Ingredients for Acylindricity

Thm 1. $d(a,b) \geq 3 \implies |\mathrm{Stab}_{MCG}(a) \cap \mathrm{Stab}_{MCG}(b)| \leq N_0 = N_0(S)$

Pf idea. $a \cup b \rightsquigarrow$ cell decomp of S

topological lemma: any $f \in \mathrm{Stab}(a) \cap \mathrm{Stab}(b)$ has a rep that preserves the cell decomp.

The resulting auto. of the cell decomp is determined by where it sends one 2-cell.

But the number of nonrectangular 2-cells is at least one and is bounded by a fn of S. $\boxtimes$

$G(a,b;r)$ = curves that lie on some tight geod. from a' to b' where $d(a,a') \leq r$, $d(b,b') \leq r$.

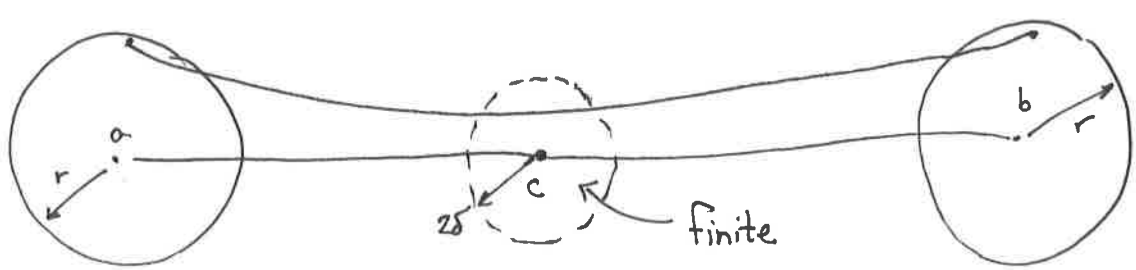
Thm 2. Fix $r \geq 0$.

$a, b \in C(S)$ with $d(a,b) \geq 2r + 2(10\delta+1) + 1$

$c \in \gamma$ = geod. from a to b.

$c \notin N_{r+10\delta+1}(a) \cup N_{r+10\delta+1}(b)$

$\rightsquigarrow \quad |G(a,b;r) \cap N_{2\delta}(c)| \leq D = D(S)$



Proof of Acylindricity

$R = 4r + 24\delta + 7$

$N = N_0 (2r + 4\delta + 1)(8\delta + 7) D$

Say $d(a,b) \geq R$

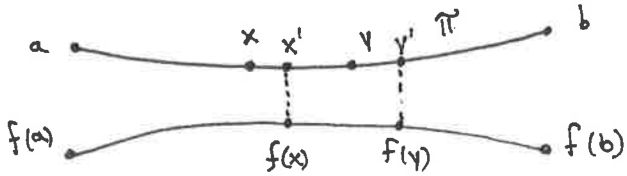
Pick $x, y \in \pi$ = tight geod from a to b.

s.t. ① $d(x,y) = 3$

② $d(\{x,y\}, \{a,b\}) \geq r + (10\delta + 1) + (2\delta + r) + 1$

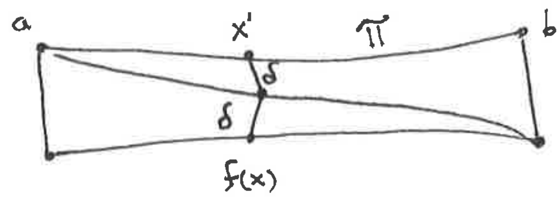
Say $f \in MCG(S)$ with $d(a, f(a)) \leq r$, $d(b, f(b)) \leq r$

Let x', y' proj's of $f(x), f(y)$ to π.



Claim 1. $d(f(x), \pi) \leq 2\delta$, $d(f(y), \pi) \leq 2\delta$

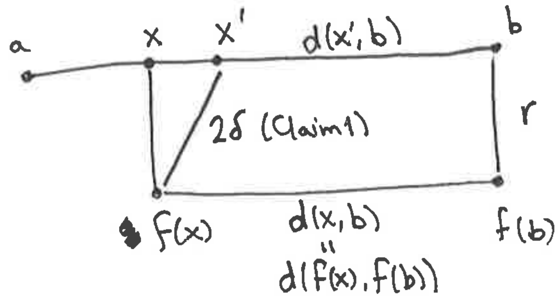
Pf.



Use δ-thinness plus fact that $f(x)$ is far from the vertical sides.

Claim 2. $d(x,x') \le r+2\delta \qquad d(y,y') \le r+2\delta$

Pf. Assume x' to right of x:

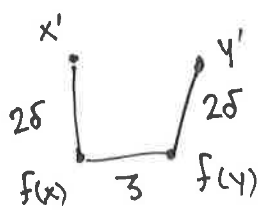


$$\begin{aligned} d(x,x') &= d(x,b) - d(x',b) \\ &\le (2\delta + d(x',b) + r) - d(x',b) \\ &= 2\delta + r \end{aligned}$$

If x' to left of x, replace b with a.

Claim 3. $d(x',y') \le 4\delta+3$

Pf.



Claim 4. $d(x',a)\,,\ d(y',b) \ge r+10\delta+2$

Pf. Immediate from Claim 2 & choice of x,y.

Claim 5. At most $2r+4\delta+1$ choices for x'.

Pf. Immediate from Claim 2.

Claim 6. Given x', at most:

- $(2r+4\delta+1)D$ choices for $f(x)$. Claim 4 + Thm 2
- $8\delta+7$ choices for y' (Claim 3)
- $(8\delta+7)D$ choices for $f(y)$ Claim 4 + Thm 2

Acylindricity now follows from Thm 1, with N as above. ∎

BOTTLENECKS

Remains to prove Thm 2. Here is a simpler version.

<u>Thm.</u> $a, b \in C(S)$

$c \in \pi$ = geod. from a to b.

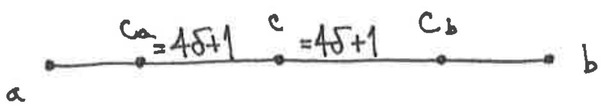
$\rightsquigarrow \; |G(a,b) \cap N_\delta(c)| \leq D.$

$G(a,b) = G(a,b;0) =$ set of curves lying on some tight geod from a to b.

Pf. For simplicity, assume c is far from a,b:

$$d(c, \{a,b\}) \geqslant 4\delta + 1.$$

Choose c_a, c_b:



Enough to show that each elt of $G(a,b) \cap N_\delta(c)$ also lies on a tight filling multipath* from c_a to c_b of length at most $12\delta + 2$.

Indeed, when we gave the algorithm for distance we showed there is a constant $B = B(S,L)$ s.t. the number of curves that can lie on a tight filling multipath of length $\leq L$ is bdd above by B.

* A tight path (v_i) where $|i-j| \geqslant 3 \Rightarrow v_i, v_j$ fill.