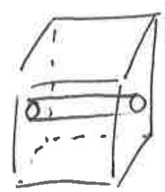
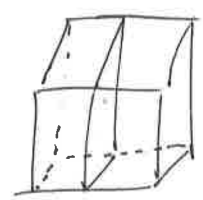
Torus Decompositions

Last time: cut M along spheres $\rightsquigarrow$ prime pieces
This time: cut irred M along tori $\rightsquigarrow$ atoroidal pieces
Next time: uniqueness

Incompressible surfaces

M = closed, conn, or 3-man
$S \subseteq M$ closed, conn, or surface. $S \neq S^2$.
S is incompressible if $\forall\, D \subseteq M$ with $D \cap S = \partial D$
$\exists\, D' \subset S$ with $\partial D' = \partial D$.
e.g. $T^2 \subseteq T^3$:


compressible


incompressible

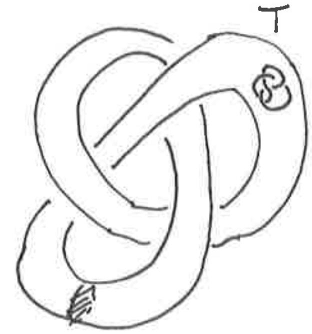
Some facts:

① $\pi_1(S) \hookrightarrow \pi_1(M) \implies S$ incompressible
(converse also true but harder).

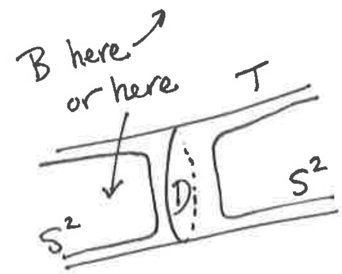
② No incompressible surfaces in S^3.

③ $T \subseteq M$ irred, or.
T compressible $\iff$ T bounds a solid torus
or lies in a ball.

example of 2^{nd} type :



<u>Pf</u>. T compressible along D
$\rightsquigarrow$ surger T along D to produce S^2
$\rightsquigarrow$ ball B bounded by S^2 (irreducibility)



Case 1. $B \cap D = \emptyset$
$\rightsquigarrow$ reverse surgery to get solid torus.

Case 2. $D \subseteq B$
$\rightsquigarrow$ $T \subseteq B$.

④ $T \subseteq S^3$ bounds a solid torus on one side or other.
Use ②+③. In Proof of ③ have a ball on both sides
by Alexander, so suffices to consider Case 1.

Exercise. $S^3 \setminus K$ toroidal $\Rightarrow$ K satellite.

⑤ $S \subseteq M$ incompressible. M irred $\iff$ $M|S$ irred
($M|S$ = M cut along S.)

⑥ $S \subseteq M$ incomp or S^2. $T \subseteq M$ incompressible $\iff$ $T \subseteq M|S$ incompressible.
$T \cap S = \emptyset$.

Existence of Torus Decomps

Irreducible M is <u>atoroidal</u> if every incompressible torus is ∂-parallel.

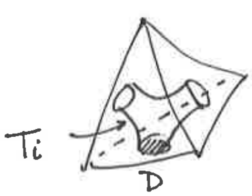
<u>Thm.</u> M = closed, conn, or, irred 3-man

There is a finite collection T of disjoint incompressible tori s.t. $M|T$ is atoroidal.

<u>Pf.</u> Want a bound on # components in a system $T = T_1 \cup \cdots \cup T_n$ of disjoint, non-parallel incomp. tori in M (similar to prime decomp).

Make T transverse to triangulation. Two simplifications

① Make each intersection of T with 3-cell union of disks.

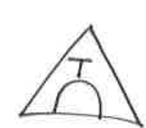
If see 

incompressibility $\to$ disk $D' \subseteq T_i$

irreducibility $\longrightarrow$ ball with $\partial = D \cup D'$

$\rightsquigarrow$ can push this intersection away (no surgery needed!).

