THE DISTANCE FORMULA

σ = finite set of vertices of $C(S)$ that fill S

$$[x]_M = \begin{cases} 0 & x \le M \\ x & x > M \end{cases}$$

<u>Thm</u> (Masur-Minsky) Let $f \in MCG(S)$

$$|f| \asymp \sum_{Y \subseteq S} [d_Y(\sigma, f(\sigma))]_M$$

word length

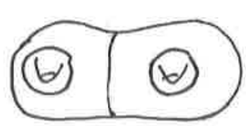
up to bounded mult. & add. error

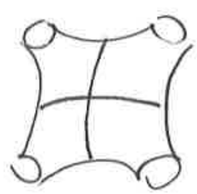
To prove this:

words in MCG $\longleftrightarrow$ moves on pants/markings $\longleftrightarrow$ hierarchies of geodesics in $C(S)$

Idea of hierarchy: a geodesic in $C(S)$ can be thickened to a path in pants complex or marking complex

<u>Pants complex</u>

Vertices  pants decomposition
= max simplex in $C(S)$

edges  or  elementary move

Marking complex: add twisting info

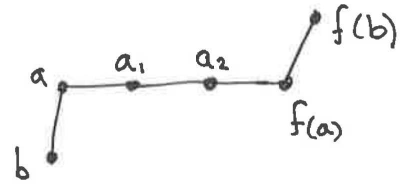
Example : $S_{0,5}$

pants dec. = edge in $C(S_{0,5})$

Let $f \in MCG(S_{0,5})$

a — b = pants decomp

and geod from a to $f(a)$:



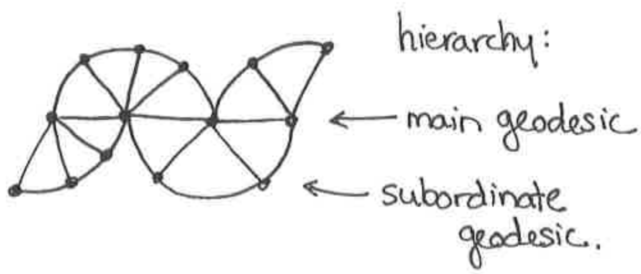
Key idea: can connect b to a_1 in $C(S_{0,5} \setminus a)$ = Farey graph

$b = c_0, \ldots, c_m = a_1$

Each $(a, c_i) \longrightarrow (a, c_{i+1})$ is an edge in pants complex

Repeat for a_1, etc.

Get this picture:



subordinacy of geods $\approx$ nesting of domains.

A hierarchy can be <u>resolved</u> into a seq of parts decomps (or markings) each of which can be thought of as a <u>slice</u> of the hierarchy.

<u>Thm.</u> Any resolution of a hierarchy into a seq of complete markings is a quasigeod. in the marking complex

In general we construct hierarchies inductively as above.
Hyperbolicity $\Rightarrow$ choices of geodesics at each stage are essentially unique.
But more is true.

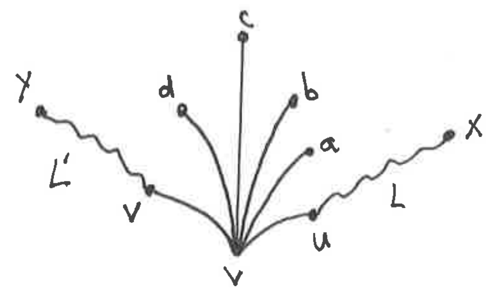
Common Links: If two hierarchies connect nearby pants dec/markings then they have (essentially) the same (long) geodesics (in the same domains).

Large Links: If two markings m_1, m_2 have $d_Y(m_1, m_2)$ large then any hierarchy connecting m_1 to m_2 has Y as a domain. The length of the corresp. geod is roughly $d_Y(m_1, m_2)$.

Both follow from Bounded Geodesic Image Thm.

<u>Example</u>: Genus 1 (Farey graph)

<u>Prop</u>. If a geodesic $x, \dots, u, v, w, \dots, y$ has $d_v(u, w) \geq 5$ then any geod from x to y must pass thru v.



