_3) = 0$.
$\rightsquigarrow$ need $i(\alpha, \sigma_1') \neq 0$.

$i(\alpha, \sigma_3) = 0 \Rightarrow$ $i(\alpha, \sigma_1) \neq 0$ since these pairs fill
$i(\alpha, \sigma_0) \neq 0$ $F(\sigma_1, \sigma_3)$ and S resp.

But $\sigma_1 \notin F(\sigma_0, \sigma_2)$
$\rightsquigarrow$ α must cross $\underbrace{\partial F(\sigma_0, \sigma_2)}_{= \sigma_1'}$ to get from σ_1 to σ_0.

$\blacksquare$

Prop. There are finitely many tight geodesics between two vertices v, w.

Pf. Say $d(v,w) = n$.

Suffices to show $\exists$ finitely many choices for σ_1 on a tight

$$v = \sigma_0, \sigma_1, \ldots, \sigma_n = w$$

Cut S along v.

$\sigma_n = w \rightsquigarrow$ filling simplex of arc complex τ_n

σ_{n-1} also gives filling simplex τ_{n-1}

Note: $i(\tau_n, \tau_{n-1}) = 0$.

Fact: Given a filling simplex τ in arc complex $\exists$ only finitely many simplices τ' with $i(\tau, \tau') = 0$.

By induction, finitely many choices for τ_2.

By tightness, one choice of σ_1 for each choice of τ_2. ∎

In the above argument, we can algorithmically list all the τ_i & σ_i's.

Cor. $\exists$ algorithm to compute distance in $C(S)$.

Pf. Assume have algorithm to distinguish distances $1, \ldots, n-1$ and $> n-1$.

Want an alg to dist. distances $1, \ldots, n$ and $> n$.

Let $v, w \in C(S)$. By induction we can tell if $d(v,w)$ is $1, \ldots, n-1$ or $> n-1$.

If it is $1, \ldots, n-1$ we are done so assume $d(v,w) \geq n$.

Need to tell if $d(v,w)$ is n or $> n$.

List all candidate σ_1's on a tight path of length n as above.

If any such σ_1 has $d(\sigma_1, w) = n-1$ (using induction), $d(v,w) = n$.

Otherwise $d(v,w) > n$. ∎

Applications of tight geodesics

Thm. Any pA in MCG(S) has a power with an honest geodesic axis. ← not. nec. tight!

Pf Sketch. Say f is pA with limit pts $a, b \in \partial C(S)$.

$\mathcal{L}_T$ = set of all tight geodesics from a to b.

$\mathcal{L}$ = set of all geodesics from a to b. ← locally finite!

G = subgraph of $C(S)$ given by union of elts of $\mathcal{L}_T$.

$\mathcal{L}_G$ = set of geodesics contained in G. Note $\mathcal{L}_T \subsetneq \mathcal{L}_G$!

$G/\langle f \rangle$ is finite

$\rightsquigarrow$ label the directed edges $1, \ldots, p$.

Say $\gamma \in \mathcal{L}_G$ is lexicographically least if $\forall\ x, y \in \gamma$ the sequence of labels along γ is lex. least among all geodesics from x to y in G.

$\mathcal{L}_L$ = set of lex. least geods $\subseteq \mathcal{L}_G$.

$\rightsquigarrow$ this is f-invariant.

Claim 1. $\mathcal{L}_L \neq \emptyset$.

Pf. Take longer and longer lex. least geods

local finiteness $\Longrightarrow$ some subseq. converges.

Claim 2. $|\mathcal{L}_L| < \infty$.

Now take any $g \in \mathcal{L}_L$. The finitely many elts are permuted by f so some power of f fixes a geodesic. ▨

Cor. Stable translation length for a pA on $C(S)$ is rational.

$$\tau(f) = \liminf_{n \to \infty} \frac{d(f^n(x), x)}{n}$$