Note: ① $\Rightarrow$ no intersection of T with 2-cell is circle (would get disk on both sides, hence sphere) ~~[illegible]~~

② Eliminate intersections of T with 2-cells like this: 

again, by pushing off.

On each 2-cell, have:

Regions of $M \setminus T$ that only intersect 2-cells in strips are I-bundles.

Trivial bundles $\leftrightarrow$ parallel tori ruled out

For nontrivial bundle bounded by T_i, let $T_i' =$ 0-section (Klein bottle)

$T' = T$ with T_i replaced by T_i'.

$M' = M \setminus Nbd(T')$

$\quad = M$ with nontrivial I-bundles deleted.

components of $M' \leq 4(\#\text{2-cells}) = N$

Have:

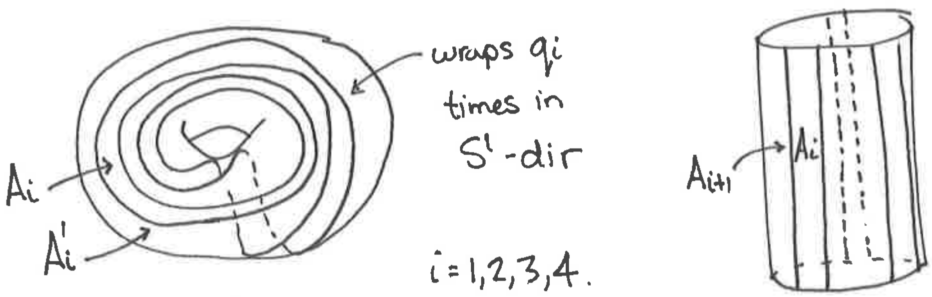
$$\begin{tikzcd}
H_3(M,T';\mathbb{Z}/2) \arrow[r] \arrow[d, phantom, "\cong \text{ excision}"] & H_2(T';\mathbb{Z}/2) \arrow[r] \arrow[d, phantom, "\cong"] & H_2(M;\mathbb{Z}/2) \\
H_3(M',\partial M';\mathbb{Z}/2) & H_2(T;\mathbb{Z}/2) \arrow[d, phantom, "\shortparallel"] & \text{only depends on } M \arrow[u] \\
\text{bounded by } N \text{ i.e. only depends on } M \arrow[u] & \lvert T\rvert &
\end{tikzcd}$$

Thus $|T|$ is bounded by a # only depending on M. $\blacksquare$

NON-UNIQUENESS OF TORUS DECOMPS.

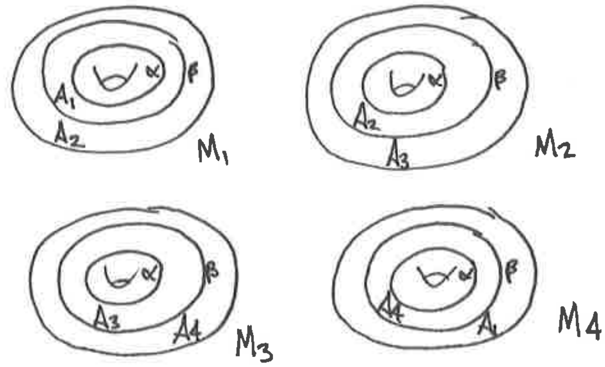
Will construct M with two very different torus decomps.

M_i =



Glue A_i' to A_{i+1} mod 4.

Simplified picture:



$T_1 = A_1 \cup A_3 \qquad M \mid T_1$ is $M_1 \cup M_2 \amalg M_3 \cup M_4$

$T_2 = A_2 \cup A_4 \qquad M \mid T_2$ is $M_2 \cup M_3 \amalg M_4 \cup M_1$

Can show: M irred

T_i incompressible

$M \mid T_i$ atoroidal.

But: the two decompositions are very different.