<u>Pf</u>. Key: every edge of Farey graph separates.
Say h is a path x to y avoiding v.
Key $\Rightarrow$ h passes thru a, b, c, d
Also: $d(x, a) \geq L$ (otherwise original path not geod).
$\Rightarrow$ length$(h) \geq (L+2) + (L'+1) >$ length of original geod. $\blacksquare$

Exercises: ① Still true if h connects x', y' adjacent to x, y
② Also h must enter $Lk(v)$ within 1 of u, w $\Bigg\}$ (Large/Common Links)

Example: Genus 2

$g = \ldots, u, v, w, \ldots$ geodesic in $C(S_2)$

g' = fellow traveler - say endpts are distance ≤ 1 from those of g.

Say distance from u, v, w to endpts of g is $\geq 2\delta + 2$

Hyperbolicity $\Rightarrow$ g & g' are $2\delta+1$ fellow travelers

Suppose $d_Y(u,w) > 32\delta + 28$ $\quad Y = S_2 \setminus v$.

Want to show : · g' must pass through/near v

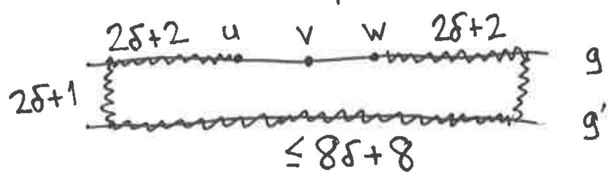
· there is a geod in the g' hierarchy close to the geod in the g-hierarchy corresp. to Y.

Case 1. v nonsep.

We claim g' must pass thru v.

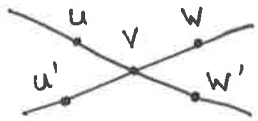
Shortcut argument: If not, each vertex of g' intersects Y.

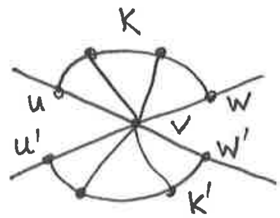
Consider this path:



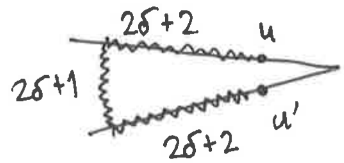
Each pt on the path intersects Y except u, w and length of path $\leq 16\delta + 14$

$\rightsquigarrow$ path in $C(Y)$ of length $\leq 32\delta + 28$ (project), a contradiction.

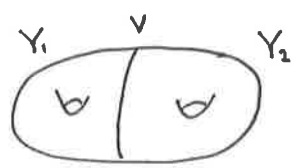
So we have: 

$\rightsquigarrow$ can continue the hierarchy: 

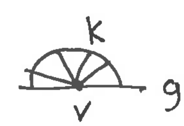
<u>Claim.</u> $d_Y(u,u') \leq 6\delta + 10$

<u>Pf.</u> similar shortcut argument: 

Since u,u' and v,v' close, the geodesics K,K' are close.

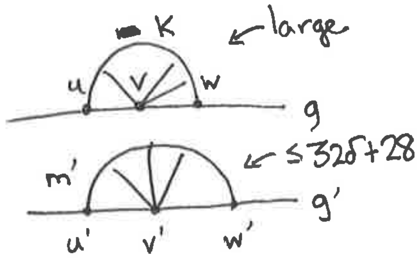
<u>Case 2</u> v separating. 

u, w must lie in same side, say Y_1.
Shortcut argument $\Rightarrow$ some curve v' of g' must miss Y_1
(still assuming $d_{Y_1}(u,w) > 32\delta + 28$).
$\Rightarrow$ $v' = v$ or v' essential in Y_2 (and is nonsep).
Suppose the latter.

Set $Y' = S_2 \setminus v'$ Goal: find geod in g'-hierarchy close to 

Shortcut argument $\Rightarrow$ $d_{Y'}(u',w') \leq 32\delta + 28$
(otherwise, by Case 1 g must pass thru v';
this is a contradiction since $d(v,v') = 1$,
$v' \neq u,v,w$ and this would mean g
not geodesic).

