Seifert Manifolds

$S^1$-bundles

A manifold $M$ is an $S^1$-bundle over a manifold $B$ if there is $p: M \to B$ and $B$ covered by $U$ with $p^{-1}(U) \cong U \times S^1$.

Prop. $B$ = orientable surface

$\forall \kappa \in \mathbb{Z}$ \exists! $S^1$-bundle $M_\kappa \to B$

s.t. $k = i(B, B)$ in $M_\kappa$.

(\text{so } k = 0 \iff M_\kappa \text{ has section})

Construction of $M_\kappa$. Let $B^0 = B \setminus \text{open disk}$

$M_\kappa = B^0 \times S^1$

$s: B^0 \to M_\kappa$ any section.

Glue $D^2 \times S^1$ so $s(\partial B^0)$ wraps $k$ times around $S^1$-dir.

\text{e.g. } B = S^2, \ k = \pm 1 \to \text{Hopf fibration of } S^3.$

Model Seifert manifolds

$B$ = compact surface, maybe not orient.

$B^0 = B \setminus \text{several open disks}$

$M^0 = \text{orientable } S^1$-bundle over $B^0$ (twisted over 1-sided loops).

$s$ = section \ (regard $M^0$ as two orientable $I$-bundles glued on $\partial I$ by $i^d$).

On each $T^2$ boundary, $s(\partial B^0) = 0$-curve fiber = $\infty$-curve

Glue $S^1 \times D^2$ to $i^{th}$ $T^2$ sending meridian to $S^1$-curve.
The $S^3$-fibration extends to Seifert fibering

Note: $\sigma_i \in \mathbb{Z}$ means the meridian hits $\partial\Sigma \sigma_{b_0}$ $\sigma_i$ times fiber $1$ time.

as in construction of $M_k$.

so $\sigma_i \in \mathbb{Z}$ $\iff$ locally have $S^3$-bundle (as opposed to Seifert).

$\rightarrow$ model $M(\pm \sigma, b; \sigma_1, \ldots, \sigma_k)$

$\uparrow$

$\Rightarrow$ gluing slopes

$\downarrow$

$\#$ boundary

$\circ$ genus

orientable or not

Prop. Every orientable Seifert manifold is $\cong$ to one of the models.

Further $M(\pm \sigma, b; \sigma_1, \ldots, \sigma_k) \cong M(\pm \sigma, b; \sigma'_1, \ldots, \sigma'_k)$

iff the following hold

1. $\sigma_i \equiv \sigma'_i \mod 1 \quad \forall i$

2. $b > 0$ or $\Sigma \sigma_i = \Sigma \sigma'_i$ (euler number).

Prop. $M(\pm \sigma, b; \sigma_i)$ has a section iff $b > 0$ or $\Sigma \sigma_i = 0$.

Examples: Lens spaces

$T, T'$ solid tori

meridian of $T = \infty$-curve, longitude $\sigma$ $0$-curve.

glue meridian of $T'$ to $plq$ curve in $T$

$\cong$ Lens space $L_{plq}$

As quotient of $S^3$:

slope $p$ curves invariant

$\rightarrow$ longitudes on quotient.
Proof of classification of Seifert man's in terms of models

\[ M = \text{Seifert} \]
\[ M^o = M \setminus \text{nbds of special fibers} \]
\[ \rightarrow S^1 \rightarrow M^o \rightarrow B^o \]
Let \( s : B^o \rightarrow M^o \) section.
\[ \rightarrow s(\partial B^o) = \text{circles of slope 0 in } \partial M^o = \mathbb{R}^2 \]
fibers = circles of slope \( \infty \).
\[ \rightarrow \text{slopes } s_i \text{ for gluing the Seifert fibred pieces back.} \]

Changing the \( s_i \) by twisting:
\[ a = \text{arc connecting } \partial B^o \]
\[ \text{replace } \]
\[ \text{with } \]
\[ \text{m times} \]
\[ \text{changes } s_i \rightarrow s_i + m \text{ at one end} \]
\[ s_j \rightarrow s_j - m \text{ at other} \]

So if \( b \neq 0 \) can connect one end of \( a \) to \( \partial M \), modifying
one \( s_i \) by \( m \).

Remains to check: any two sections differ by these twist moves. Indeed, cut \( \partial B^o \) along arcs to get a disk.
Away from arcs, one choice of section. Near arcs, only have twisting.

\[ \Box \]
Classification of Seifert Fiberings

Thus. Seifert fiberings of orientable Seifert manifolds are unique up to isomorphism, except:
(a) $M(0, 1; \alpha/\beta)$ the fiberings of $S^1 \times D^2$
(b) $M(0, 1; 1/2, 1/2) = M(-1, 1; )$ fiberings of $S^1 \times S^1 \times I$
(c) $M(0, 0; s_1, s_2)$ various fiberings of $S^3, S^1 \times S^2$, lens sp
(d) $M(0, 0; 1/2, -1/2, \alpha/\beta) = M(-1, 0; \beta/\alpha) \quad \alpha, \beta \neq 0.$
(e) $M(0, 0; 1/2, 1/2, -1/2, -1/2) = M(-2, 0)$ fiberings of $S^1 \times S^1 \times S^1$

The two fiberings of $S^1 \times S^1 \times I$.

Let $f: S^1 \times I \to S^1 \times I$ reflection in both factors.

$f$ has 2 fixed pts

$S^1 \times S^1 \times I$ is mapping torus:

fibring by horizontals has two special fibers.
fibring by verticals has no special fibers.

Note $c, d, e$ come from $a, b$: specifically, the fiberings in $c$ come from different fiberings in $a$, $d$ comes from gluing a model solid torus to $b$ and $e$ is the double of $b$. 