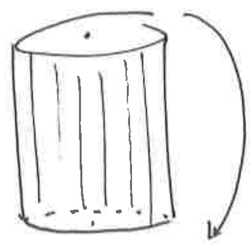
Van Kampen $\Longrightarrow$ $\pi_1(M_i \cup M_{i+1}) = \langle x_i, x_{i+1} \mid x_i^{q_i} = x_{i+1}^{q_{i+1}} \rangle$

These groups all different. The center is $\langle x_i^{q_i} \rangle$ and if we mod out we get $\mathbb{Z}/q_i * \mathbb{Z}/q_{i+1}$

Turns out: these are the only types of counterexamples!

Seifert Manifolds

A model Seifert fibering of $S^1 \times D^2$ is the decomp. into circles given by:



A Seifert fibering of a 3-man is a decomp. into disjoint circles so each circle has a nbd that is a model Seifert fibering.

A Seifert manifold is one with a Seifert fibering

↳ multiplicity of a fiber is q.

Collapsing each circle to a pt, get a map $M \to S$ = surface.

Thm. M = closed, or, irred 3-man.
$\exists$ collection T of disjoint incomp. tori s.t.
each component of $M \setminus T$ is either ① atoroidal, or
② Seifert

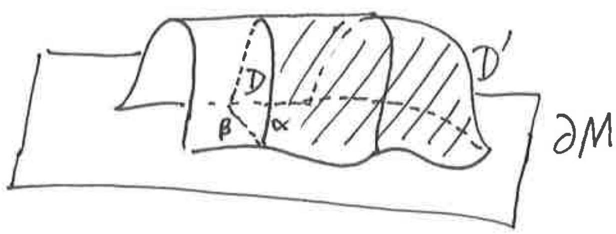
A minimal such collection is unique up to isotopy.

Uniqueness of Torus Decomps

∂-incompressible surfaces

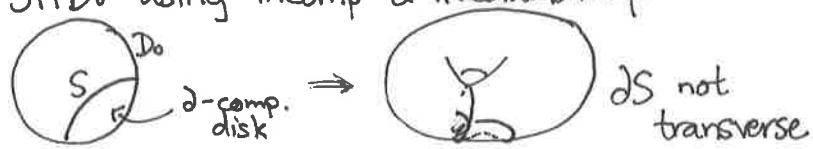
$S \subset M$ is ∂-incomp. if $\forall\, D \subseteq M$ s.t. $\partial D = \alpha \cup \beta$, $D \cap S = \alpha$, $D \cap \partial M = \beta$

$\exists\, D' \subset S$ with $\alpha \subseteq \partial D'$, $\partial D' - \alpha \subset \partial S$.



Warmup. The only ∂-incomp, incomp surfaces in $S^1 \times D^2$ are disks isotopic to meridional disks.

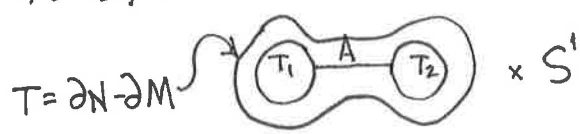
Pf. Let S = connected, incomp, ∂-incomp.
Modify S so ∂S either meridians or transverse to meridians
Make S transverse to D_0 = fixed merid. disk.
Eliminate circles of $S \cap D_0$ using incomp & irreducibility.
Eliminate/rule out



$\Rightarrow S \cap D_0 = \emptyset$.
$\Rightarrow \partial S$ = union of meridian circles.
S incomp. in $M | D_0 = B^3 \Rightarrow S$ = union of disks
By Alexander's thm, a disk with meridional ∂ is isotopic to merid. disk with same ∂. ▨

Key Lemma. M = compact, conn, or., irred, atoroidal, torus boundary
If M contains an incomp., ∂-incomp annulus A
then M is Seifert.

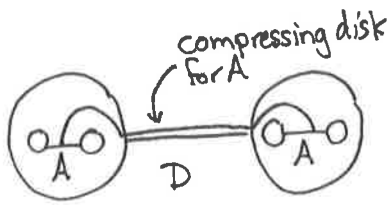
Pf. Assume ∂A in two different tori (other case similar), say T_1 & T_2
Let $N = \mathrm{Nbd}(A \cup T_1 \cup T_2)$:



Seifert fibered!