Now have:

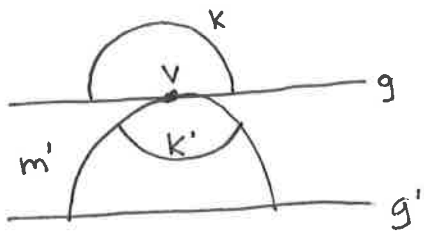


Claim that m' must have a vertex z missing Y_1.

Suppose not. $\rightsquigarrow$ can find a path u to w missing v' and of (small) bounded length and so each vertex intersects Y_1, contradicting largeness of k.

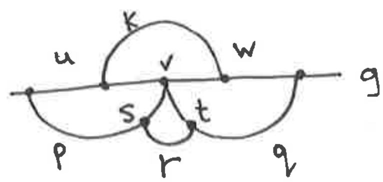
z misses $v', Y_1 \Rightarrow z = v$.

$\rightsquigarrow$ have:



$\rightsquigarrow$ construct k'. Similar arguments as before $\Rightarrow$ k close to k'. $\boxtimes$

None of k, k', m' have Y_2 as domain. But if we continue the g hierarchy, we will see Y_2:



The geodesic r lies in Y_2.

[say: r is forward subord. to q, backwards subord to p]

<u>Resolving the hierarchies</u>

g' : v' (bottom level), v (next level), any $x \in k'$ form a pants decomp. = slice.

If x' is successor of x along k' then $(v'; v, x) \longrightarrow (v'; v, x')$ is elem. move.

g : v, $a \in k$, $b \in r$ $\rightsquigarrow$ (v, a, b) = pants decomp.

Again: to really understand MCG, need markings (pants + twisting data).

AN MCG ACTION ON QUASI-TREES.

Bestvina–Bromberg–Fujiwara: We have subsurface projections that behave like closest point projections in a δ-hyp space?
So is there an ambient δ-hyp space lurking?

Setup: $\mathcal{Y}$ = collection of metric spaces
$\pi_X(Y)$ = projection of X to Y $\qquad \forall X, Y \in \mathcal{Y}$
$M \geq 0$

Axioms:
0. $\forall X, Y \in \mathcal{Y}$ diam $\pi_X(Y) \leq M$
1. $\forall X, Y, Z \in \mathcal{Y}$ at most one of
$$d_X(Y,Z) \quad d_Y(X,Z) \quad d_Z(X,Y)$$
is $> M$.

$d_A(B,C) =$
diam $\pi_A(B) \cup \pi_A(C)$

2. $\forall X, Y \in \mathcal{Y}$
$$\{Z \in \mathcal{Y} : d_Z(X,Y) > M\}$$
is finite.

Examples.
① $\mathcal{Y}$ = set of horizontal lines in $F_2 = \langle a, b \rangle$
= axes for conjugates of a
② $\mathcal{Y}$ = set of lifts to $\mathbb{H}^2$ of geodesic $\gamma \subseteq S_g$.
③ $\mathcal{Y}$ = set of $\mathcal{C}(Y)$ $\quad Y \subseteq S_g$
(really a subset where all Y pairwise intersect).

In example 3, what is the ambient space?

Thm (BBF) $\exists$ geodesic metric space $\mathcal{C}(\mathcal{Y})$
that contains isometric, totally geodesic, pairwise disjoint
copies of the $Y \in \mathcal{Y}$.
and so $\forall$ $X, Y \in \mathcal{Y}$ the nearest pt proj of Y to X
in $\mathcal{C}(\mathcal{Y})$ is uniformly close to $\pi_X(Y)$.

There's more...

Quasi-trees

A quasi-tree is a geod. metric space quasi-isometric to a tree.

Asymptotic dimension

How to assign dim to a gp? Want $\dim(F_n) = 1$, $\dim \pi_1(S_g) = 2$, etc.

A metric space X has $\operatorname{asdim}(X) \le n$ if $\forall$ $R > 0$ $\exists$ covering of X by unif. bdd sets s.t. every metric R-ball intersects at most $n+1$ of the sets.
(large-scale analog of covering dim).

examples:
1. $\operatorname{asdim} \mathbb{Z}^n = n$
2. $\operatorname{asdim} F_n = 1$
3. $\operatorname{asdim} \pi_1 S_g = 2$
4. $\operatorname{asdim} F = \infty$ (Thompson's gp F contains $\mathbb{Z}^\infty$).

