SEIFERT MANIFOLDS

S^1 - bundles

A manifold M is an S^1-bundle over a manifold B if there is $p: M \to B$ and B covered by U with $p^{-1}(U) \cong U \times S^1$.

e.g. T^2, Klein bottle

<u>Prop</u>. B = closed orientable surface

$\forall\, k \in \mathbb{Z}$ $\exists!$ S^1-bundle $M_k \to B$

s.t. $k = i(B,B)$ in M_k.

(so $k = 0 \iff M_k$ has section)

Construction of M_k. Let $B^\circ = B \setminus$ open disk

$M_k^\circ = B^\circ \times S^1$

$s: B^\circ \to M_k^\circ$ any section.

Glue $D^2 \times S^1$ so $s(\partial B^\circ)$ wraps k times around S^1-dir.

e.g. $B = S^2$, $k = \pm 1 \rightsquigarrow$ Hopf fibration of S^3.

Model Seifert manifolds

B = compact surface, maybe not orient.

$B^\circ = B \setminus$ several open disks

M° = orientable S^1-bundle over B° (twisted over 1-sided loops).

s = section (regard M° as two orientable I-bundles glued on ∂I by id).

On each T^2 boundary, $s(\partial B^\circ) = 0$-curve fiber $= \infty$-curve

Glue $S^1 \times D^2$ to i^{th} T^2 sending meridian to s_i-curve.

The S^1 - fibering extends to Seifert fibering

[Note: $s_i \in \mathbb{Z}$ means the meridian hits $s(\partial B_0)$ s_i times
fiber 1 time.

as in construction of M_K.

so $s_i \in \mathbb{Z} \iff$ locally have S^1-bundle (as opposed to Seifert).]

$\rightsquigarrow$ model $M(\pm g, b\,;\, s_1, \ldots, s_k)$

- $s_1, \ldots, s_k$: gluing slopes
- b: # boundary
- g: genus
- $\pm$: orientable or not

Prop. Every orientable Seifert manifold is $\cong$ to one of the models.
Further $M(\pm g, b\,;\, s_1, \ldots, s_k) \overset{o.p.}{\cong} M(\pm g, b\,;\, s_1', \ldots, s_k')$
iff the following hold ① $s_i \equiv s_i' \mod 1 \quad \forall\, i$
② $b > 0$ or $\sum s_i = \sum s_i'$ (euler number).

Prop. $M(\pm g, b\,;\, s_i)$ has a section iff $b > 0$ or $\sum s_i = 0$.

Examples: Lens spaces

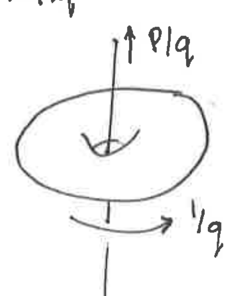
T, T' solid tori

meridian of T = ∞ - curve, longitude = 0-curve.

glue meridian of T' to p/q curve in T

$\rightsquigarrow$ Lens space $L_{p/q}$

As quotient of S^3:



slope p curves invariant
$\rightsquigarrow$ longitudes on quotient.

Proof of classification of Seifert man's in terms of models

M = Seifert

$M^\circ = M \setminus$ nbds of special fibers

$\rightsquigarrow S^1 \to M^\circ \to B^\circ$

Let $s : B^\circ \to M^\circ$ section.

$\rightsquigarrow s(\partial B^\circ)$ = circles of slope 0 in $\partial M^\circ = \coprod T^2$

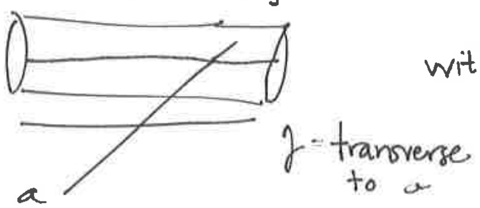
fibers = circles of slope ∞.

$\rightsquigarrow$ slopes s_i for gluing the Seifert fibered pieces back.

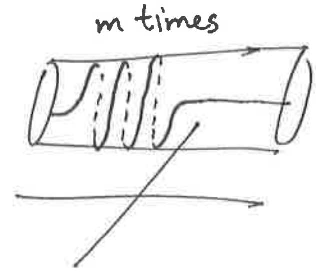
Changing the s_i by twisting:

a = arc connecting ∂B°

replace



with



Changes $s_i \to s_i + m$ at one end (the basis $(1,0), (0,1)$ gets replaced with $(1,m), (0,1)$)

$s_j \to s_j - m$ at other.

So if $b \neq 0$ can connect one end of a to ∂M, modifying one s_i by m.

Remains to check: any two sections differ by these twist moves. Indeed, cut ∂B° along arcs to get a disk. Away from arcs, one choice of section. Near arcs, only have twisting. $\blacksquare$

Classification of Seifert Fiberings

Thm. Seifert fiberings of orientable Seifert man's are unique up to isomorphism, except:

(a) $M(0,1;\alpha/\beta)$ the fiberings of $S^1 \times D^2$

(b) $M(0,1;1/2,1/2) = M(-1,1;)$ fiberings of $S^1 \tilde{\times} S^1 \tilde{\times} I$

(c) $M(0,0;s_1,s_2)$ various fiberings of S^3, $S^1 \times S^2$, lens sp

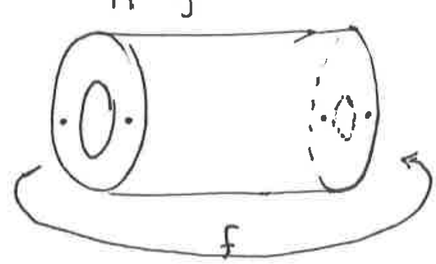
(d) $M(0,0;1/2,-1/2,\alpha/\beta) = M(-1,0;\beta/\alpha)$ $\alpha,\beta \neq 0$.

(e) $M(0,0;1/2,1/2,-1/2,-1/2) = M(-2,0)$ fiberings of $S^1 \tilde{\times} S^1 \tilde{\times} S^1$

The two fiberings of $S^1 \tilde{\times} S^1 \tilde{\times} I$.

Let $f: S^1 \times I \to S^1 \times I$ reflection in both factors.
f has 2 fixed pts

$S^1 \tilde{\times} S^1 \tilde{\times} I$ is mapping torus:



fibering by horizontals has two special fibers.
fibering by verticals has no special fibers.

Note c,d,e come from a,b: specifically the fiberings in c come from different fiberings in a, d comes from gluing a model solid torus to b and e is the double of b.