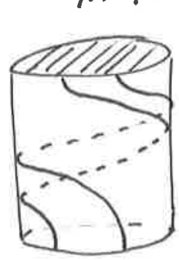
M atoroidal $\Rightarrow$ T either ① ∂ parallel, or
② compressible

In case ① $M \cong T$, so M is Seifert.
Now case ②. Let D = compressing disk
$\rightsquigarrow$ ∂D = nontrivial loop in T
Clearly $D \not\subseteq N$ (look at picture, or use π_1,
or Prop 1.13(a) in AH).
$\Rightarrow D \cap N = \partial D$.
Surgering T along D $\rightsquigarrow$ sphere
$\rightsquigarrow$ ball B (irreducibility)
B outside N since $N \neq$ solid torus.



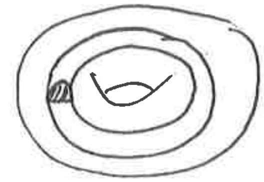
$\Rightarrow M - N$ = solid torus
Claim: ∂D not ~~meridional~~ fibers in $T \subseteq N$
Pf. If it were, would give compressing disk for A.
Thus, S^1-fibers of N wrap at least once around
S^1-dir of $M - N = D^2 \times S^1$
$\rightsquigarrow$ can extend Seifert fibering from N to $M-N$.
$\Rightarrow$ M Seifert fibered. ∎

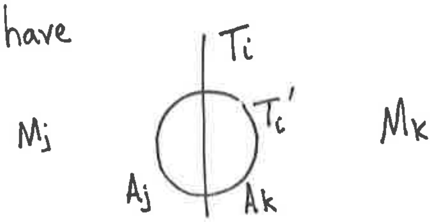
M-N



Thm (Uniqueness of Torus decomp) M = closed, or., irred. 3-man.
$\exists$ collection T of disjoint incomp tori s.t.
each component of $M \setminus T$ is either ① atoroidal or
② Seifert
A minimal such collection is unique up to isotopy.

Pf of uniqueness.
Say $T = T_1 \cup \cdots \cup T_m \longrightarrow$ split into M_j $\quad m, n \neq 0$.
$T' = T_1' \cup \cdots \cup T_n' \longrightarrow$ split into M_j'
Make transverse
Eliminate:  push off.

So components of $T_i' \cap M_j$ are tori, annuli.
Annuli. Annulus components are incomp since the T_i' are.
If have ∂-incomp annulus: 
the annulus is ∂-parallel
$\rightsquigarrow$ push off. (AH Lemma 1.10)
Now have



where A_j, A_k incomp. ∂-incomp.

(assume $M_j \neq M_k$ for simplicity).
Key Lemma $\Rightarrow M_j, M_k$ Seifert.
To show: can make the Seifert fiberings agree along T_i
$\rightsquigarrow T_i$ can be removed.

So $T \cap T' = \emptyset$.

Now assume $T \cap T' = \emptyset$.

If any T_i lies in M_j' then M_j' toroidal, hence Seifert fibered.

<u>Fact</u>. A surface in a Seifert man. is either isotopic to a horizontal one or a vertical one.

$\partial M_j' \neq \emptyset \Rightarrow T_i$ <u>vertical</u>.

Suppose $T_1' \subseteq M_j$. Want to argue the two sides of T_1' have compatible fiberings, so T_1' can be deleted.

Call the two sides M_k', M_ℓ'.

- If $\exists\, T_i \subseteq M_k'$ then $M_k' =$ Seifert as above $\Longrightarrow M_j \cap M_k'$ has two Seifert fiberings, from M_j & M_k'

  Seifert fiberings are (almost always) unique, so fibering of M_k' compatible with M_j.

- If no $T_i \subseteq M_k'$ then $M_k' \subseteq M_j$ and so M_k' again has fibering from M_j.

Same for M_ℓ'. So $M_k' \cup M_\ell'$ has fibering from M_j $\rightsquigarrow T_1'$ can be deleted. $\blacksquare$