$\operatorname{asdim} G < \infty \Rightarrow G \hookrightarrow$ Hilbert space $\Rightarrow$ Novikov higher signature conj:
$\exists$ invariant of smooth type of $K(G,1)$
(defined in terms of pi)
which is really a homotopy invt.

Thm (BBF). $\mathcal{C}(\mathcal{Y})$ also satisfies:

(i) the construction is equivariant wrt any group action on $\amalg Y$ that respects projections

(ii) if each Y is isometric to $\mathbb{R}$, $\mathcal{C}(\mathcal{Y})$ is quasi-tree

(iii) if $\amalg Y$ is δ-hyp, $\mathcal{C}(\mathcal{Y})$ is δ'-hyp.

(iv) if asdim $\amalg Y \leq n$ then asdim $\mathcal{C}(\mathcal{Y}) \leq n+1$.

(ii) $\Rightarrow$ $\mathcal{C}(\mathcal{Y})$ is a quasi-tree in example ② above, not $\mathbb{H}^2$!

Projection Complex

$\mathcal{P}(\mathcal{Y}) = \mathcal{C}(\mathcal{Y})/\mathcal{Y}$ space obtained by collapsing each $Y \in \mathcal{Y}$ to pt.

Thm (BBF). $\mathcal{P}(\mathcal{Y})$ is a quasi-tree.

Example

M^3 = closed hyp. 3-man

$\gamma \subseteq M$ geod.

$\mathcal{Y}$ = lifts of γ to $\mathbb{H}^3$.

$\rightsquigarrow$ action of $\pi_1(M)$ on quasitree where γ acts loxodromically.

Note: Any action of $\pi_1(M^3)$ on actual tree has a global fixed pt.

The Construction

Basic idea: Say Y is between X and Z if

$$d_Y(X,Z) \geq D$$

We connect each pt of X to each point of Z by a segment of length 1 if $\nexists$ Y between.

Mapping Class Groups

Goal: $MCG(S)$ equivariantly quasi-isometrically embeds in a finite product of quasitrees:

$$\mathcal{P}(\mathcal{Y}_1) \times \cdots \times \mathcal{P}(\mathcal{Y}_n)$$

For all $Y, Y' \in \mathcal{Y}_i$ $\pi_Y(Y')$ is defined, i.e. need to color the subsurfaces of S by finitely many colors s.t. disjoint subsurfs have diff colors.

Cor: $\operatorname{asdim} MCG(S) < \infty$.

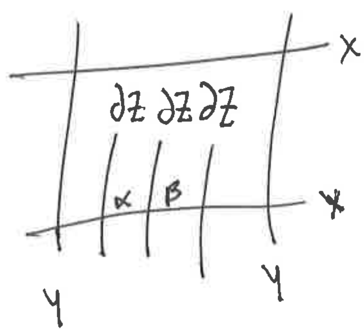
To get the qi embedding use the fact that each ∞-order elt of MCG acts loxodromically on ~~the~~ $C(Y)$ for some $Y \subseteq S$.

Axiom 2 for MCG

We'll prove something more general.

Lemma. $x, y \in C(S) \rightsquigarrow \exists$ finitely many $Z \subseteq S$ s.t.
$$d_Z(x,y) > 3.$$

Pf. Assume first x, y fill.
If $i(x, \partial Z) + i(y, \partial Z)$ large, see:



$\Rightarrow \exists$ arc of $x-y$ (or $y-x$) lying in Z and disjoint from y (namely α or β).
$\Rightarrow d_Z(x,y) \leq 3$
$\rightsquigarrow$ finite list of Z.

In general, let $R \subseteq S$ be subsurf filled by $x \cup y$.
If $Z \nsubseteq R$ $\exists$ curve in Z disjoint from $x \cap Z$, $y \cap Z$.
$\Rightarrow d_Z(x,y) \leq 2$.
If $Z \subseteq R$ we are in filling case with S replaced by R. $\blacksquare$